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Stability analysis of relaxed Dirichlet boundary control problems

by

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Abstract: We study the relaxation by Young measures of a Dirichlet control problem with pointwise state constraints. We give a necessary and sufficient condition for the properness of the relaxation. This condition is expressed in terms of stability properties, for the original control problem, with respect to geometrical perturbations of state constraints.

Keywords: Control problems, pointwise state constraints, Dirichlet boundary control, relaxation, Young measures, stability properties

1. Introduction

This paper is concerned with the relaxation by Young measures of Dirichlet boundary control problems for semilinear parabolic equations. For problems with pointwise state constraints, it is now well known that properness of the relaxation by Young measures is closely related to some stability property of the original problem, with respect to perturbations of the state constraints (Arada, Raymond, 1998, Casas, 1996, Roubiček, 1990, 1997, Dontchev, Mordukhovich, 1983). More precisely, consider the following abstract control problem

$$\inf\{J(y,\pi) \mid (y,\pi) \in Y \times \Pi_{ad},\ G(y,\pi) = 0,\ g(y) \in \mathcal{C}\}, \qquad (\mathcal{P})\ (1)$$

where Y is the space of state variables, Π_{ad} is the set of admissible controls, G is a nonlinear operator and $G(y,\pi) = 0$ stands for the state equation, g is a mapping from Y into a Banach space Z, and $\mathcal{C}$ is a nonempty set in Z. Typically, for problems considered here $Y \equiv C_b(Q)$, where $Q = \Omega\times]0,T[$, Ω is a bounded domain in $I\!R^N$. The relaxed problem corresponding to $(\mathcal{P})$ is defined by Young measures, and it may be written in the form

$$\inf\{J(y,r) \mid (y,r) \in Y \times \mathcal{R}_{ad},\ G(y,r) = 0,\ g(y) \in \mathcal{C}\}, \qquad (R\mathcal{P})(2)$$

where $\mathcal{R}_{ad}$ is the set of relaxed controls. For $\delta > 0$, we also consider the problem

$$\inf\{J(y,\pi) \mid (y,\pi) \in Y \times \Pi_{ad},\ G(y,\pi) = 0,$$

In Arada, Raymond (1998), Casas (1996), it is proven that the relaxation is proper (that is $\inf(\mathcal{P}) = \inf(R\mathcal{P})$) if, and only if,

$$\inf(\mathcal{P}) = \lim_{\delta \searrow 0} \inf(\mathcal{P}_\delta).$$

This result is based on the continuity of the mapping $r \longrightarrow (y_r, J(y_r, r))$ from $(\mathcal{R}_{ad}, \|\cdot\|_w)$ into $Y \times I\!R$, where $\|\cdot\|_w$ is the norm associated with the weak-star topology, Warga (1972) (y_r is the relaxed trajectory corresponding to r, that is the solution of $G(y, r) = 0$). For Dirichlet control problems considered here, the mapping $r \longrightarrow y_r$ is not continuous from $(\mathcal{R}_{ad}, \|\cdot\|_w)$ into $Y \equiv C_b(Q)$, but is continuous for topologies weaker than the one of Y. We exploit this behavior in the following manner. Consider a family of seminorms $(\|\cdot\|_\tau)_{\tau>0}$ defined on Y. For each $\tau > 0$, we define the semidistance to $\mathcal{C}$: $d_{\mathcal{C},\tau} : \phi \longrightarrow d_{\mathcal{C},\tau}(\phi) = \inf_{z\in\mathcal{C}} \|z - \phi\|_\tau$. Suppose that the family $(\|\cdot\|_\tau)_{\tau>0}$ satisfies the following properties:

– The mapping $r \longrightarrow y_r$ is continuous from $(\mathcal{R}_{ad}, \|\cdot\|_w)$ into $(Y, \|\cdot\|_\tau)$, for all $\tau > 0$,

– $\lim_{\tau \searrow 0} d_{\mathcal{C},\tau}(\phi) = d_{\mathcal{C}}(\phi)$ for all $\phi \in Y$.

We introduce a family of perturbed problems

$$\inf\{J(y, \pi) \mid (y, \pi) \in Y \times \Pi_{ad}, G(y, \pi) = 0, d_{\mathcal{C},\tau}(g(y)) \le \delta\}. \qquad (\mathcal{P}_{\delta,\tau})(4)$$

In our case, $\|\cdot\|_\tau = \|\cdot\|_{C(\bar{Q}^\tau)}$, where Q^τ is a subcylinder strictly contained into the full cylinder $Q = \Omega \times]0, T[$ (see the precise definition in Section 2.1). We prove (in Theorem 7.2) that the relaxation of (P) is proper if, and only if, one of the two following equivalent properties are satisfied

$$\inf(\mathcal{P}) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(\mathcal{P}_{\delta,\tau}),\ \inf(\mathcal{P}) = \lim_{\tau \searrow 0} \lim_{\delta \searrow 0} \inf(\mathcal{P}_{\delta,\tau}).$$

The idea of the proof is the following. We first prove the compactness of the sets of relaxed trajectories and admissible relaxed trajectories, for the topology of $C(\bar{Q}^\tau)$, for every $\tau > 0$ (Propositions 5.2 and 5.3). With this compactness property, we prove that the relaxed control problem $(R\mathcal{P})$ admits solutions (Theorem 5.5) and satisfies some interesting stability conditions (Theorem 5.6 and Theorem 5.7).

Next, we prove that the set of classical trajectories is dense in the set of relaxed trajectories, for the topology of $C(\bar{Q}^\tau)$, for every $\tau > 0$ (Proposition 6.1). We establish that the set of admissible relaxed trajectories for $(R\mathcal{P})$ is included in the closure in $C(\bar{Q}^t)$ of the set of classical admissible trajectories for $(\mathcal{P}_{\delta,\tau})$, for every $t > 0$ (Proposition 6.2). The connection of these compactness and denseness results gives the necessary and sufficient conditions for the properness

