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Portfolio selection model with information cost

by

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Abstract: A portfolio planning model which takes into account a cost of purchase of market information is considered. In the presented model, the objective of the investor is to maximize probability of attaining or exceeding the required return z. It is shown that the presented stochastic model reduces to a nonlinear programming problem that can be solved efficiently.

Keywords: portfolio, information cost, Fischer information

1. Introduction

Optimization under uncertainty deals with the situation where the information available is limited at the moment the decision is taken. If the uncertainty is stochastic, the decision-maker may only have a probabilistic distribution of some unknown quantities. In some cases the decision-maker can invest in getting more precise information about the probabilities of the uncertain values in his problem. This leads to new, interesting and important questions which turn out to be difficult to solve. What does the more precise information really mean? When does a trade-off between the cost of buying information and future returns exist? Since, "buying information" is a kind of irreversible investment, when is the decision-maker to stop it?

The importance of these questions and the problem as a whole was noticed by researchers. In the series of papers, Z. Artstein and R. J-B Wets (see Artstein, Wets, 1993, 1994, 1995) developed a new concept, called *sensors* and illustrated how it works in some simple examples. Portfolio selection was also recognized as a natural source of these kind of questions, and so, the potential field of futures applications. Using a computer-based set of experiments R. Bricker and M. De Bruine (see Bricker, DeBruine, 1993) analyzed the relationship between information cost and availability, and the investment strategy

However, it becomes obvious that the problem needs further studies and a more general and systematic approach. Examples of such an approach can be found in Banek (1999A, B).

Investing, and investing in stock assets in particular, is inevitably connected with risk. Various models of optimal portfolio selection, which have appeared so far in the literature of the subject, can be briefly described as "how to obtain the highest return at the lowest risk". No matter how the above statement is realized in particular models, one feature is common. Namely, these models do not take into account the cost of obtaining information about particular assets (their returns and risk), which is necessary to select the portfolio optimally, (see Haugen, 1993). Whatever is the reason of neglecting the cost of information, it results in a brutal simplification of the models when comparing with the real world.

The current paper is based on Banek (1999A, B) presenting probably the first attempt of the general approach to the problem. Assumptions from Banek (1999a, b) are used here in order to create a mathematical model of portfolio selection in which the objective of the investor is to maximize probability of attaining or exceeding the required return z. It is proved that, under realistic assumptions, the model allows to select the optimal portfolio by reducing the primary, stochastic problem to a nonlinear programming problem that can be solved efficiently.

2. Mathematical model

We consider selecting a portfolio of n assets. Let us use the following notations: x - a vector of investment, j - the n-element vector of 1's, M - the total capital, $z > 0$ - a minimal level of return, required by the investor. Let $t, 0 \leq t \leq M$ denote the amount of money spent on purchase of information.

The random vector ξ represents future unknown returns on the assets whose parameters are estimated by analysts. We assume that the investor knows a priori that the vector ξ has a normal distribution with the parameters (estimated in any way): m - a vector of expected values (returns) and Q - a covariance matrix. The job done by analysts consists in further estimations of the mean vector m and the covariance matrix Q. As a result of their studies they produce $m(t)$, $Q(t)$. In general, $m(\cdot)$, $Q(\cdot)$ are stochastic processes and $Q(\cdot) = (q_{ij}(\cdot))_{i,j=1,\dots,n}$ is a square, symmetric matrix with differentiable elements, such that

$$x^T \dot{Q}(t) x < 0 \text{ for any } x \neq 0.$$

The latter requirement comes from the fact that as work of the analysts continues, the mean square error of estimation of $m(\cdot)$ should decrease. For simplicity we make further assumptions.

(A1) The $m(\cdot)$ is constant, i.e.

(A2) The matrix $Q(\cdot)$ is deterministic and is of the form

$$Q(t) = Q(I + tHQ)^{-1}. \tag{1}$$

where H is some square symmetric matrix with trace of H, Sp $H > 0$. As it is easy to see, the inequality $x^T \dot{Q}(t)x < 0$ for any $x \neq 0$ is a consequence of (1).

We shall explain below our motivation for adopting (A2). Let $G_n(z, Q)$ denote an n-dimensional Gaussian density with the mean z and the covariance matrix Q.

