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On the evaluation of compatibility with gradual rules in information systems: a topological approach

by

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Abstract: The paper presents an approach to detecting dependence of a decision-maker's actions (or decisions) on conditions with respect to a certain type of fuzzy rules, namely gradual rules. Gradual rules are of the form "*the more c is $\mathcal{C}$, the more d is $\mathcal{D}$*", where $\mathcal{C}$ and $\mathcal{D}$ are L-fuzzy sets on the domain of the attributes c and d with membership values possibly in different lattices, respectively. The decision-maker's actions and conditions for those are found in a data base-like system, called an information system. We use the information system to define relations corresponding to L-fuzzy sets, and we define weakening modifiers by these relations. These modifiers are closure operators in the corresponding topologies, and we can define fineness relation between L-fuzzy sets. This relation is useful when comparing different decision-makers' actions. We define strong degree and topological degree of dependence of the attributes (with respect to a gradual rule), and we study some properties of those. Finally, we present a small application.

Keywords: data analysis, L-fuzzy sets, knowledge acquisition, modifier logic, rough sets, topology

1. Introduction

The main idea of the paper is to present an approach to detecting the dependence of a decision-maker's actions (or decisions) on conditions with respect to gradual rules. Gradual rules are studied e.g. in Dubois and Prade (1991) and they are of the form "the more c is $\mathcal{C}$, the more d is $\mathcal{D}$", where $\mathcal{C}$ and $\mathcal{D}$ are L-fuzzy sets with membership values possibly in different lattices. The fuzzy sets were introduced by L. A. Zadeh (1965) and they are generalized to the L-fuzzy sets by J. A. Goguen (1967). In this paper we denote L-fuzzy sets by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$. As the least element 0. The membership function of $\mathcal{A} \subset U$, $U \neq \emptyset$, is a mapping $\mu_{\mathcal{A}} : U \to L$ and we write L-fuzzy sets with ordered pairs as follows:

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \,|\, x \in U,\, \mu_{\mathcal{A}}(x) \in L\}. \tag{1}$$

The class of all L-fuzzy sets in U is denoted by L^U. We denote ordinary subsets of U by $A, B, C, \ldots$ (if needed). Classes of ordinary subsets of U are denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ and especially the ordinary power set of U is denoted by $\mathbf{P}(U)$.

The elements of a non-empty finite set U are called objects in this paper, and they are described by attributes. We define L-fuzzy sets on the domain of the attributes and after data is collected, L-fuzzy sets on U can be determined. Z. Pawlak has introduced a data base-like system, called an information system (see e.g. Pawlak, 1986, 1992), where the objects are described by the attributes, and the objects need not to be distinguished by values of the attributes. So, we use information systems to define L-fuzzy sets on U. For example, an information system S might describe a school, where the teacher (expert) gives grades (decisions) to the students (objects) by test results and other activities (conditions). We might then define a L-fuzzy set "good" on the domain of the attribute "grade". When the grades are given to the students, we might then determine the L-fuzzy set "students with good grades" or "good students" on U in S. No doubt, many real life phenomena can be described by information systems, and data can be represented by simple tables where row labels describe the objects and column labels describe the attributes.

An *information system* S is understood to be a quadruple (U, Q, V, δ), where U is a non-empty finite set and its elements are called *objects* of S, $Q = C \cup D$ is a set of *attributes*, where C is a non-empty finite set, its elements are called *conditions* of S, and D is also a non-empty finite set, its elements are called *decisions* of S, and $C \cap D = \emptyset$. $V = \bigcup_{q \in Q} V_q$ is a non-empty set and V_q ($\neq \emptyset$) is the set of values of the attribute $q \in Q$, called the *domain* of the attribute $q \in Q$. A mapping $\delta : U \times Q \to V$ is called a *description function* of S such that $\forall (x, q) \in U \times Q$, $\delta(x, q) \in V_q$. The values $\delta(x, q)$ are called the *data* in S. In this paper we keep $q \in Q$ fixed more likely than $x \in U$. It is then reasonable to define for each $q \in Q$ a function $\delta_q : U \to V_q$ such that $\forall x \in U$, $\delta_q(x) = \delta(x, q)$. So, we understand the information systems in just the same way as in e.g. Grzymala-Busse (1988), but the domain V_q of the attribute $q \in Q$ may be an infinite set.