2. Setting of the control problem

Consider the semilinear parabolic equation:

$$\frac{\partial y}{\partial t} + Ay + \Phi(\cdot, y, u) = 0 \text{ in } Q,\ y = v \text{ on } \Sigma,\ y(0) = y_0 \text{ in } \Omega, \tag{5}$$

where $Q = \Omega \times]0, T[$, Ω is a bounded domain in $I\!R^N$, $\Sigma = \Gamma \times]0, T[$, Γ is the boundary of Ω. The distributed control u belongs to $L^\infty(Q)$, the boundary control v belongs to $L^\infty(\Sigma)$, the initial condition y_0 is fixed and belongs to $L^\infty(\Omega)$, and A is a second order elliptic operator of the form $Ay(x) = -\sum_{i,j=1}^N D_i(a_{ij}(x) D_j y(x))$ (where D_i denotes the partial derivative with respect to x_i). Set the following state and control constraints

$$g(y) \in \mathcal{C}, \tag{6}$$

$$\begin{aligned} u \in \mathcal{U}_{ad} &= \{u \in L^\infty(Q) \mid u(x,t) \in K_U(x,t) \subset U \text{ for almost all } (x,t) \in Q\}, \\ v \in \mathcal{V}_{ad} &= \{v \in L^\infty(\Sigma) \mid v(s,t) \in K_V(s,t) \subset V \text{ for almost all } (s,t) \in \Sigma\}, \end{aligned}$$

where g is a mapping from $C_b(Q \cup \Omega_T)$ into $C_b(Q \cup \Omega_T)$, $\mathcal{C} \subset C_b(Q \cup \Omega_T)$ is a closed convex subset with a nonempty interior in $C_b(Q \cup \Omega_T)$, $K_U(\cdot)$ and $K_V(\cdot)$ are measurable multimappings with nonempty and closed values in $\mathcal{P}(I\!R)$, U and V are compact subsets in $I\!R$. The paper is concerned with the following control problem

$$\begin{aligned} &\inf\{J(y,u,v) \mid (y,u,v) \text{ belongs to } C_b(Q \cup \Omega_T) \times \mathcal{U}_{ad} \times \mathcal{V}_{ad}, \\ &\text{and satisfies (5) and (6)}\}, \end{aligned} \tag{P) (7}$$

where the cost functional is defined by

$$\begin{aligned} J(y,u,v) &= \int_Q F(x,t,y(x,t),u(x,t))\,dx\,dt + \int_\Sigma G(s,t,v(s,t))\,ds\,dt \\ &+ \int_\Omega L(x, y(x,T))\,dx. \end{aligned}$$

The set $\mathcal{U}^r_{ad}$ of distributed relaxed controls consists of weak-star measurable functions from Q into the space of Radon probability measures on U (see the precise definition in Section 3). The set $\mathcal{V}^r_{ad}$ of boundary relaxed controls consists of weak-star measurable functions from Σ into the space of Radon probability measures on V. The relaxed state equation is given by

$$\frac{\partial y}{\partial t} + Ay + \Phi(\cdot, y, \sigma^{(\cdot)}) = 0 \text{ in } Q,\ y = I_V(\omega^{(\cdot)}) \text{ on } \Sigma,\ y(0) = y_0 \text{ in } \Omega, \tag{8}$$

where $\Phi(x,t,y,\sigma^{(x,t)}) = \int_U \Phi(x,t,y,\lambda) d\sigma^{(x,t)}(\lambda)$, and $I_V(\omega^{(s,t)}) = \int_V \lambda d\omega^{(s,t)}(\lambda)$. The relaxed control problem is defined in the following way

$$\inf\{J(y,\sigma,\omega) \mid (y,\sigma,\omega) \in C_b(Q \cup \Omega_T) \times \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}, \tag{9}$$

where the relaxed cost functional is

$$
\begin{aligned}
&J(y,\sigma,\omega) = \\
&\int_Q F(x,t,y,\sigma^{(x,t)})\,dx\,dt + \int_\Sigma G(s,t,\omega^{(s,t)})\,ds\,dt + \int_\Omega L(x,y(T))\,dx, \\
&= \int_Q \int_U F(x,t,y(x,t),\lambda)d\sigma^{(x,t)}(\lambda)\,dx\,dt + \int_\Sigma \int_V G(s,t,\lambda)d\omega^{(s,t)}(\lambda)\,ds\,dt \\
&+ \int_\Omega L(x,y(x,T))\,dx.
\end{aligned}
$$

2.1. Assumptions

Throughout the sequel, Ω is a bounded open and connected subset in $I\!R^N (N \geq 2)$ of class $C^{2+\bar{\gamma}}$, for some $0 < \bar{\gamma} \leq 1$. The coefficients a_{ij} of the operator A belong to $C^{1+\bar{\gamma}}(\overline{\Omega})$ and satisfy the condition:

$$
a_{ij}(x) = a_{ji}(x) \text{ for every } i,j \in \{1,...,N\},\ m_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j,
$$

for every $\xi \in I\!R^N$ and every $x \in \overline{\Omega}$, with $m_0 > 0$. The conormal derivative of y with respect to A is denoted by $\frac{\partial y}{\partial n_A}$, that is

$$
\frac{\partial y}{\partial n_A}(s,t) = \sum_{i,j} a_{ij}(s) D_j y(s,t) n_i(s),
$$

where $n = (n_1, \cdots, n_N)$ is the unit normal to Γ outward Ω. We set $\overline{\Omega}_0 = \overline{\Omega} \times \{0\}$, $\overline{\Omega}_T = \overline{\Omega} \times \{T\}$, $Q_{\tau T} = \Omega \times]\tau, T[$, $\Omega_\tau = \{x \in \Omega \mid d(x,\Gamma) > \tau\}$ (d is the Euclidean distance), $Q^\tau = \Omega_\tau \times]\tau, T[$, for every $\tau > 0$. Denote by y_{uv} (resp. $y_{\sigma\omega}$) the solution of (5) corresponding to (u,v) (resp. the solution of (8) corresponding to (σ,ω)), and let

$$
\mathcal{Y} = \{y_{uv} \mid (u,v) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}\} \text{ and } \mathcal{Y}^r = \{y_{\sigma\omega} \mid (\sigma,\omega) \in \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}\}
$$

be the sets of classical and relaxed trajectories. For $\tau \in]0,T[$, $\delta \geq 0$ and $\phi \in C_b(Q \cup \Omega_T)$, set

$$
d_{\mathcal{C}}(\phi) = \inf_{z \in \mathcal{C}} ||\phi - z||_{C_b(Q \cup \Omega_T)}, d_{\mathcal{C},\tau}(\phi) = \inf_{z \in \mathcal{C}} ||\phi - z||_{C(\bar{Q}^\tau)},
$$

and denote by

$$
\begin{aligned}
&\mathcal{Y}_{ad}(\delta) = \{y_{uv} \mid d_{\mathcal{C}}(g(y_{uv})) \leq \delta, (u,v) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}\}, \\
&\mathcal{Y}_{ad}(\delta,\tau) = \{y_{uv} \mid d_{\mathcal{C},\tau}(g(y_{uv})) \leq \delta, (u,v) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}\},
\end{aligned}
$$

the sets of admissible classical trajectories for (P_δ) and for $(P_{\delta,\tau})$. In the same manner, $\mathcal{Y}^r_{ad}(\delta)$ (resp. $\mathcal{Y}^r_{ad}(\delta,\tau)$) stands for the set of admissible relaxed trajecin place of $\mathcal{Y}_{ad}(0)$ (resp. $\mathcal{Y}^r_{ad}(0)$). As usually, we define the set of solutions for (P) by

$$\operatorname{Arg\,inf}(P) = \{(y, u, v) \in \mathcal{Y}_{ad} \times \mathcal{U}_{ad} \times \mathcal{V}_{ad} \mid J(y, u, v) = \inf(P)\}.$$

The notation $\operatorname{Arg\,inf}(RP)$ stands for the set of solutions for (RP).