The Fischer information on the vector ξ contained in the distribution function with Gaussian density G_n is the following

$$\begin{aligned} I_t &= \int_{R^n} \frac{\|\nabla G_n(z-m, Q(t))\|^2}{G_n(z-m, Q(t))} dz = \\ &= \text{Sp } Q^{-1}(t) \left[\int_{R^n} [z-m][z-m]^T G_n(z-m, Q(t)) \right] Q^{-1}(t) = \\ \text{Sp } Q^{-1}(t) &= \text{Sp } (Q^{-1} + tH) \equiv a + b \cdot t \end{aligned}$$

what shows that for $Q(\cdot)$ given by (1), I_t increases linearly with t. Thus, if the cost of purchasing information is proportional to I_t, which we adopt as our next assumption, i.e.

(A3) $c(t) = c \cdot (I_t - I_0)$,

then as a consequence we obtain

(A3') $c(t) = \alpha \cdot t$, $\alpha = c \cdot$ Sp H.

In other words (A3') is in agreement with (A3) if $Q(\cdot)$ is of the form (1).

In order to simplify calculations, we will carry out further considerations for the particular case of $H = I$.

Let us denote by ξ_t a random variable representing a vector of returns obtained by analysts and purchased by the investor, who pays $c(t)$ for it.

The amount t cannot be negative because short-selling of information is impossible. It cannot exceed M either, otherwise all the money invested in assets and a part spent on information would come from short-selling. We assume unlimited short-selling on all the assets.

Since $\xi_t \sim N(m, Q(t))$, then

$$\langle x, \xi_t \rangle \sim N\left(\langle x, m \rangle, x^T Q(t) x \right).$$

Under above assumptions we can formulate the following stochastic model of the decision problem of the investor: given $z > 0$, M, m, $Q(t)$, find (t_{opt}, x_{opt}) - a solution of the following stochastic programming problem

$$\max_{t + <x, j> = M \; 0 \le t \le M} P\left(\langle x, \xi_t \rangle \ge z + M \right). \tag{2}$$

By Lemma 1 (see Appendix) problem (2) reduces to the following nonlinear programming problem

$$\max \frac{\langle x, m \rangle - z - M}{\quad}$$

Before formulating the theorem being the main result of the paper, we have to introduce some necessary notations.

Let

$$p(t) \hat{=} j^T Q^{-1}(t) \left(m - \frac{M+z}{M-t} \cdot j \right) = \frac{j^T Q^{-1}(t) \left((M-t)m - (M+z)j \right)}{M-t}.$$

Function $p(t)$ can have at most two real roots because its numerator

$$r(t) \hat{=} j^T Q^{-1}(t) \left((M-t)m - (M+z)j \right)$$

is a square function of one variable. Let us denote roots of $r(t)$ (if they exist) by $\underline{t}, \overline{t}$, respectively. Set, $T = \{t : r(t) > 0\} = (\underline{t}, \overline{t})$.

Let $h \hat{=} M - t$ and $\nu(h)$ denote the following polynomial

$$\begin{aligned} v(h) &= \|m\|^2 h^3 - \\ & h \left((M+z)^2 \|j\|^2 + 2(M+z) j^T Q^{-1} m + 2M(M+z) \langle m, j \rangle \right) + \\ & +2M(M+z)^2 \|j\|^2 + 2(M+z)^2 j^T Q^{-1} j. \end{aligned}$$

By $h_{\max}$ we denote the biggest real root of this polynomial.

Because, by the assumption, $t \in [0, M]$, in further reasoning we will consider $P \hat{=} [0, M] \cap T$. By Lemma 2 we can see that

$$P = \begin{cases} [0, \overline{t}) & \text{if } \underline{t} < 0 \\ (\underline{t}, \overline{t}) & \text{if } \underline{t} \geq 0. \end{cases}$$

THEOREM 2.1 *If assumptions (A1), (A2), (A3) are satisfied and $P \neq \emptyset$, the solution of the problem (2) is a pair (x_{opt}, t_{opt}), where*

$$x_{opt} = \frac{M - t_{opt}}{j^T Q^{-1}(t_{opt}) \left(m - \frac{z+M}{M-t_{opt}} \cdot j \right)} Q^{-1}(t_{opt}) \left(m - \frac{z+M}{M-t_{opt}} \cdot j \right)$$

$$t_{opt} = M - h_0$$

and

$$h_0 = \begin{cases} h_{\max}, & \text{if } \underline{h} < h_{\max} \leq M \text{ and } v(h_{\max}) < v(M) \\ M, & \text{elsewhere} \end{cases}.$$