In information systems, the attributes may depend on each other in many different ways, and the notion of gradual rules is one way to express dependence of the attributes. In Section 2 we define strong degree and topological degree of dependence of the attributes (with respect to a gradual rule) by means of weakening modifiers and topologies corresponding to L-fuzzy sets. The following

DEFINITION 1.1 *Let L be a complete lattice with ordering "$\geq$", U a non-empty set and $\mathcal{A} \in L^U$. We say that $R^{\mathcal{A}} \subset U \times U$ is relation corresponding to $\mathcal{A}$, if*

$$\forall x, y \in U, (x, y) \in R^{\mathcal{A}} \Leftrightarrow \mu_{\mathcal{A}}(x) \geq \mu_{\mathcal{A}}(y). \tag{2}$$

Clearly, relations corresponding to L-fuzzy sets are ordinary reflexive and transitive relations. However, there might be such reflexive and transitive relations which are not definable by any $\mathcal{A} \in L^U$.

We use relations corresponding to L-fuzzy sets to define weakening modifiers. Modifier logic is studied by J. K. Mattila (1992), and we define weakening modifiers (corresponding to L-fuzzy sets) as follows:

DEFINITION 1.2 *Let $\mathbf{P}(U)$ be the class of all ordinary subsets of a non-empty set U, L a complete lattice, $\mathcal{A} \in L^U$ and $R^{\mathcal{A}} \subset U \times U$. We say that mapping $H^{\mathcal{A}} : \mathbf{P}(U) \to \mathbf{P}(U)$ is weakening modifier (corresponding to $\mathcal{A}$), if for all $A \in \mathbf{P}(U)$,*

$$H^{\mathcal{A}}(A) = \left\{ y \in U \,\middle|\, \exists x \in U, (x, y) \in R^{\mathcal{A}} \text{ and } x \in A \right\}. \tag{3}$$

Notice that at first we define the relation corresponding to $\mathcal{A} \in L^U$, and then we keep it fixed when operating ordinary subsets by $H^{\mathcal{A}}$. Because the relation $R^{\mathcal{A}}$ is reflexive and transitive, the weakening modifier $H^{\mathcal{A}}$ is then the closure operator in the induced topology (see Kortelainen 1994,1997).

This topology consists of ordinary subsets of U and we denote it by $\mathbf{T}^{\mathcal{A}}$ in this paper. The topology $\mathbf{T}^{\mathcal{A}}$ is then called the *topology corresponding* to $\mathcal{A}$ (or topology induced by $\mathcal{A}$). Clearly, the dual operator, for all $A \in \mathbf{P}(U)$,

$$\left(H^{\mathcal{A}}\right)^* (A) = \overline{H^{\mathcal{A}}(\overline{A})}, \tag{4}$$

called the substantiating modifier (corresponding to $\mathcal{A}$), is the interior operator in the induced topology $\mathbf{T}^{\mathcal{A}}$. Notice that the overbar denotes the complementation.

Now, two L-fuzzy sets can be compared topologically as follows:

DEFINITION 1.3 *Let L be a complete lattice, U a non-empty set and $\mathcal{A}, \mathcal{B} \in L^U$. We say that $\mathcal{A}$ and $\mathcal{B}$ are topologically similar, denoted by $\mathcal{A} \approx \mathcal{B}$, if $R^{\mathcal{A}} = R^{\mathcal{B}}$, and we say that $\mathcal{A}$ is finer than $\mathcal{B}$, denoted by $\mathcal{A} \preceq \mathcal{B}$, if $R^{\mathcal{A}} \subset R^{\mathcal{B}}$.*

Clearly, the relation "$\approx$" is an equivalence relation and "$\preceq$" determines a partial ordering on $L^U/\approx$ in a natural way.