A1 - Φ is a Carathéodory function from $Q \times I\!R^2$ into $I\!R$. For almost every $(x, t) \in Q$ and every $u \in I\!R$, $\Phi(x, t, \cdot, u)$ is of class C^1. The following estimates hold

$$|\Phi(x, t, 0, u)| \leq \Phi_1(x, t) + C_1|u|, \; 0 \leq \Phi'_y(x, t, y, u) \leq (\Phi_1(x, t) + C_1|u|)\eta(|y|),$$

where $\Phi_1 \in L^\infty(Q)$, $C_1 > 0$, η is a nondecreasing function from $I\!R^+$ into $I\!R^+$.

A2 - F is a Carathéodory function from $Q \times I\!R^2$ into $I\!R$. The following estimates hold

$$|F(x, t, y, u)| \leq F_1(x, t)\eta(|u|)\eta(|y|),$$
$$|F(x, t, y, u) - F(x, t, z, u)| \leq F_1(x, t)\eta(|u|)\eta(|y|)\eta(|z|)\zeta(|y - z|),$$

where $F_1 \in L^1(Q)$, η is defined as in **A1**, and ζ is an increasing continuous function from $I\!R^+$ into $I\!R^+$ such that $\zeta(0) = 0$.

A3 - G is a Carathéodory function from $\Sigma \times I\!R$ into $I\!R$. The following estimate holds

$$|G(s, t, v)| \leq G_1(s, t)\eta(|v|)$$

where

$$G_1 \in L^1(\Sigma).$$

A4 - L is a Carathéodory function from $\Omega \times I\!R$ into $I\!R$. The following estimate holds

$$|L(x, y)| \leq L_1(x)\eta(|y|),$$

where $L_1 \in L^1(\Omega)$, and η is as in **A1**.

A5 - The infimum of (P) is finite (that is, there exists at least one admissible triplet (y, u, v)).

3. Relaxed controls

Recall that U is a compact subset of $I\!R$. Let $\mathcal{M}(U)$ be the space of Radon measures on U and $L^\infty_w(Q; \mathcal{M}(U))$ be the space of weak-star measurable functions Q from Σ into $\mathcal{M}(U)$, satisfying

$$\operatorname{ess\ sup}\ |\sigma^{(x,t)}|(U) < \infty.$$

where $|\sigma^{(x,t)}|$ denotes the total variation of the measure $\sigma^{(x,t)}$. The space $L^\infty_w(Q;\mathcal{M}(U))$ is a Banach space for the norm $||\sigma||_Q = \text{ess}\sup_{(x,t)\in Q}\left(|\sigma^{(x,t)}|(U)\right)$ and can be identified with the dual space of $L^1(Q;C(U))$ by associating with each $\sigma \in L^\infty_w(Q;\mathcal{M}(U))$ the continuous linear form on $L^1(Q;C(U))$ defined by:

$$\sigma : \phi \longrightarrow \int_Q \phi(x,t,\sigma^{(x,t)})\,dx\,dt = \int_Q\int_U \phi(x,t,\lambda)d\sigma^{(x,t)}(\lambda)\,dx\,dt.$$

The set of admissible distributed relaxed controls is defined by

$$\mathcal{U}^r_{ad} = \{\sigma \in L^\infty_w(Q;\mathcal{M}^+(U)) \mid \int_U d\sigma^{(x,t)}(\lambda) = 1,$$
$$\text{supp}\,\sigma^{(x,t)} \subset K_U(x,t) \text{ a.e. in } Q\},$$

where supp $\sigma^{(x,t)}$ denotes the support of the measure $\sigma^{(x,t)}$. Observe that $\mathcal{U}_{ad}$ can be considered as a subset of $\mathcal{U}^r_{ad}$. Indeed every $u \in \mathcal{U}_{ad}$ can be identified with the distributed relaxed control $\delta_{u(\cdot)}$, where δ_u denotes the Dirac measure concentrated at u. Since $K_U : Q \longrightarrow U$ is a measurable multimapping with nonempty and compact values, $\mathcal{U}^r_{ad}$ is convex, compact and sequentially compact for the weak-star topology of $L^\infty_w(Q;\mathcal{M}(U))$. Moreover, $\mathcal{U}^r_{ad}$ is the closure of $\mathcal{U}_{ad}$ for this topology (see Warga, 1972, and Ball, 1989). In the same way, we define the set of admissible boundary relaxed controls

$$\mathcal{V}^r_{ad} = \{\omega \in L^\infty_w(\Sigma;\mathcal{M}^+(V)) \mid \int_V d\omega^{(s,t)}(\lambda) = 1,$$
$$\text{supp}\,\omega^{(s,t)} \subset K_V(s,t) \text{ a.e. in } \Sigma\}.$$

The set $\mathcal{V}^r_{ad}$ is convex, sequentially compact for the weak-star topology of $L^\infty_w(\Sigma;\mathcal{M}(V))$, and is the closure of $\mathcal{V}_{ad}$ for this topology. We also set

$$||\omega||_\Sigma = \text{ess}\sup_{(s,t)\in\Sigma}\left(|\omega^{(s,t)}|(V)\right).$$

4. Relaxed state equation

In this section, we recall some existence, uniqueness and regularity results for the relaxed state.

DEFINITION 4.1 *A function $y \in L^1(Q)$ is a weak solution of (8) if, and only if, $\Phi(\cdot,y,\sigma) \in L^1(Q)$ and*

$$\int_Q y(-\frac{\partial z}{\partial t} + Az)\,dx\,dt + \int_Q \Phi(x,t,y,\sigma^{(x,t)})z\,dx\,dt =$$
$$\int_\Omega y_0 z(0)dx - \int_\Sigma I_V(\omega^{(s,t)})\frac{\partial z}{\partial n_A}\,ds\,dt,$$

The proofs of Theorem 4.2 and Theorem 4.3 can be adapted from the ones given for Theorem 3.9 and Theorem 3.10 in Arada, Raymond (1997A, B).