Proof. Let us consider problem (3) (equivalent to (2), by Lemma 1) now taking into account only the constraint $\langle x, j \rangle + t = M$. The Lagrange function is

$$F(x, t, \lambda) = \frac{\langle x, m \rangle - z - M}{\sqrt{x^T Q(t) x}} - \lambda \left(t + \langle x, j \rangle - M \right).$$

By differentiating it by x^T, t and λ, we obtain a system of $n+2$ equations

$$m\sqrt{x^T Q(t) x} + (M + z - \langle x, m \rangle) \frac{Q(t)x}{\sqrt{x^T Q(t) x}}$$

$$\frac{\frac{1}{2}\left(\langle x,m\rangle - z - M\right)\frac{x^TQ^2(t)x}{\sqrt{x^TQ(t)x}}}{x^TQ(t)x} - \lambda = 0 \tag{5}$$

$$t + \langle x,j\rangle - M = 0.$$

Let us multiply both sides of (4) by x^T, find λ and put it in (3). There are the following cases

1. If $M \neq t$, then
$$\frac{M+z}{(M-t)\sqrt{x^TQ(t)x}} = \lambda$$
so
$$\frac{m\sqrt{x^TQ(t)x} - \frac{\langle x,m\rangle - z - M}{\sqrt{x^TQ(t)x}}Q(t)x}{x^TQ(t)x} = \frac{M+z}{(M-t)\sqrt{x^TQ(t)x}}\cdot j, \tag{6}$$
For simplification, let us denote $w \hat{=} \frac{\langle x,m\rangle - z - M}{\sqrt{x^TQ(t)x}}$. We obtain
$$\frac{w}{\sqrt{x^TQ(t)x}}x = Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right). \tag{7}$$
Next, multiply both sides of (6) by j^T. We have
$$w(M-t) = \sqrt{x^TQ(t)x}\, j^TQ^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right).$$
The following cases can occur.

   (a) If $j^TQ^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right) > 0$ (i.e. $t \in T$), then w is positive for those x, for which the gradient of the Lagrange function $\nabla_x F$ vanishes.

   (b) If $j^TQ^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right) < 0$, then w is negative for x for which the gradient of the Lagrange function vanishes. Nevertheless, for arbitrary t one can always choose such a vector x, $\langle x,j\rangle = M - t$ that the numerator of the objective function is equal to a fixed a. The sign of w depends only on the sign of the numerator, because the denominator is positive (except for $x = 0$). It is obvious that the maximum of w also must be positive, independently of the choice of t.

   (c) If $j^TQ^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right) = 0$, then $w = 0$. Analogously as in 1.2 (the existence of positive w for arbitrary t), one can see that this case inessential with respect to looking for the maximum (2).

2. If $M = t$, then $\langle x,j\rangle = 0$ and the gradient of the Lagrange function is equal to the gradient of the objective function (3). By comparing it to zero, we have
$$m\sqrt{x^TQ(M)x} + (M + z - \langle x,m\rangle)\frac{Q(M)x}{\sqrt{x^TQ(M)x}}$$

For $x = 0$ the objective function is undefined and it tends to minus infinity, so that its gradient does not exist. If $x \neq 0$ then, multiplying both sides of (12) by x^T, we obtain
$$\frac{M+z}{x^T Q(M) x} = 0.$$
Because $M + z$ is positive, the equality above can be satisfied only asymptotically for x's such that $\|x\| \to \infty$ and $\langle x, j \rangle = 0$. Obviously, also in this case, one can choose such x that $\langle x, j \rangle = 0$ and the value of the objective w corresponding to this x is positive. Nevertheless, there always exists a better solution for $t < M$, so we do not obtain the maximal value of the objective

Taking into account 1.1, 1.2, 1.3 and 2, we can limit our further considerations only to the case 1.1 ($w > 0$). For $t \in T$ we have

$$x(t) = \frac{1}{w}\sqrt{x^T Q(t) x}\, Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right).$$

Again multiply both sides by j^T. We obtain

$$\frac{\sqrt{x^T Q(t) x}}{w} = \frac{x^T Q(t) x}{\langle x, m\rangle - z - M} = \frac{M-t}{j^T Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right)}. \tag{9}$$