2. On dependence of the attributes

We are given an information system $S = (U, Q, V, \delta)$. For each attribute $q \in Q$ we connect a lattice L_q, by which we can define the L_q-fuzzy sets on V_q. The denoted by 1_q and 0_q, respectively. Now, suppose the membership function of $\mathcal{Q} \subset V_q$ is a mapping $\mu_{\mathcal{Q}} : V_q \to L_q$. Because $\forall x \in U$, $\delta_q(x) \in V_q$, we think that the composition function $(\mu_{\mathcal{Q}} \circ \delta_q)(x) = \mu_{\mathcal{Q}}(\delta_q(x))$ can be understood as a membership function from U to L_q. So, L_q-fuzzy sets on U in an information system S can be defined as follows:

DEFINITION 2.1 *We are given an information system $S = (U, Q, V, \delta)$, $q \in Q$, L_q a complete lattice, $\mathcal{Q} \in (L_q)^{V_q}$ and $\mathcal{A} \in (L_q)^U$. We say that $\mathcal{A}$ is L_q-fuzzy set on U in S, if*

$$\forall x \in U, \mu_{\mathcal{A}}(x) = \mu_{\mathcal{Q}}(\delta_q(x)). \tag{5}$$

Now we like to use the previously defined notations and concepts to interpret gradual rules and to define dependence of the attributes by this interpretation. Suppose $\mathcal{C} \in (L_c)^{V_c}$, $\mathcal{D} \in (L_d)^{V_d}$ and gradual rules are written as follows:

$$\text{“the more } c \text{ is } \mathcal{C}, \text{ the more } d \text{ is } \mathcal{D}\text{”.} \qquad (\Re)$$

D. Dubois and H. Prade have studied gradual rules, as a special kind of if-then rules, from the possibility theory point of view in, e.g., Dubois and Prade (1991). We think that L-fuzzy sets are one way to "order" the elements in the universe, and we interpret the rule $(\Re)$ in the following way: the rule $(\Re)$, connecting the attributes c and d, describes how the attribute d depends on the attribute c. So, we interpret the rule $(\Re)$ as an order-preserving condition and, in an information system S, the intended meaning of this interpretation is

"if the membership in $\mathcal{C}$ of the value of the attribute c for object x is at least as high as for object y, then the membership in $\mathcal{D}$ of the value of the attribute d for object x is also at least as high as for object y".

If we define $\forall x \in U, \mu_{\mathcal{A}}(x) = \mu_{\mathcal{C}}(\delta_c(x))$ and $\mu_{\mathcal{B}}(x) = \mu_{\mathcal{D}}(\delta_d(x))$, then for $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$ we interpret $(\Re)$ as follows:

$$\text{“}\forall x, y \in U, \text{ if } \mu_{\mathcal{A}}(x) \geq_c \mu_{\mathcal{A}}(y) \text{ then } \mu_{\mathcal{B}}(x) \geq_d \mu_{\mathcal{B}}(y)\text{”}. \qquad (\Im)$$

The interpretation $(\Im)$ is now one way to analyse the compatibility with the rule $(\Re)$ in S. This means that if $(\Im)$ is true then we can say that the data in S is compatible with the rule $(\Re)$, or we can say that the attribute $d \in Q$ depends on the attribute $c \in Q$ in S (with respect to the rule $(\Re)$).

Now, let the rule $(\Re)$ be given. From now on in this Section, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$ are understood to be L_c - and L_d - fuzzy sets on U in S, respectively. Thus, $\forall x \in U$,

$$\mu_{\mathcal{A}}(x) = \mu_{\mathcal{C}}(\delta_c(x)) \text{ and } \mu_{\mathcal{B}}(x) = \mu_{\mathcal{D}}(\delta_d(x)) \tag{6}$$

DEFINITION 2.2 *We are given an information system $S = (U, Q, V, \delta)$, L_c and L_d ($c, d \in Q$) complete lattices, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$. We say that the attribute $d \in Q$ depends on the attribute $c \in Q$ in S (with respect to the rule ($\Re$)), if $R^{\mathcal{A}} \subset R^{\mathcal{B}}$.*

Because $R^{\mathcal{A}}$ and $R^{\mathcal{B}}$ are ordinary reflexive and transitive relations, we use them to define $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$ which are closure operators in the induced topologies $\mathbf{T}^{\mathcal{A}}$ and $\mathbf{T}^{\mathcal{B}}$, respectively. However, if the lattices L_c and L_d are equal, then by the Definition 1.3, we can compare $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$ by the fineness relation. Thus, we can say that the attribute $d \in Q$ depends on the attribute $c \in Q$ in S (with respect to the rule ($\Re$)), if $\mathcal{A} \preceq \mathcal{B}$.