THEOREM 4.2 *If $\sigma \in L^\infty_w(Q; \mathcal{M}(U))$ and $\omega \in L^\infty_w(\Sigma; \mathcal{M}(V))$, then equation (8) admits a unique weak solution in $L^1(Q)$. This solution belongs to $C_b(Q \cup \Omega_T)$ and satisfies*

$$||y_{\sigma\omega}||_{\infty,Q} \leq C(||\sigma||_Q + ||\omega||_\Sigma + 1),$$

where $C \equiv C(T, \Omega, N)$. Moreover, the mapping $(\sigma, \omega) \longrightarrow y_{\sigma\omega}$ is continuous from $L^\infty_w(Q; \mathcal{M}(U)) \times L^\infty_w(\Sigma; \mathcal{M}(V))$ (endowed with its strong topology) into $C_b(Q \cup \Omega_T)$.

THEOREM 4.3 *Let (σ, ω) be in $L^\infty_w(Q; \mathcal{M}(U)) \times L^\infty_w(\Sigma; \mathcal{M}(V))$ such that $||\sigma||_Q + ||\omega||_\Sigma \leq M$. For all $\tau > 0$, the weak solution $y_{\sigma\omega}$ of (8) corresponding to (σ, ω) is Hölder continuous on $\bar{Q}^\tau$ and satisfies*

$$||y_{\sigma\omega}||_{C^{\nu,\nu/2}(\bar{Q}^\tau)} \leq C(\tau),$$

where $C(\tau) \equiv C(T, \Omega, N, M, \nu, \tau)$.

REMARK 4.4 *If (σ, ω) belongs to $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, then $||\sigma||_Q = ||\omega||_\Sigma = 1$. From Theorem 4.2 and Theorem 4.3, it follows that the solution of (8), corresponding to (σ, ω), satisfies*

$$||y_{\sigma\omega}||_{\infty,Q} \leq C_1 ||y_{\sigma\omega}||_{C^{\nu,\nu/2}(\bar{Q}^\tau)} \leq C_2(\tau),$$

where $C_1 \equiv C_1(T, \Omega, N)$ and $C_2(\tau) \equiv C_2(T, \Omega, N, \nu, \tau)$.

5. Compactness results

5.1. Compactness properties of the relaxed trajectories

In the following theorem, we state a continuity result which is fundamental for the sequel.

THEOREM 5.1 *The mapping $(\sigma, \omega) \longrightarrow y_{\sigma\omega}$ is continuous from $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, endowed with its weak-star topology, into $C(\bar{Q}^\tau)$, for all $\tau \in]0, T[$.*

Proof. Let $(\sigma_n, \omega_n)_n$ be a sequence of relaxed controls converging to (σ, ω) for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$. Let y_n and $y_{\sigma\omega}$ be the solutions of (8), corresponding to (σ_n, ω_n) and (σ, ω). Due to Theorem 4.2 and Theorem 4.3, the sequence $(y_n)_n$ is bounded in $L^\infty(Q)$ and in $C^{\nu,\nu/2}(\bar{Q}^\tau)$, for every $\tau \in]0, T[$ and for some $\nu > 0$. Since the imbedding from $C^{\nu,\nu/2}(\bar{Q}^\tau)$ into $C(\bar{Q}^\tau)$ is compact, converges to y for the weak-star topology of $L^\infty(Q)$, and $(y_n)_n$ converges to y uniformly on $\bar{Q}^\tau$, for every $\tau > 0$. On the other hand, observe that y_n satisfies

$$\int_Q y_n(-\frac{\partial z}{\partial t} + Az)\,dxdt + \int_Q \Phi(x,t,y_n,\sigma_n)z\,dxdt =$$
$$-\int_\Sigma I_V(\omega_n)\frac{\partial z}{\partial n_A}\,dsdt + \int_\Omega y_0 z(0)dx,$$

for every $z \in C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z_{|\Sigma} = 0$. From assumptions on Φ and due to Theorem 4.2, there exists $C > 0$ (independent of n) such that $||\Phi(\cdot, y_n, \sigma_n)z||_{\infty,Q} \leq C$, for every n. Let z be in $C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z_{|\Sigma} = 0$, and let $\epsilon > 0$. It follows that

$$\Big|\int_Q (y_n(-\frac{\partial z}{\partial t} + Az)\,dxdt + \int_{Q^\epsilon} \Phi(x,t,y_n,\sigma_n)z\,dxdt + \int_\Sigma I_V(\omega_n)\frac{\partial z}{\partial n_A}\,dsdt$$
$$-\int_\Omega y_0 z(0)dx\Big|$$
$$\leq \Big|\int_{Q\setminus Q^\epsilon} \Phi(x,t,y_n,\sigma_n)z(x,t)\,dxdt\Big| \leq C\mathcal{L}^{N+1}(Q\setminus Q^\epsilon) \leq C\epsilon.$$

With the convergence of $(\sigma_n, \omega_n)_n$ to (σ, ω) for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$ and the uniform convergence of $(y_n)_n$ to y on $\bar{Q}^\epsilon$, it follows that

$$\Big|\int_Q (y(-\frac{\partial z}{\partial t} + Az)\,dxdt + \int_{Q^\epsilon} \Phi(x,t,y,\sigma)z\,dxdt +$$
$$\int_\Sigma I_V(\omega)\frac{\partial z}{\partial n_A}\,dsdt - \int_\Omega y_0 z(0)dx\Big| \leq C\epsilon.$$

By passing to the limit when ϵ tends to zero, we finally obtain

$$\int_Q y(-\frac{\partial z}{\partial t} + Az)\,dxdt + \int_Q \Phi(x,t,y,\sigma)\,dxdt =$$
$$-\int_\Sigma I_V(\omega)\frac{\partial z}{\partial n_A}\,dsdt + \int_\Omega y_0 z(0)dx,$$

for every $z \in C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z_{|\Sigma} = 0$. Therefore y is the solution of (8) corresponding to (σ, ω). ∎

The result stated below describes the dependence of the relaxed trajectories with respect to the corresponding controls, and gives valuable information about the topological structure of the set of relaxed trajectories.

PROPOSITION 5.2 *$\mathcal{Y}^r$ is sequentially compact for the usual topology of $C(\bar{Q}^\tau)$, for all $\tau \in]0, T[$.*

Proof. Let $(y_n)_n \subset \mathcal{Y}^r$ be a bounded sequence. It corresponds to a sequence there exists a subsequence, still indexed by n, and $(\sigma, \omega) \in \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, such that $(\sigma_n, \omega_n)_n$ weak-star converges to (σ, ω). Due to Theorem 5.1, the sequence $(y_n)_n$ converges to $y_{\sigma\omega}$ uniformly in $\bar{Q}^\tau$, for every $\tau \in]0, T[$. ∎

Now, we are interested in trajectories which are admissible for the relaxed control problem (RP). As shown below, the compactness property for $\mathcal{Y}^r_{ad}$ is inherited from $\mathcal{Y}^r$.