Put (9) in equation (8). Finally, we have

$$x(t) = \frac{M-t}{j^T Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right)} Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right). \tag{10}$$

In particular, if $t = 0$, then $x(t) = \frac{M}{j^T Q^{-1}\left(m - \frac{z+M}{M}\cdot j\right)} Q^{-1}\left(m - \frac{z+M}{M}\cdot j\right)$.
Now put $\frac{M+z}{(M-t)\sqrt{x^T Q(t) x}} = \lambda$ in (5). We obtain

$$\frac{\frac{1}{2}\left(\langle x, m\rangle - z - M\right)\frac{x^T Q^2(t) x}{\sqrt{x^T Q(t) x}}}{x^T Q(t) x} - \frac{M+z}{(M-t)\sqrt{x^T Q(t) x}} = 0$$

what, after multiplying both sides by $2\sqrt{x^T Q(t) x}$, yields

$$\frac{\left(\langle x, m\rangle - z - M\right) x^T Q^2(t) x}{x^T Q(t) x} - 2\frac{M+z}{M-t} x^T Q(t) x = 0. \tag{11}$$

Next, put (10) to (11).

$$\frac{M-t}{\ldots} \cdot \left[\frac{M-t}{\ldots}\right]^2$$

$$\left(m - \frac{M+z}{M-t} \cdot j\right)^T \left(m - \frac{M+z}{M-t} \cdot j\right) -$$

$$2\frac{M+z}{M-t}\left[\frac{M-t}{j^T Q^{-1}(t)\left(m - \frac{M+z}{M-t} \cdot j\right)}\right]^2$$

$$\left(m - \frac{M+z}{M-t} \cdot j\right)^T Q^{-1}(t)\left(m - \frac{M+z}{M-t} \cdot j\right) = 0$$

By dividing both sides of the above equation by

$$\frac{(M-t)^2}{\left(j^T Q^{-1}(t)\left(m - \frac{M+z}{M-t} \cdot j\right)\right)^3}$$

we obtain

$$\begin{aligned} &(M-t)\left\{m^T(M-t) - (M+z)j^T\right\}\left\{m(M-t) - (M+z)j\right\} - \\ &-2(M+z)j^T Q^{-1}(t)\left\{m(M-t) - (M+z)j\right\} = 0 \end{aligned}$$

where $Q^{-1}(t) = Q^{-1} + tH$. This is an equation of the 3rd degree with respect to the variable t. As it was stated previously, we will look for roots of (12) only among $t \in P$. By substituting these roots and 0 (the end of the interval P) in the objective we find the optimal t, i.e. maximizing (3). For reasons of simplicity, denote $M - t = h$. Then, equation (12) looks as follows

$$\begin{aligned} &v(h) \hat{=} \|m\|^2 h^3 - \\ &h\left((M+z)\|j\|^2 + 2(M+z)j^T Q^{-1} m + 2M(M+z)\langle m, j\rangle\right) + \\ &+2M(M+z)^2\|j\|^2 + 2(M+z)^2 j^T Q^{-1} j = 0. \end{aligned} \tag{12}$$

The derivative is

$$v'(h) = 3\|m\|^2 h^2 - \left((M+z)^2\|j\|^2 + 2(M+z)^2 j^T Q^{-1} m + 2M(M+z)\langle m, j\rangle\right).$$

Let us compare it to zero. If

$$(M+z)^2\|j\|^2 + 2(M+z)^2 j^T Q^{-1} m + 2M(M+z)\langle m, j\rangle \geq 0$$

then

$$h_\pm = \pm\frac{\sqrt{(M+z)^2\|j\|^2 + 2(M+z)^2 j^T Q^{-1} m + 2M(M+z)\langle m, j\rangle}}{\sqrt{3}\|m\|}$$

are roots of $v(h)$.