If the attribute $d \in Q$ does not depend on the attribute $c \in Q$ in S (with respect to the rule ($\Re$)) in the sense of the Definition 2.2, then it may be useful to consider the dependence in a weaker sense and define a degree of dependence of the attributes (with respect to the rule ($\Re$)). We follow now the ideas given e.g. in Grzymala-Busse (1988), Pawlak (1986): we say that the attribute $d \in Q$ depends in degree γ on the attribute $c \in Q$ in S (with respect to the rule ($\Re$)), if

$$\gamma = \sum_{G \in \Gamma} \frac{\left|(H^{\mathcal{A}})^*(G)\right|}{|U|}. \tag{7}$$

In the formula (7) $\Gamma = U/R^{\mathcal{B}}$ and, for all $A \subset U$, $|A|$ is the number of the elements in A. Unfortunately, if $R^{\mathcal{A}}$ and $R^{\mathcal{B}}$ are not symmetric relations then the formula (7) is not applicable in the present form.

Next we like to give the Definition of strong degree of dependence of the attributes (with respect to the rule ($\Re$)), such that we can study also those cases where the relations are not necessarily symmetric. However, let us at first introduce some useful notations:

If $\mathbf{T}$ is any topology on U then we denote the class of all $\mathbf{T}$-closed sets by $\mathbf{C}$. Thus, $\mathbf{C} = \left\{A \in \mathbf{P}(U) \,\middle|\, \bar{A} \in \mathbf{T}\right\}$, and e.g. $\mathbf{C}^{\mathcal{A}}$ denotes then the class of all $\mathbf{T}^{\mathcal{A}}$-closed sets. Moreover, let

$$\Theta = \left\{x \in U \,\middle|\, H^{\mathcal{B}}(\{x\}) \neq U,\ H^{\mathcal{B}}(\{x\}) \in \mathbf{C}^{\mathcal{A}}\right\} \tag{8}$$

and

$$\Omega = \left\{x \in U \,\middle|\, H^{\mathcal{B}}(\{x\}) \neq U\right\}. \tag{9}$$

Now, the following Definition can be given:

DEFINITION 2.3 *We are given an information system $S = (U, Q, V, \delta)$, L_c and L_d ($c, d \in Q$) complete lattices, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$. We say that the attribute $d \in Q$ depends in strong degree χ on the attribute $c \in Q$ in S (with respect to the rule ($\Re$)), if*

$$\chi = \begin{cases} 1, |\Omega| = 0, \\ \frac{|\Theta|}{|\Omega|} & otherwise \end{cases} \tag{10}$$

By the following Propositions, we evaluate the strong degree of dependence and show an essential connection between γ and χ.

PROPOSITION 2.1 *We are given an information system $S = (U, Q, V, \delta)$, L_c and L_d $(c, d \in Q)$ complete lattices, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$. Then $R^{\mathcal{A}} \subset R^{\mathcal{B}} \Leftrightarrow \chi = 1$.*

Proof. Suppose $R^{\mathcal{A}} \subset R^{\mathcal{B}}$. It is proved in Kortelainen (1994) that in this case $\mathbf{T}^{\mathcal{B}} \subset \mathbf{T}^{\mathcal{A}}$ and it is well known (see e.g. Bourbaki, 1989) that $\mathbf{T}^{\mathcal{B}} \subset \mathbf{T}^{\mathcal{A}} \Leftrightarrow \mathbf{C}^{\mathcal{B}} \subset \mathbf{C}^{\mathcal{A}}$. Especially, $\forall x \in U$, $H^{\mathcal{B}}(\{x\}) \in \mathbf{C}^{\mathcal{A}}$ and this means by the formulae (8) and (9) that $\Theta = \Omega$. Thus, $\chi = 1$.