PROPOSITION 5.3 *Let δ be in $I\!R^+$. The set $\mathcal{Y}^r_{ad}(\delta)$ is sequentially compact for the usual topology of $C(\bar{Q}^\tau)$, for all τ in $]0, T[$.*

Proof. Suppose that $\delta = 0$ (the proof is the same for $\delta > 0$). Let $(y_n)_n$ be a sequence in $\mathcal{Y}^r_{ad}$. Since $\mathcal{Y}^r$ is sequentially compact for the topology of $C(\bar{Q}^\tau)$ (for all $\tau \in]0, T[$), there exists $y_{\sigma\omega} \in \mathcal{Y}^r$ (associated with some $(\sigma, \omega) \in \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$) such that $(y_n)_n$ converges to $y_{\sigma\omega}$ uniformly in $\bar{Q}^\tau$, for all $\tau > 0$. Moreover, since $d_{\mathcal{C}}(g(y_n)) = 0$, it follows that $d_{\mathcal{C},\tau}(g(y_n)) = 0$, for all $\tau > 0$. By passing to the limit when n tends to infinity, and then when τ tends to zero, we obtain $d_{\mathcal{C}}(y_{\sigma\omega}) = 0$. Therefore $y_{\sigma\omega}$ belongs to $\mathcal{Y}^r_{ad}$. The proof is complete. ∎

5.2. Existence and stability

This section is devoted to the analysis of the relaxed control problem.

PROPOSITION 5.4 *The mapping $(\sigma, \omega) \longrightarrow J(y_{\sigma\omega}, \sigma, \omega)$ is sequentially continuous from $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, endowed with the weak-star topology, into $I\!R$ ($y_{\sigma\omega}$ is the solution of (8) corresponding to (σ, ω)).*

Proof. Let $(\sigma_n, \omega_n)_n$ be a sequence in $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$ converging to (σ, ω) for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$. Let y_n and $y_{\sigma\omega}$ be the solutions of (8), corresponding to (σ_n, ω_n) and (σ, ω). Notice that

$$
\begin{aligned}
&J(y_n, \sigma_n, \omega_n) - J(y_{\sigma\omega}, \sigma, \omega) \\
&= \int_Q \Big(F(x, t, y_{\sigma\omega}(x,t), \sigma_n^{(x,t)}) - F(x, t, y_{\sigma\omega}(x,t), \sigma^{(x,t)}) \Big)\, dx\, dt \\
&+ \int_\Sigma \Big(G(s, t, \omega_n^{(s,t)}) - G(s, t, \omega^{(s,t)}) \Big)\, ds\, dt + \int_\Omega \Big(L(x, y_n(T)) - L(x, y(T)) \Big)\, dx \\
&+ \int_Q \Big(F(x, t, y_n(x,t), \sigma_n^{(x,t)}) - F(x, t, y_{\sigma\omega}(x,t), \sigma_n^{(x,t)}) \Big)\, dx\, dt.
\end{aligned}
$$

From assumptions on F and G, and since $(\sigma_n, \omega_n)_n$ converges to (σ, ω) for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, we deduce that

$$
\lim_{n \to \infty} \int_Q F(x, t, y_{\sigma\omega}(x,t), \sigma_n^{(x,t)})\, dx\, dt = \int_Q F(x, t, y_{\sigma\omega}(x,t), \sigma^{(x,t)})\, dx\, dt, \quad (11)
$$

$$
\lim \int G(s, t, \omega^{(s,t)})\, ds\, dt = \int G(s, t, \omega^{(s,t)})\, ds\, dt \quad (12)
$$

On the other hand, due to Remark 4.4 and Theorem 5.1, the sequence $(y_n)_n$ is bounded in Q, and converges to $y_{\sigma\omega}$ uniformly on $\bar{Q}^\tau$, for all $\tau > 0$. From assumptions on L, with Lebesgue's theorem of dominated convergence applied on Ω_τ, we obtain

$$\lim_{n\to\infty} \int_\Omega \Big(L(x, y_n(T)) - L(x, y(T)) \Big)\, dx = 0. \tag{13}$$

Finally, observe that, with assumption **A2**, we have

$$\begin{aligned}
&\Big| \int_Q \Big(F(x,t,y_n(x,t),\sigma_n^{(x,t)}) - F(x,t,y_{\sigma\omega}(x,t),\sigma_n^{(x,t)}) \Big)\, dx\, dt \Big| \\
&= \Big| \int_Q \int_U \Big(F(x,t,y_n(x,t),\lambda) - F(x,t,y_{\sigma\omega}(x,t),\lambda) \Big)\, d\sigma_n^{(x,t)}(\lambda)\, dx\, dt \Big| \\
&\le \int_Q \max_{\lambda\in U} \Big| F(x,t,y_n(x,t),\lambda) - F(x,t,y_{\sigma\omega}(x,t),\lambda) \Big|\, dx\, dt \\
&\le C \int_Q F_1(x,t)\, \eta(|y_n(x,t)|)\, \eta(|y_{\sigma\omega}(x,t)|)\, \zeta(|y_n - y_{\sigma\omega}|(x,t))\, dx\, dt \\
&\le C \int_Q F_1(x,t)\, \zeta(|y_n - y_{\sigma\omega}|(x,t))\, dx\, dt,
\end{aligned}$$

where $C \equiv C(||y_n||_{\infty,Q}, ||y_{\sigma\omega}||_{\infty,Q}, U)$ is a positive constant independent of n. Therefore, with Lebesgue's theorem of dominated convergence (applied on Q^τ), it follows that

$$\lim_{n\to\infty} \int_Q \Big(F(x,t,y_n(x,t),\sigma_n^{(x,t)}) - F(x,t,y_{\sigma\omega}(x,t),\sigma_n^{(x,t)}) \Big)\, dx\, dt = 0. \tag{14}$$

The conclusion follows from (11), (12), (13), and (14). ∎

THEOREM 5.5 *Let δ be in $I\!R^+$. The relaxed problem (RP_δ) admits at least one solution.*

Proof. The proof is stated for $\delta = 0$. Similar arguments may be used for $\delta > 0$. We can easily prove that $\inf(RP) \in I\!R$. Let $(y_n, \sigma_n, \omega_n)_n$ be a minimizing sequence for (RP). From Proposition 5.3 and from the sequential compactness (for the weak-star topology) of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, we deduce the existence of a subsequence, still indexed by n, and $(y, \sigma, \omega) \in \mathcal{Y}^r_{ad} \times \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$ such that $(y_n, \sigma_n, \omega_n)_n$ converges to (y, σ, ω) in the $C(\bar{Q}^\tau)\times$ weak*- $\mathcal{U}^r_{ad}\times$ weak*-$\mathcal{V}^r_{ad}$ topology, for all $\tau > 0$. Due to the sequential continuity of J (see Proposition 5.4), it follows that

$$\lim_n J(y_n, \sigma_n, \omega_n) = J(y, \sigma, \omega) = \inf(RP).$$

In other words, (y, σ, ω) is an optimal solution for (RP). The proof is complete.