Let us denote by h_1, h_2, h_3 the roots of $v(h)$. Let $h_{\max} \hat{=} \max(h_i, i = 1, 2, 3\ h_i \in$

By the definition of h and 1.1, we will restrict our further considerations to the analysis of the polynomial $v(h)$ in the set $S\hat{=}(\underline{h}, M]$, where $\underline{h} = M - \bar{t}$. Let

$$h_0 = \begin{cases} h_{\max}, & \text{if } \underline{h} < h_{\max} \leq M \text{ and } v(h_{\max}) < v(M) \\ M, & \text{elsewhere} \end{cases}.$$

For

$$x(t) = \frac{M-t}{j^T Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right)} Q^{-1}(t)\left(m - \frac{z+M}{M-t}\cdot j\right)$$

the gradient of the objective function vanishes. By substituting $x(t)$ in the objective we obtain

$$\begin{aligned}
\psi(t) &= \frac{\langle x(t), m\rangle - z - M}{\sqrt{x^T(t)Q(t)x(t)}} = \\
&= \frac{\frac{M-t}{j^T Q^{-1}(t)\left(m-\frac{M+z}{M-t}\cdot j\right)} m^T Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right) - M - z}{\frac{M-t}{j^T Q^{-1}(t)\left(m-\frac{M+z}{M-t}\cdot j\right)}\sqrt{\left(m^T - \frac{M+z}{M-t}\cdot j^T\right)Q^{-1}(t)Q(t)Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right)}} = \\
&= \frac{m^T Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right) - \frac{M+z}{M-t} j^T Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right)}{\sqrt{\left(m^T - \frac{M+z}{M-t}\cdot j^T\right)Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right)}} = \\
&= \sqrt{\left(m^T - \frac{M+z}{M-t}\cdot j^T\right)Q^{-1}(t)\left(m - \frac{M+z}{M-t}\cdot j\right)}.
\end{aligned}$$

Thus the objective in terms of the variable h is expressed by

$$\psi(h) = \sqrt{\left(m^T - \frac{M+z}{h}\cdot j^T\right)Q^{-1}(M-h)\left(m - \frac{M+z}{h}\cdot j\right)}.$$

After differentiating over h we obtain

$$\begin{aligned}
\psi'(h) &= -\frac{1}{2h^2\sqrt{(hm^T - (z+M)\cdot j^T)Q^{-1}(M-h)(hm - (z+M)\cdot j)}} \cdot \\
&\quad \left[\|m\|^2 h^3 - h\left((M+z)^2\|j\|^2 + 2(M+z)^2 j^T Q^{-1} m + \right.\right. \\
&\quad \left. 2M(M+z)\langle m, j\rangle\right) + \\
&\quad \left. +2M(M+z)^2\|j\|^2 + 2(M+z) j^T Q^{-1} j\right] = \\
&= -\frac{1}{2h^3\psi(h)}\nu(h)
\end{aligned}$$

Thus $\psi'(h)$ changes the sign in the following way. The first factor is negative for

Thus the sign of $\psi'(h)$ depends only on the sign of the second factor (that in the square brackets). By Lemma 3 it is easy to see that for h_0 the objective function attains its maximum.

2. If $P = \emptyset$, then for each $t \in [0, M]$ we have $j^T Q^{-1}(t) \left(m - \frac{M+z}{M-t} \cdot j \right) < 0$. That is why the solution of the problem (3) does not depend on the value of t, what was described in 1.2 ∎

3. Conclusions

The results obtained in the paper show a possibility of an essential improvement in portfolio selection which can be achieved by taking into account an option of purchase of information. Despite the fact that purchase of information is costly, the probability of exceeding the required level of return can be higher when using this option. In this case, our results can be a good illustration of the commonly known fact that information is the most precious commodity. What is important, computational complexity of the portfolio selection model presented in the paper is slightly higher than that of the classical Roy model. In particular, the most time-consuming computation, namely reversing the covariance matrix remains unchanged. The only computation, which does not occur in the classical model, is calculation of the roots of a polynomial of the 3rd degree. The time of this computation can obviously be neglected, as it does not depend on the number of portfolio components.

Further investigations focused on testing the model by numerical simulations, including usage of real-world data. The results of these simulations confirmed usefulness of the model and will be published soon. Obviously, the model considered is not a mathematical description of any existing information-seller but just a proposal how such an information-seller might work. This is why these simulations are a kind of "what-if" analysis rather than performance of real portfolios.