Now, suppose $\chi = 1$. If $|\Omega| = 0$ then clearly $\forall x \in U, H^{\mathcal{B}}(\{x\}) = U$ by the formula (9). This means that $R^{\mathcal{B}} = U \times U$ by the formula (3) and, for all $\mathcal{A} \in (L_c)^U$, $R^{\mathcal{A}} \subset R^{\mathcal{B}}$.

Also, if $|\Theta| = |\Omega|$ then $\Theta = \Omega$, because U is finite and $\Theta \subset \Omega$ by the formulae (8) and (9). So, especially, $\forall x \in U$, $H^{\mathcal{B}}(\{x\}) \in \mathbf{C}^{\mathcal{A}}$. Because U is finite, then any $A \in \mathbf{P}(U)$ is representable by a finite union of singletons $\{x\}$, $x \in U$. Also, $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$ are closure operators and this means that for all $A \in \mathbf{P}(U)$, $H^{\mathcal{B}}(A) \in \mathbf{C}^{\mathcal{A}}$. Thus, $\mathbf{C}^{\mathcal{B}} \subset \mathbf{C}^{\mathcal{A}}$ and $\forall x \in U$, $H^{\mathcal{A}}(\{x\}) \subset H^{\mathcal{B}}(\{x\})$. So, $\forall x, y \in U$, $(x, y) \in R^{\mathcal{A}} \Rightarrow (x, y) \in R^{\mathcal{B}}$. Thus, $R^{\mathcal{A}} \subset R^{\mathcal{B}}$.

LEMMA 2.1 *We are given an information system $S = (U, Q, V, \delta)$, L_d $(d \in Q)$ complete lattice and $\mathcal{B} \in (L_d)^U$ such that $R^{\mathcal{B}}$ is an equivalence relation. Then $\Omega = \emptyset$ or $\Omega = U$.*

Proof. Let $R^{\mathcal{B}}$ be an equivalence relation and $\Omega \neq U$. Then $\exists x \in U, H^{\mathcal{B}}(\{x\}) = U$ and this means that $R^{\mathcal{B}} = U \times U$. Thus, $\Omega = \emptyset$.

PROPOSITION 2.2 *We are given an information system $S = (U, Q, V, \delta)$, L_c and L_d $(c, d \in Q)$ complete lattices, $\mathcal{A} \in (L_c)^U$, $\mathcal{B} \in (L_d)^U$ such that $R^{\mathcal{A}}$ and $R^{\mathcal{B}}$ are equivalence relations, we have $\chi \leq \gamma$.*

Proof. Suppose $R^{\mathcal{B}}$ is an equivalence relation. Now, by the previous Lemma, $\Omega = \emptyset$ or $\Omega = U$. If $\Omega = \emptyset$ then clearly $R^{\mathcal{B}} = U \times U$ and $\chi = \gamma = 1$, because $\Gamma = \{U\}$. So, if $\Omega = U$ then it is enough to prove that $\Theta \subset \bigcup_{G \in \Gamma} (H^{\mathcal{A}})^*(G)$, where $\Gamma = U/R^{\mathcal{B}}$. Now, let $x \in \Theta$. Then $x \in H^{\mathcal{B}}(\{x\})$, $H^{\mathcal{B}}(\{x\}) \neq U$ and $H^{\mathcal{B}}(\{x\}) \in \mathbf{C}^{\mathcal{A}}$. If $R^{\mathcal{A}}$ is an equivalence relation then $H^{\mathcal{B}}(\{x\})$ is also $\mathbf{T}^{\mathcal{A}}$-open set and $(H^{\mathcal{A}})^*(H^{\mathcal{B}}(\{x\})) = H^{\mathcal{B}}(\{x\})$. This means that $x \in \bigcup_{G \in \Gamma} (H^{\mathcal{A}})^*(G)$, because $H^{\mathcal{B}}(\{x\}) \in \Gamma$. Thus, $\Theta \subset \bigcup_{G \in \Gamma} (H^{\mathcal{A}})^*(G)$ and $\chi \leq \gamma$.

