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CONTENTS

3. The problem

We are interested in the minimal controlling time

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : U(T, \beta, c) \neq \emptyset\}.$$

The number T^* is the infimum of the set of points $T \in [\underline{T}, \overline{T}]$ for which there exists a control $u \in Z(0,T)$ that satisfies the moment equations, i.e. such that $\langle u, z_j \rangle_{(0,T)} = c_j$ for all $j \in I\!N$ and for which $\|S_T u - b\|^2_{(0,T)} \leq \beta^2$.

The lower bound $\underline{T}$ is introduced since only for $T \geq \underline{T}$, (A1) implies that $U(T, \infty, c)$ is nonempty (see Guerre-Delabriere, 1992, Lemma I.6.2, where a result for the more general case of reflexive spaces is given).

For $T \in [\underline{T}, \overline{T}]$ define the parametric optimization problem $P_\infty(T)$:

$$\min \|S_T u - b\|^2_{(0,T)} - \beta^2 \;\text{ s.t. }\; \langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in I\!N.$$

Let $\omega(T)$ denote the value of $P_\infty(T)$.

Note that in the theory of moment problems (e.g. in Vasin and Ageev, 1995), usually instead of $\|S_T u - b\|^2_{(0,T)}$ the objective function $\|u\|^2_{(0,T)}$ is considered that yields so called normal solutions. For the special case of the control of a rotating beam with S_T as in (2), Krabs considers an objective function of the form $\|S_T \cdot -b\|^2_{(0,T)}$ (see Krabs, 1993), that is equal to the L^2 norm of the momentum at the axis of the beam.

In problem $P_\infty(T)$, the controlling time is fixed and the constraint function that is used to define the problem of time-minimal control is taken as the objective function.

4. The discretized problem

Since $P_\infty(T)$ has an infinite number of equality constraints, for numerical purposes it is necessary to examine a discretized problem $P_N(T)$, where only the first N equality constraints of problem $P_\infty(T)$ are considered.

For $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$ define the parametric optimization problem $P_N(T)$:

$$\min \|S_T u - b\|^2_{(0,T)} - \beta^2 \;\text{ s.t.}$$

$$\langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in \{1, ..., N\}.$$

Let $\omega_N(T)$ denote the value of $P_N(T)$. Then for all $T \in [\underline{T}, \overline{T}]$, the inequality $\omega_{N+1}(T) \geq \omega_N(T)$ is valid.

In the following Lemma, the solution of problem $P_N(T)$ is characterized.

LEMMA 4.1 *Let $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$. For $j \in \{1, ..., N\}$, define $H_j(T) = (S_T^*)^{-1} z_j$. Define $\eta_N(T) = (\eta_i^N(T))_{i=1}^N \in I\!R^N$ as the solution of the linear system*

Then $u_N(T) = S_T^{-1}(\sum_{i=1}^N \eta_i^N(T)H_i(T) + b)$ *is the unique solution of problem* $P_N(T)$.

For the proof of Lemma 4.1, we need the following trivial statement.

STATEMENT 4.1 *Let* $S, T \in [0, \overline{T}]$. *Let* $v, w \in Z(S,T)$ *and* $\langle v - w, w\rangle_{(S,T)} = 0$. *Then* $\|w\|_{(S,T)} \leq \|v\|_{(S,T)}$.

Proof of Lemma 4.1. Define the symmetric matrix

$$G_N(T) = \left(\langle H_i(T), H_j(T)\rangle_{(0,T)}\right)_{i,j=1}^N.$$

Assumption (A1) implies that $G_N(T)$ is positive definite.

Define $\eta_N(T)$ as the solution of the linear system given in Lemma 4.1 and $v_N(T)$ by the equation

$$v_N(T) = \sum_{j=1}^N \eta_j^N(T)H_j(T).$$

Then, for $i \in \{1, ..., N\}$ the following equation holds:

$$\begin{aligned} \langle H_i(T), v_N(T)\rangle_{(0,T)} &= \sum_{j=1}^N \langle H_i(T), H_j(T)\rangle_{(0,T)}\eta_j^N(T) \\ &= c_i - \langle b, H_i(T)\rangle_{(0,T)}. \end{aligned}$$

Define the set $B_N(T) = \{v \in Z(0,T) :$

$$\langle v, H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)},\ i \in \{1, ..., N\}\}.$$

Since $v_N(T) \in \operatorname{span}\{H_1(T), ..., H_N(T)\}$, for all $v \in B_N(T)$ we have

$$\langle v - v_N(T), v_N(T)\rangle_{(0,T)} = 0.$$

Thus, Statement 4.1 implies that $v_N(T)$ is the element of $B_N(T)$ with minimal norm.