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Appendix

LEMMA 3.1 *Problem (1) reduces to the following non-linear programming problem*

$$\max_{t+<x,j>=M\ 0\le t\le M} \frac{\langle x,m\rangle - z - M}{\sqrt{x^TQ(t)x}}$$

Proof.

$$\max_{t+<x,j>=M\ 0\le t\le M} P\left(\langle x,\xi_t\rangle \ge z+M\right) =$$

$$= \max_{t+<x,j>=M\ 0\le t\le M} \frac{1}{\sqrt{2\pi x^TQ(t)x}} \int_{z+M}^{\infty} \exp\left(-\frac{(s-\langle x,m\rangle)^2}{2x^TQ(t)x}\right) ds$$

Let us make a substitution $u = \frac{s-\langle x,m\rangle}{\sqrt{x^TQ(t)x}}$. Then we obtain

$$\max_{t+<x,j>=M\ 0\le t\le M} \frac{1}{\sqrt{2\pi x^TQ(t)x}} \int_{z+M}^{\infty} \exp\left(-\frac{(s-\langle x,m\rangle)^2}{2x^TQ(t)x}\right) ds =$$

$$\max_{t+<x,j>=M\ 0\le t\le M} \frac{1}{\sqrt{2\pi}} \int_{\frac{z+M-<x,m>}{\sqrt{x^TQ(t)x}}}^{\infty} \exp\left(-\frac{u^2}{2}\right) du =$$

$$\max_{t+<x,j>=M\ 0\le t\le M} \left\{1 - G\left(\frac{z+M-\langle x,m\rangle}{\sqrt{x^TQ(t)x}}\right)\right\},$$

where $G(\cdot)$ is a distribution function of $N(0,1)$. Now our stochastic problem reduces to the following nonlinear problem

$$\max_{t+<x,j>=M\ 0\le t\le M} \frac{z+M-\langle x,m\rangle}{\sqrt{x^TQ(t)x}} \Leftrightarrow \max_{t+<x,j>=M\ 0\le t\le M} \frac{\langle x,m\rangle - z - M}{\sqrt{x^TQ(t)x}}.$$

■

LEMMA 3.2 *If $T \neq \emptyset$, then $\bar{t} < M$.*

Proof.

$$\begin{aligned} r(t) &= j^TQ^{-1}(t)\left((M-t)m - (M+z)j\right) = \\ &\quad j^T\left[Q^{-1}+tI\right]\left((M-t)m - (M+z)j\right) = \\ &\quad -\langle m,j\rangle t^2 + \left(-j^TQ^{-1}m + M\langle m,j\rangle - \langle j,j\rangle (M+z)\right)t + \end{aligned}$$

For all $z \geq 0$

$$r(M) = j^T \left[Q^{-1} + MI\right]((M - M)m - (M + z)j) = -(M + z)j^T Q^{-1}(M)j < 0.$$

The mean of roots

$$t_w = \frac{-j^T Q^{-1} m + M \langle m, j\rangle - \langle j, j\rangle (M + z)}{2 \langle m, j\rangle}$$

decreases as z increases. We can see that $t_w < M$. It implies that for all z, for which $\underline{t}, \overline{t}$ exist the inequality $\overline{t} < M$. ■

LEMMA 3.3 *1. For arbitrary values of the polynomial $v(h)$ there exists a real negative root h_1.*

2. If h_+ exists and $v(h_+) = 0$, then there exists also one double real root $h_2 = h_3$

3. If h_+ exists and $v(h_+) < 0$, then there exist two different real positive roots h_2, h_3, $h_2 < h_3$.

Proof.

1. For arbitrary values of parameters of the polynomial $v(h)$ we have
   $$v(0) = 2M(M + z)^2 \|j\|^2 + 2(M + z)j^T Q^{-1} j > 0.$$
   Because $v(h)$ is of the 3rd degree with a positive coefficient at the highest power, its value tends to $-\infty$ when h tends to $-\infty$ and it must have one real, negative root h_1. If
   $$(M + z)^2 \|j\|^2 + 2(M + z)^2 j^T Q^{-1} m + 2M(M + z) \langle m, j\rangle \leq 0,$$
   then $v(h)$ >0 for all h, so the only solution of the equation $v(h) = 0$ is $h_1 < 0$. If
   $$(M + z)^2 \|j\|^2 + 2(M + z)^2 j^T Q^{-1} m + 2M(M + z) \langle m, j\rangle > 0$$
   and $v(h_+) > 0$, then positive roots do not exist.
2. If $v(h_+) = 0$, then $h_2 = h_+$ is the only positive root.
3. If $v(h_+) < 0$, then there exist two different positive roots h_2, h_3, $h_2 < h_3$.

■