It is important to notice the difference when computing χ and γ: when computing χ we insist that $x \in \Theta$, thus $\{x\}$ is not dense and $H^{\mathcal{B}}(\{x\}) \in \mathbf{C}^{\mathcal{A}}$, but this is not necessarily the case when computing γ.

Next we like to define topological degree of dependence of the attributes (with respect to the rule $(\Re)$), such that we compute a degree of "$\mathbf{T}_*^{\mathcal{B}}$ is a subclass

(1997) that the α-level sets of $\mathcal{A} \in L^U$, $\mathcal{A}_\alpha = \{x \in U \mid \mu_{\mathcal{A}}(x) \geq \alpha, \alpha \in L\}$ form a base for $\mathbf{T}^{\mathcal{A}}$, and we think that $|\mathbf{T}^{\mathcal{A}}_*|$ might be quite easy to compute (at least in the case of L being a complete linear lattice).

We give now the following Definition:

DEFINITION 2.4 *We are given an information system $S = (U, Q, V, \delta)$, L_c and L_d $(c, d \in Q)$ complete lattices, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$. We say that the attribute $d \in Q$ depends in topological degree κ on the attribute $c \in Q$ in S (with respect to the rule $(\Re)$), if*

$$\kappa = \begin{cases} 1, & |\mathbf{T}^{\mathcal{B}}_*| = 0, \\ \frac{|\mathbf{T}^{\mathcal{A}}_* \cap \mathbf{T}^{\mathcal{B}}_*|}{|\mathbf{T}^{\mathcal{B}}_*|}, & \textit{otherwise.} \end{cases} \tag{11}$$

The idea for this kind of computation comes from Kosko (1990). To complete this Section we study some properties of the topological degree of dependence.

PROPOSITION 2.3 *Given an information system $S = (U, Q, V, \delta)$, L_c and L_d $(c, d \in Q)$ complete lattices, $\mathcal{A} \in (L_c)^U$ and $\mathcal{B} \in (L_d)^U$, we have $R^{\mathcal{A}} \subset R^{\mathcal{B}} \Leftrightarrow \kappa = 1$.*

Proof. Suppose $R^{\mathcal{A}} \subset R^{\mathcal{B}}$. Clearly, $\mathbf{T}^{\mathcal{B}} \subset \mathbf{T}^{\mathcal{A}}$ and $\kappa = 1$.

Now, let $\kappa = 1$. If $|\mathbf{T}^{\mathcal{B}}_*| = 0$ then clearly $R^{\mathcal{B}} = U \times U$. This means that for all $\mathcal{A} \in (L_c)^U$, $R^{\mathcal{A}} \subset R^{\mathcal{B}}$. Suppose now that $|\mathbf{T}^{\mathcal{A}}_* \cap \mathbf{T}^{\mathcal{B}}_*| = |\mathbf{T}^{\mathcal{B}}_*|$.

Then $\mathbf{T}^{\mathcal{A}}_* \cap \mathbf{T}^{\mathcal{B}}_* = \mathbf{T}^{\mathcal{B}}_*$, because they are finite classes of sets and $\mathbf{T}^{\mathcal{A}}_* \cap \mathbf{T}^{\mathcal{B}}_* \subset \mathbf{T}^{\mathcal{B}}_*$. Thus, $\mathbf{T}^{\mathcal{B}} \subset \mathbf{T}^{\mathcal{A}}$ and this means that $R^{\mathcal{A}} \subset R^{\mathcal{B}}$, because $\forall x \in U, H^{\mathcal{A}}(\{x\}) \subset H^{\mathcal{B}}(\{x\})$.