THEOREM 5.6 *The relaxed problem (RP) is weakly stable on the right, that is:*

$$\lim_{\delta \searrow 0} \inf(RP_\delta) = \inf(RP). \tag{15}$$

Proof. Let $\delta > 0$, and let $(y_\delta, \sigma_\delta, \omega_\delta)$ be a solution of (RP_δ). It is clear that

$$\inf(RP_\delta) \leq \inf(RP).$$

Without loss of generality, we can suppose that the sequence $(\sigma_\delta, \omega_\delta)_\delta$ converges to some (σ, ω) for the weak-star topology of $\mathcal{U}_{ad}^r \times \mathcal{V}_{ad}^r$. It follows that $(y_\delta)_\delta$ converges to $y_{\sigma\omega}$ in $C(\bar{Q}^\tau)$ for every $\tau \in]0, T[$. Since $d_{\mathcal{C},\tau}(g(y_\delta)) \leq \delta$ for all $\tau > 0$, by passing to the limit for δ and then for τ, we obtain

$$d_{\mathcal{C},\tau}(g(y_{\sigma\omega})) \leq 0 \text{ and } d_{\mathcal{C}}(g(y_{\sigma\omega})) \leq 0.$$

Therefore $(y_{\sigma\omega}, \sigma, \omega)$ is admissible for (RP), and

$$\min(RP) \leq J(y_{\sigma\omega}, \sigma, \omega) = \lim_{\delta \searrow 0} \min(RP_\delta) \leq \min(RP).$$

■

The following result shows that (RP) is closely related to the relaxed control problem $(RP_{\delta,\tau})$.

THEOREM 5.7 *If* **A1**-**A5** *are fulfilled, then*

$$\lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) = \inf(RP_\delta) \text{ for all } \delta > 0, \tag{16}$$

$$\lim_{\tau \searrow 0} \lim_{\delta \searrow 0} \inf(RP_{\delta,\tau}) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) = \inf(RP). \tag{17}$$

Proof. Let δ be positive and τ be in $]0, T[$. By arguments similar to those used for Theorem 5.5, we can prove that the problem $(RP_{\delta,\tau})$ admits at least one solution $(\bar{y}_{\delta,\tau}, \bar{\sigma}_{\delta,\tau}, \bar{\omega}_{\delta,\tau})$. Moreover, it is obvious that

$$\min(RP_{\delta,\tau}) \leq \min(RP_\delta).$$

• The sequence $(\bar{\sigma}_{\delta,\tau}, \bar{\omega}_{\delta,\tau})_{\tau>0}$ is sequentially compact for the weak-star topology of $\mathcal{U}_{ad}^r \times \mathcal{V}_{ad}^r$. Then there exist a subsequence, still indexed by τ for simplicity, and $(\sigma_\delta, \omega_\delta)$ in $\mathcal{U}_{ad}^r \times \mathcal{V}_{ad}^r$, such that $(\bar{\sigma}_{\delta,\tau}, \bar{\omega}_{\delta,\tau})_{\tau>0}$ converges to $(\sigma_\delta, \omega_\delta)$ for the weak-star topology of $\mathcal{U}_{ad}^r \times \mathcal{V}_{ad}^r$. It follows that the sequence $(\bar{y}_{\delta,\tau})_{\tau>0}$ converges to $y_\delta \equiv y_{\sigma_\delta,\omega_\delta}$ for the usual topology of $C(\bar{Q}^\epsilon)$, for all $\epsilon \in]0, T[$. Let us observe that for $\epsilon > \tau$, we have $d_{\mathcal{C},\epsilon}(g(\bar{y}_{\delta,\tau})) \leq d_{\mathcal{C},\tau}(g(\bar{y}_{\delta,\tau})) \leq \delta$. By passing to the limit on τ, and next on ϵ, we obtain

$$d_{\mathcal{C},\epsilon}(g(y_\delta)) \leq \delta \text{ and } \lim_{\epsilon \searrow 0} d_{\mathcal{C},\epsilon}(g(y_\delta)) = d_{\mathcal{C}}(g(y_\delta)) \leq \delta.$$

Therefore $(y_\delta, \sigma_\delta)$ is admissible for (RP_δ) and

$$\min(RP_\delta) \leq J(y_\delta, \sigma_\delta) = \lim J(\bar{y}_{\delta,\tau}, \bar{\sigma}_{\delta,\tau}) = \lim \min(RP_{\delta,\tau}) \leq \min(RP_\delta)$$

• It remains to prove (17). By passing to the limit in (16), when δ tends to zero, and taking (15) into account, we obtain

$$\lim_{\delta\searrow 0}\lim_{\tau\searrow 0}\inf(RP_{\delta,\tau}) = \lim_{\delta\searrow 0}\inf(RP_\delta) = \inf(RP).$$

Let us prove the equality $\lim_{\tau\searrow 0}\lim_{\delta\searrow 0}\inf(RP_{\delta,\tau}) = \inf(RP)$. Using the same arguments as for (16), we have

$$d_{\mathcal{C},\epsilon}(g(\bar{y}_{\delta,\tau})) \leq d_{\mathcal{C},\tau}(g(\bar{y}_{\delta,\tau})) \leq \delta \text{ for all } \epsilon > \tau.$$

Since $(\bar{\sigma}_{\delta,\tau}, \bar{\omega}_{\delta,\tau})_{\delta>0}$ converges (up to a subsequence) to some $(\sigma_\tau, \omega_\tau)$ for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, we deduce that the sequence $(\bar{y}_{\delta,\tau})_\delta$ converges to $y_\tau \equiv y_{\sigma_\tau\omega_\tau}$ uniformly on $\bar{Q}^\epsilon$, for all $\epsilon \in]0, T[$. It follows that

$$\lim_{\delta\searrow 0} d_{\mathcal{C},\epsilon}(g(\bar{y}_{\delta,\tau})) = d_{\mathcal{C},\epsilon}(g(y_\tau)) \leq 0.$$