For a point $u \in Z(0,T)$ the statement $\langle u, z_j\rangle_{(0,T)} = c_j$ $(j = 1, ..., N)$ holds if and only if $S_T u - b \in B_N(T)$. Hence $u_N(T) = S_T^{-1}(v_N(T) + b)$ is the solution of $P_N(T)$. The fact that the solution of $P_N(T)$ is uniquely determined follows from the strict convexity of $\|S_T \cdot -b\|_{(0,T)}$. □

5. Solvability of problem $P_\infty(T)$

To analyse the solvability of problem $P_\infty(T)$, we need an additional assumption. Assume that in the sequel, the following statement (A2) is valid:

(A2) For all $N \in \mathbb{N}$, $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S < T$ the functions $z_1|_{[S,T]}, ..., z_N|_{[S,T]}$

LEMMA 5.1 *For all $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S \le T$, $u \in Z(S,T)$ the following inequality holds:*

$$\sum_{i=1}^{\infty} \left(\langle u, H_i(T) \rangle_{(S,T)} \right)^2 \le \hat{P}^2 \|u\|^2_{(S,T)}.$$

Proof. If $S = T$, the assertion is trivial.

Assume now that $S < T$. For $N \in \mathbb{N}$ we define the symmetric matrix

$$G_N(S,T) = \left(\langle H_i(T), H_j(T) \rangle_{(S,T)} \right)_{i,j=1}^{N}.$$

Due to Assumption (A2), the functions $H_1(T)|_{[S,T]}, ..., H_N(T)|_{[S,T]}$ are linearly independent. Hence the matrix $G_N(S,T)$ is positive definite.

Let $u \in Z(S,T)$. Define

$$\begin{aligned} U_N &= \left(\langle u, H_i(T) \rangle_{(S,T)} \right)_{i=1}^{N}, \\ \alpha_N &= (G_N(S,T))^{-1} U_N \text{ and} \\ u_N &= \sum_{i=1}^{N} \alpha_i^N H_i(T). \end{aligned}$$

Then we have $\langle u_N - u, u_N \rangle_{(S,T)} = 0$. Thus, Statement 4.1 implies

$$\|u_N\|_{(S,T)} \le \|u\|_{(S,T)}.$$

Lemma 2.2 implies that for all $N \in \mathbb{N}$, $(a_1, ..., a_N) \in \mathbb{R}^N$, the following inequality holds:

$$\begin{aligned} \| \sum_{i=1}^{N} a_i H_i(T) \|_{(S,T)} &\le \| \sum_{i=1}^{N} a_i H_i(T) \|_{(0,T)} \\ &\le \hat{P} \left(\sum_{i=1}^{N} a_i^2 \right)^{1/2}. \end{aligned}$$

This implies that for all $y \in \mathbb{R}^N$, we have

$$y^T y \le \hat{P}^2 y^T \left(G_N(S,T) \right)^{-1} y.$$

Thus the following statement is valid:

$$\begin{aligned} \sum_{i=1}^{N} \left(\langle u, H_i(T) \rangle_{(S,T)} \right)^2 &= U_N^T U_N \\ &\le \hat{P}^2 U_N^T \left(G_N(S,T) \right)^{-1} U_N \\ &= \hat{P}^2 \alpha_N^T G_N(S,T) \alpha_N \\ &= \hat{P}^2 \|u_N\|^2_{(S,T)} \end{aligned}$$

Since this inequality holds for all $N \in I\!N$, the assertion follows. □

LEMMA 5.2 *For all $T \in [\underline{T}, \overline{T}]$ there exists an element $v_*(T)$ of the closure of span$\{H_i(T) : i \in I\!N\}$ such that for all $i \in I\!N$ the equality*

$$\langle v_*(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)} \tag{4}$$

is valid. Moreover, $u_*(T) = S_T^{-1}(v_*(T) + b)$ is the unique solution of problem $P_\infty(T)$.