We noticed previously that we like to compare different decision-makers' actions. Suppose there are two persons, who define the information systems $S_1 = (U, Q, V, \delta_1)$ and $S_2 = (U, Q, V, \delta_2)$. If $\forall q \in Q \setminus \{d\}$, $\forall x \in U$, $(\delta_1)_q(x) = (\delta_2)_q(x)$, then the data may differ only by one attribute, namely $d \in Q$. In this case we denote the two L_d-fuzzy sets on U in S_1 and S_2 by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Also the two topological degrees (with respect to the rule $(\Re)$) are denoted by κ_1 and κ_2, respectively. So,

$$\forall x \in U,\ \mu_{\mathcal{B}_1}(x) = \mu_{\mathcal{D}}((\delta_1)_d(x)) \text{ and } \mu_{\mathcal{B}_2}(x) = \mu_{\mathcal{D}}((\delta_2)_d(x)),$$

and the following Proposition can be presented:

PROPOSITION 2.4 *Suppose we are given information systems $S_1 = (U, Q, V, \delta_1)$ and $S_2 = (U, Q, V, \delta_2)$. Let L_c and L_d $(c, d \in Q)$ be complete lattices, $\mathcal{A} \in (L_c)^U$, $\mathcal{B}_1 \in (L_d)^U$ and $\mathcal{B}_2 \in (L_d)^U$. If $R^{\mathcal{B}_1} \subset R^{\mathcal{A}}$ and $\mathcal{B}_2 \preceq \mathcal{B}_1$, then $\kappa_2 \leq \kappa_1$.*

Proof. Now suppose $\mathcal{B}_2 \preceq \mathcal{B}_1$. This means that $R^{\mathcal{B}_2} \subset R^{\mathcal{B}_1}$ and because also $R^{\mathcal{B}_1} \subset R^{\mathcal{A}}$, it must be then $\mathbf{T}^{\mathcal{A}} \subset \mathbf{T}^{\mathcal{B}_1} \subset \mathbf{T}^{\mathcal{B}_2}$. This means that $|\mathbf{T}^{\mathcal{B}_1}_*| \leq |\mathbf{T}^{\mathcal{B}_2}_*|$

			Teacher 1	Teacher 2
Student	Test results t	Other acts. o	Grades g	Grades g
x_1	30	2	0	0
x_2	59	0	1	1
x_3	78	3	4	3
x_4	89	5	5	5
x_5	67	2	3	2
x_6	50	2	2	2
x_7	48	3	2	2
x_8	66	4	3	4
x_9	74	1	2	2

Table 1.

So, we can use the fineness relation and the topological degree of dependence together when comparing different decision-makers' actions. An interesting special case occurs when $\kappa_1 = \kappa_2 = 1$: we can still try to compare $\mathcal{B}_1$ and $\mathcal{B}_2$ by "$\preceq$". Also if $R^{\mathcal{A}} = I$ (the identity relation) then for all $\mathcal{B} \in (L_d)^U$, $\kappa = 1$. Finally, if $R^{\mathcal{A}} = U \times U$ then $\kappa = 0$ for all $\mathcal{B} \in (L_d)^U$, $R^{\mathcal{B}} \neq U \times U$. We should remind the reader that we use the interpretation ($\Im$) just as one of the ways to analyse dependencies of the attributes in S. However, we think that this approach could be used as a tool for data analysis (see e.g. Bandemer and Näther, 1992).

3. An example

Suppose there is a small group of students to be evaluated by test results and other activities. There are nine students, who are the objects of two information systems S_1 and S_2 defined by two teachers, so $U = \{x_1, x_2, \ldots, x_9\}$. There are three attributes, namely test results t, other activities o and grades g. So, $C = \{o, t\}$, $D = \{g\}$ and $Q = C \cup D$. The domains of the attributes are as follows: $V_t = [0, 100]$ (the percentage of the max points in the test) and $V_o = V_g = \{0, 1, 2, 3, 4, 5\}$.

Two teachers were asked to give grades to the students, and the data is presented by Table 1.

Clearly, the two information systems S_1 and S_2 are written in the same table, for simplicity. Let us now evaluate the teachers' decisions.

If $\mathcal{T} \in (L_t)^{V_t}$ is "good", $\mathcal{O} \in (L_o)^{V_o}$ is "good" and $\mathcal{G} \in (L_g)^{V_g}$ is "good", then we think that the grades should depend on the test results and the other activities at least by the rule

	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
$\mu_{\mathcal{G}}(w)$	0	0.1	0.5	1	0.7	0.5

Table 2.