Since $(\sigma_\tau, \omega_\tau)_\tau$ converges (up to a subsequence) to some $(\sigma, \omega) \in \mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, for the weak-star topology of $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$, and since $(y_\tau)_\tau$ converges to $y_{\sigma\omega}$ uniformly on $\bar{Q}^\epsilon$, we obtain

$$\lim_{\tau\searrow 0} d_{\mathcal{C},\epsilon}(g(\bar{y}_\tau)) = d_{\mathcal{C},\epsilon}(g(y_{\sigma\omega})) \leq 0,$$

and thus

$$\lim_{\epsilon\searrow 0} d_{\mathcal{C},\epsilon}(g(y_{\sigma\omega})) \leq 0.$$

Therefore the pair $(y_{\sigma\omega}, \sigma, \omega)$ is admissible for (RP) and finally

$$\min(RP) \leq J(y_{\sigma\omega}, \sigma, \omega) = \lim_{\tau\searrow 0}\lim_{\delta\searrow 0} J(\bar{y}_{\delta,\tau}, \bar{\sigma}_{\delta,\tau}, \bar{\omega}_{\delta,\tau}) = \lim_{\tau\searrow 0}\lim_{\delta\searrow 0}\min(RP_{\delta,\tau})$$
$$\leq \min(RP).$$

The proof is complete. ∎

6. Denseness results

In the sequel we state the results on connection between the set of classical trajectories $\mathcal{Y}$ and the set of relaxed trajectories $\mathcal{Y}^r$.

PROPOSITION 6.1 *$\mathcal{Y}$ is dense in $\mathcal{Y}^r$ with respect to the usual topology of $C(\bar{Q}^\tau)$, for all $\tau \in]0, T[$.*

Proof. Let (σ, ω) be in $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$ and $y_{\sigma\omega}$ be the corresponding solution of (8). Since $\mathcal{U}^r_{ad} \times \mathcal{V}^r_{ad}$ is the closure of $\mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (for the weak-star topology), it follows that there exists a sequence $(u_n, v_n)_n \subset \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ converging to (σ, ω) for the weak-star topology. From Theorem 5.1, the sequence $(y_n)_n$ (corresponding to (u_n, v_n)) converges to $y_{\sigma\omega}$ uniformly in $\bar{Q}^\tau$, for every $\tau \in]0, T[$. The proof is complete. ∎

The next result links together the admissible relaxed trajectories and the

PROPOSITION 6.2 *Let ϵ be in $I\!R^+$ and τ be in $]0,T[$. Then,*

$$\mathcal{Y}^r_{ad}(\epsilon,\tau) \subset \mathbf{cl}_{\mathbf{C}(\bar{\mathbf{Q}}\mathbf{t})}\mathcal{Y}_{ad}(\delta+\epsilon,\tau) \text{ for all } \delta > 0, \text{ and all } t \in]0,T[,$$

where $\mathbf{cl}_{\mathbf{C}(\bar{\mathbf{Q}}\mathbf{t})}$ denotes the closure for the usual topology of $C(\bar{Q}^t)$.

Proof. Observe that $\mathcal{Y}_{ad}(\epsilon,\tau) \subset \mathcal{Y}^r_{ad}(\epsilon,\tau) \subset \mathcal{Y}^r$. Let $y \in \mathcal{Y}^r_{ad}(\epsilon,\tau)$. Due to Proposition 6.1, there exists $(u_n, v_n) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ such that $(y_n)_n$ (the solution of (5) corresponding to (u_n, v_n)) converges to y uniformly in $\bar{Q}^t$, for all $t \in]0,T[$. On the other hand, since $\lim_n d_{\mathcal{C},\tau}(g(y_n)) = d_{\mathcal{C},\tau}(g(y)) \le \epsilon$, it follows that, for every $\delta > 0$, there exists $n_\delta > 0$ such that if $n > n_\delta$, then we have

$$d_{\mathcal{C},\tau}(g(y_n)) \le \delta + \epsilon.$$

In other words, for n bigger than n_δ, we have

$$y_n \in \mathcal{Y}_{ad}(\delta+\epsilon,\tau).$$

By the uniform convergence of $(y_n)_n$ to y in $\bar{Q}^t$, we deduce that

$$\lim_n y_n = y \in \mathbf{cl}_{\mathbf{C}(\bar{\mathbf{Q}}\mathbf{t})}\mathcal{Y}_{\mathbf{ad}}(\delta+\epsilon,\tau),$$

and thus $\mathcal{Y}^r_{ad}(\epsilon,\tau) \subset \mathbf{cl}_{\mathbf{C}(\bar{\mathbf{Q}}\mathbf{t})}\mathcal{Y}_{\mathbf{ad}}(\delta+\epsilon,\tau)$. The proof is complete. ∎

7. Properness of the relaxation

Using stability results stated in Casas (1996), we have proved in Arada, Raymond (1998), that the relaxation of control problems (P^N) with Robin boundary condition gives some information on the limit behavior of the perturbed problem (P^N_δ):

$$\inf(RP^N) = \lim_{\delta \searrow 0} \inf(P^N_\delta).$$

A necessary condition to get such a result is that $\mathcal{Y}$ be dense in $\mathcal{Y}^r$ for the usual topology of $L^\infty(Q)$. A necessary and sufficient condition to get properness of the relaxation (i.e. to ensure that $\inf(P^N) = \inf(RP^N)$) is that the original problem be weakly stable on the right, that is, $\inf(P^N) = \lim_{\delta \searrow 0} \inf(P^N_\delta)$.

For problems with Dirichlet boundary conditions, this result is not true. However, other properties enable us to give necessary and sufficient conditions for properness of the relaxation.

LEMMA 7.1 *For every $\tau \in]0,T[$ and every $\delta > 0$, we have*

Proof. We have to prove that $\inf(P_{\delta,\tau}) \leq \inf(RP)$. If $(\bar{y}, \bar{\sigma}, \bar{\omega})$ is a solution of (RP), then for every $\tau \in]0, T[$, $\bar{y}$ belongs to $\mathcal{Y}_{ad}(0, \tau)$. Due to Proposition 6.2, $\bar{y}$ belongs to $cl_{C(\bar{Q}^t)}\mathcal{Y}_{ad}(\delta, \tau)$, for all $\delta > 0$ and all $t \in]0, T[$. In other words, there exists a sequence $(y_n)_n \subset \mathcal{Y}_{ad}(\delta, \tau)$ (y_n is the solution of (5) corresponding to some $(u_n, v_n) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$), converging to $\bar{y}$ for the usual topology of $C(\bar{Q}^t)$, for all $t \in]0, T[$. Hence,

$$\inf(P_{\delta,\tau}) \leq J(y_n, u_n, v_n) \text{ for } n \text{ big enough,}$$

and thus

$$\inf(P_{\delta,\tau}) \leq \lim_n J(y_n, u_n, v_n) = \min(RP).$$

The proof is complete. ∎

THEOREM 7.2 *Consider the following statements:*

$$(1) \inf(RP) = \inf(P),$$

$$(2) \inf(P) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}),$$

$$(3) \inf(P) = \lim_{\tau \searrow 0} \lim_{\delta \searrow 0} \inf(P_{\delta,\tau}),$$

$$(4) (Arg\inf(RP)) \cap (\mathcal{Y}_{ad} \times \mathcal{U}_{ad} \times \mathcal{V}_{ad}) = Arg\inf(P).$$