Proof. Let $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$ be given and $G_N(T)$, $v_N(T)$ as in the proof of Lemma 4.1. Define

$$V_N = (c_i - \langle b, H_i(T)\rangle_{(0,T)})_{i=1}^N \in I\!R^N.$$

As in Lemma 4.1, let $\eta_N(T)$ be defined as

$$\eta_N(T) = (G_N(T))^{-1} V_N.$$

On account of Lemma 4.1 and Lemma 2.2 we have the inequality

$$\begin{aligned} \|v_N(T)\|^2_{(0,T)} &= \eta_N(T)^T G_N(T) \eta_N(T) \\ &= V_N^T (G_N(T))^{-1} V_N \\ &\le \hat{M}^2 V_N^T V_N \\ &\le \hat{M}^2 \gamma(T), \end{aligned}$$

with $\gamma(T) = \sum_{i=1}^{\infty} (c_i - \langle b, H_i(T)\rangle_{(0,T)})^2$.

Due to Lemma 5.1, $\gamma(T)$ is finite. Hence the sequence $(v_N(T))_{N\in I\!N}$ is bounded, and thus contains a weakly convergent subsequence. Let $v_*(T)$ denote a weak cluster point of $(v_N(T))_{N\in I\!N}$. For all i, $N \in I\!N$ with $i \le N$ the following equation holds:

$$\langle v_N(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

Due to the definition of weak convergence, this implies for all $i \in I\!N$ the equation

$$\langle v_*(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

For all $N \in I\!N$, the function $v_N(T)$ is in span$\{H_1(T), ..., H_N(T)\}$ (see the proof of Lemma 4.1). Hence $v_*(T)$ is in the closure of span$\{H_i(T), i \in I\!N\}$.

Define the set $B(T)$

$$= \{v \in Z(0,T) : \langle v, H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)},\ i \in I\!N\}.$$

Since $v_*(T)$ is in the closure of span$\{H_i(T), i \in I\!N\}$, for all $w \in B(T)$ we have $\langle w - v_*(T), v_*(T)\rangle_{(0,T)} = 0$. Thus Statement 4.1 implies that $v_*(T)$ is the element of $B(T)$ with minimal norm.

For a point $u \in Z(0,T)$ the equation $\langle u, z_j\rangle_{(0,T)} = c_j$ ($j \in I\!N$) holds if and only if $S_T u - b \in B(T)$. Hence $u_*(T) = S_T^{-1}(v + b)$ is the solution of $P_\infty(T)$.

6. Continuity of the value function for the original problem

In this section, we demonstrate the continuity of the optimal value function ω.

First we prove that the solutions of $P_\infty(T)$ for $T \in [\underline{T}, \overline{T}]$ are uniformly bounded. Then we use this fact to show that ω is lower semicontinuous.

We introduce a dual problem for $P_\infty(T)$ and show that the corresponding dual solutions are also uniformly bounded on $[\underline{T}, \overline{T}]$. We use this fact to show that ω is upper semicontinuous.

LEMMA 6.1 (UNIFORM BOUNDEDNESS OF PRIMAL SOLUTIONS)
The solutions of $P_\infty(T)$ are uniformly bounded on $[\underline{T}, \overline{T}]$, that is there exists $r \in I\!R$, such that for all $T \in [\underline{T}, \overline{T}]$

$$\|u_*(T)\|_{(0,T)} \le r.$$

Proof. Let $T \in [\underline{T}, \overline{T}]$. Let $\gamma(T)$ and $v_*(T)$ be defined as in the proof of Lemma 5.2. Then due to Lemma 5.1 we have

$$\begin{aligned}
\sqrt{\gamma(T)} &= \left(\sum_{i=1}^{\infty}(c_i - \langle b, H_i(T)\rangle_{(0,T)})^2\right)^{1/2} \\
&\le \|c\|_{l^2} + \left(\sum_{i=1}^{\infty}\langle b, H_i(T)\rangle^2_{(0,T)}\right)^{1/2} \\
&\le \|c\|_{l^2} + \hat{P}\|b\|_{(0,\overline{T})} =: R.
\end{aligned}$$