Student	$\mu_{\mathcal{A}\wedge\mathcal{B}}$	$\mu_{\mathcal{C}_1}$	$\mu_{\mathcal{C}_2}$
x_1	0	0	0
x_2	0	0.1	0.1
x_3	0.88	0.7	1
x_4	0.44	0.5	0.5
x_5	0.5	1	0.5
x_6	0	0.5	0.5
x_7	0	0.5	0.5
x_8	0.64	1	0.7
x_9	0.1	0.5	0.5

Table 3.

For simplicity, all membership values are taken from the unit interval $[0,1]$ with natural ordering "$\geq$". The membership functions are then as follows: If $w \in V_g$ then we give Table 2.

Because $V_o = V_g$, we define $\forall w \in V_g$, $\mu_{\mathcal{O}}(w) = \mu_{\mathcal{G}}(w)$, and the membership function of $\mathcal{T} \subset V_t$ is defined by the following formula:

$$\mu_{\mathcal{T}}(z) = \begin{cases} 0, & 0 \leq z \leq 50 \\ \frac{z-50}{25}, & 50 < z \leq 75 \\ \frac{100-z}{25}, & 75 < z \leq 100. \end{cases}$$

We define now the fuzzy sets on U in S_j, $j = 1, 2$, as follows: $\forall x \in U$, $\mu_{\mathcal{A}}(x) = \mu_{\mathcal{T}}((\delta_j)_t(x))$, $\mu_{\mathcal{B}}(x) = \mu_{\mathcal{O}}((\delta_j)_o(x))$ and $\mu_{\mathcal{C}_j}(x) = \mu_{\mathcal{G}}((\delta_j)_g(x))$. We compute the expression (t is $\mathcal{T}$ and o is $\mathcal{O}$) by the min-operator, for simplicity. Thus, $\forall x \in U$, $\mu_{\mathcal{A}\wedge\mathcal{B}}(x) = \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}$. We can now present Table 3.

Now, the level sets are open sets in the induced topologies and

$$\begin{aligned} \mathbf{T}_*^{\mathcal{A}\wedge\mathcal{B}} &= \{\{x_3\},\{x_3,x_8\},\{x_3,x_5,x_8\},\{x_3,x_4,x_5,x_8\},\{x_3,x_4,x_5,x_8,x_9\}\}, \\ \mathbf{T}_*^{\mathcal{C}_1} &= \{\{x_5,x_8\},\{x_3,x_5,x_8\},\{x_3,x_4,x_5,x_6,x_7,x_8,x_9\},U\setminus\{x_1\}\}, \\ \mathbf{T}_*^{\mathcal{C}_2} &= \{\{x_3\},\{x_3,x_8\},U\setminus\{x_1,x_2\},U\setminus\{x_1\}\}. \end{aligned}$$

So, finally the two topological degrees of dependence (with respect to the rule $(\mathfrak{R}^*)$) are

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Clearly, the grades given by the Teacher 2 depend more on the conditions than the grades given by the Teacher 1 (with respect to the rule ($\Re^*$)). We can also say that the data given by the Teacher 2 are more compatible with the rule ($\Re^*$) than the data given by the Teacher 1. However, it seems that the grades depend on the test results and the other activities more likely with respect to a set of rules than by one rule ($\Re^*$) only. Also, $C_1 \not\preceq C_2$ and $C_2 \not\preceq C_1$.

4. Conclusion

The paper presents a topological approach to detection of dependence of a decision-maker's actions on conditions with respect to a gradual rule. We defined the strong degree of dependence to study the cases where the relations are not necessarily symmetric. It should be noted that, in the case of equivalence relations, the formula (10) gives more "pessimistic" values than the formula (7). Then, by the ideas presented in Kosko (1990), we defined the topological degree of dependence. We found out that we can compare different decision-makers' actions also by the fineness relation and the topological degree of dependence together.

As an application, two teachers were asked to give grades to the students, in Section 3. We analysed the given data by one rule ($\Re^*$), but in general, the data should be analysed by all those rules that affect decisions. It is not clear whether we can compute the topological degree of dependence with respect to a set of many parallel rules, and it might be difficult to find all those rules. However, we think that our approach could be used as a tool for data analysis. Especially, this approach is useful when comparing different decision-makers' actions.

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