The statements (1), (2) *and* (3) *are equivalent. If problem* (P) *admits a solution, then* (1), (2), (3) *and* (4) *are equivalent.*

Proof. By taking (17) into account, and by passing to the limit in (18), when τ tends to zero and then when δ tends to zero, we obtain

$$\inf(RP) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) \leq \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \leq \inf(RP).$$

In the same way, by passing to the limit in (18), when δ tends to zero and then when τ tends to zero, we obtain

$$\inf(RP) = \lim_{\tau \searrow 0} \lim_{\delta \searrow 0} \inf(RP_{\delta,\tau}) \leq \lim_{\tau \searrow 0} \lim_{\delta \searrow 0} \inf(P_{\delta,\tau}) \leq \inf(RP).$$

Therefore,

$$\inf(RP) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) = \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}),$$

and the equivalence between (1), (2), and (3) is direct. Let us prove the equivalence between (1) and (4). Suppose that (1) is satisfied. First, observe that $\operatorname{Arg}\inf(P) \subset (\operatorname{Arg}\inf(RP)) \cap (\mathcal{Y}_{ad} \times \mathcal{U}_{ad} \times \mathcal{V}_{ad})$ (in other words, a solution for (P) is also a solution for (RP)). Let $(\bar{y}, \bar{u}, \bar{v}) \in (\operatorname{Arg}\inf(RP)) \cap (\mathcal{Y}_{ad} \times \mathcal{U}_{ad} \times \mathcal{V}_{ad})$, then $J(\bar{y}, \bar{u}, \bar{v}) = \min(RP) = \inf(P)$ and $(\bar{y}, \bar{u}, \bar{v})$ is admissible for (P). Thus (4) is established.

Conversely, if (P) admits a solution and if $\operatorname{Arg}\inf(P) = (\operatorname{Arg}\inf(RP)) \cap (\mathcal{Y}_{ad} \times \mathcal{U}_{ad} \times \mathcal{V}_{ad})$, then it is clear that $\inf(P) = \min(RP)$. The proof is complete.

8. Strong stability conditions

We say that (P) is strongly stable on the right, if

- There exist $\bar{\delta} > 0$ and $\bar{r} > 0$ such that for every $\delta \in [0, \bar{\delta}]$, we have

$$\inf(P) - \inf(P_\delta) \leq \bar{r}\delta.$$

The condition of strong stability on the right has been introduced by Bonnans (1991) to obtain necessary optimality conditions in qualified form for some control problems (see also Burke, 1991, and Bonnans, Casas, 1995). In Arada, Raymond (1998), for control problems with Robin boundary conditions, we have proven that the strong stability condition on the right for the classical problem is equivalent to the strong stability condition on the right for the relaxed problem paired with the properness property. More precisely, for the problems considered in Arada, Raymond (1998), Bonnans, Casas (1995), Casas (1996), the following assertions are equivalent:

- (P) is strongly stable on the right.
- (RP) is strongly stable on the right and $\inf(RP) = \inf(P)$.

For the problem we consider here, we have the following result:

PROPOSITION 8.1 *The following statements are equivalent*

(C1) (RP) is strongly stable on the right and $\inf(RP) = \inf(P)$.

(C2) There exist $\bar{\delta} > 0$ and $\bar{r} > 0$ such that

$$\inf(P) - \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \leq \bar{r}\delta \textit{ for every } \delta \in [0, \bar{\delta}].$$

Proof. (C1) implies (C2). From (C1) we obtain the existence of $\bar{\delta} > 0$ and $\bar{r} > 0$ such that:

$$\min(RP) - \min(RP_\delta) \leq \bar{r}\delta \text{ for every } \delta \in [0, \bar{\delta}].$$

Due to Theorem 5.7, with the previous inequality we obtain

$$\min(RP) - \lim_{\tau \searrow 0} \min(RP_{\delta,\tau}) \leq \bar{r}\delta \text{ for every } \delta \in [0, \bar{\delta}].$$

On the other hand, due to Lemma 7.1 and due to (C1), we have

$$\lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) \leq \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \leq \inf(RP) = \inf(P) \text{ for every } \delta > 0.$$

Therefore, we can write

$$\begin{aligned} &\inf(P) - \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \\ &= (\inf(P) - \lim_{\tau \searrow 0} \min(RP_{\delta,\tau})) + (\lim_{\tau \searrow 0} \min(RP_{\delta,\tau}) - \lim_{\tau \searrow 0} \inf(P_{\delta,\tau})) \end{aligned}$$

(C2) implies (C1). Conversely, if (C2) is satisfied, then

$$0 \leq \inf(P) - \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \leq \bar{r}\delta \text{ for every } \delta \in [0, \bar{\delta}].$$

Therefore,

$$0 \leq \inf(P) - \lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) \leq 0.$$

Since $\lim_{\delta \searrow 0} \lim_{\tau \searrow 0} \inf(P_{\delta,\tau}) = \min(RP)$, we have $\inf(P) = \min(RP)$. For $\delta \in [0, \bar{\delta}[$, consider the sequence $(\delta_n)_n$ defined by $\delta_n = \delta + \frac{\bar{\delta}-\delta}{n} \in [0, \bar{\delta}]$. Since $\delta_n > \delta$, the same arguments as in Lemma 7.1 give:

$$\lim_{\tau \searrow 0} \inf(P_{\delta_n,\tau}) \leq \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}).$$

Since (C2) is satisfied, we have

$$\inf(P) - \lim_{\tau \searrow 0} \inf(P_{\delta_n,\tau}) \leq \bar{r}\delta_n.$$

Then,

$$\inf(P) - \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) \leq \bar{r}\delta_n.$$

By passing to the limit when n tends to infinity, we obtain:

$$\begin{aligned}
&\inf(RP) - \inf(RP_\delta) \\
&= \inf(RP) - \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) = \inf(P) - \lim_{\tau \searrow 0} \inf(RP_{\delta,\tau}) \leq \bar{r}\delta.
\end{aligned}$$

The proof is complete. ■

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