The fact that $v_*(T)$ is a weak cluster point of the sequence $(v_N(T))_{N\in I\!N}$ implies

$$\|v_*(T)\|^2_{(0,T)} \le \hat{M}^2\gamma(T) \le \hat{M}^2 R^2.$$

According to Lemma 5.2, we have $u_*(T) = S_T^{-1}(v_*(T) + b)$. By Lemma 2.1, this yields the inequality

$$\begin{aligned}
\|u_*(T)\|_{(0,T)} &\le \|S_T^{-1}\|\left(\|v_*(T)\|_{(0,T)} + \|b\|_{(0,T)}\right) \\
&\le \|S_{\overline{T}}^{-1}\|\left(\hat{M}R + \|b\|_{(0,\overline{T})}\right) =: r,
\end{aligned}$$

and the assertion follows. □

LEMMA 6.2 (LOWER SEMICONTINUITY) *The function ω is lower semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_l)_{l\in I\!N} \in [\underline{T}, \overline{T}]^{I\!N}$ converging to T be given. For $k \in I\!N$, let $u_k = u_*(T_k)$. Due to Lemma 6.1 there is $r \in I\!R$ such that

Define $\tilde{u}_k(\cdot) = u_k(\cdot\, T_k/\overline{T}) \in Z(0,\overline{T})$. Then

$$\|\tilde{u}_k\|_{(0,\overline{T})} = (\overline{T}/T_k)^{1/2}\|u_k\|_{(0,T_k)} \leq (\overline{T}/T_k)^{1/2} r.$$

Hence the sequence $(\tilde{u}_k)_{k\in I\!N}$ is bounded. Thus there exists a subsequence that converges weakly to a point $\tilde{u}_* \in Z(0,\overline{T})$. Assume without restriction that the whole sequence $(\tilde{u}_k)_{k\in I\!N}$ is weakly convergent.

The definition of $\tilde{u}_j$ implies $u_j(\cdot) = \tilde{u}_j(\cdot\overline{T}/T_j)$. Define $w_*(\cdot) = \tilde{u}_*(\cdot\overline{T}/T)$. Let $\tilde{z}_l^j = z_l(\cdot T_j/\overline{T})$. For all $l \in I\!N$ we have $c_l = \langle u_j, z_l\rangle_{(0,T_j)} = (T_j/\overline{T})\langle \tilde{u}_j, \tilde{z}_l^j\rangle_{(0,\overline{T})}$.

Let $\tilde{z}_l^*(\cdot) = z_l(\cdot T/\overline{T})$. Then

$$\lim_{j\to\infty} \|\tilde{z}_l^j - \tilde{z}_l^*\|_{(0,\overline{T})} = 0.$$

Therefore for all $l \in I\!N$ the following equation holds:

$$\langle \tilde{u}_*, \tilde{z}_l^*\rangle_{(0,\overline{T})} = \lim_{j\longrightarrow\infty} \langle \tilde{u}_j, \tilde{z}_l^*\rangle_{(0,\overline{T})}$$

$$= \lim_{j\longrightarrow\infty} \langle \tilde{u}_j, \tilde{z}_l^j\rangle_{(0,\overline{T})} = \lim_{j\longrightarrow\infty} (\overline{T}/T_j)c_l = (\overline{T}/T)c_l.$$

Hence we get

$$\langle w_*, z_l\rangle_{(0,T)} = (T/\overline{T})\langle \tilde{u}_*, \tilde{z}_l^*\rangle_{(0,\overline{T})} = (T/\overline{T})(\overline{T}/T)c_l = c_l.$$

Thus we have $w_* \in U(T,\infty,c)$ and so $\omega(T) \leq \|S_T w_* - b\|^2_{(0,T)} - \beta^2$.

The function $u \mapsto \|u\|_{(0,T)}$, $Z(0,T) \to I\!R$ is sequentially weakly lower semi-continuous (as the supremum of sequentially weakly continuous functions, see Pedersen, 1988, Prop. 1.5.12).

Let $\tilde{b}(\cdot) = b(\cdot T/\overline{T})$. Let $v_j = S_{T_j}u_j - b \in Z(0,T_j)$ and $\tilde{v}_j(\cdot) = v_j(\cdot\, T_j/\overline{T}) \in Z(0,\overline{T})$. Let $v_* = S_T w_* - b$ and $\tilde{v}_*(\cdot) = v_*(\cdot\, T/\overline{T}) \in Z(0,\overline{T})$. For $f \in Z(0,\overline{T})$, let $\hat{f}_j(\cdot) = f(\cdot\, \overline{T}/T_j) \in Z(0,T_j)$ and $\hat{f}(\cdot) = f(\cdot\, \overline{T}/T) \in Z(0,T)$. Then

$$\langle f, \tilde{v}_j\rangle_{(0,\overline{T})} = \langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} - \langle f, b(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})}.$$

Our definitions imply the equation

$$\begin{aligned}\langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} &= \langle \hat{f}_j, (S_{T_j}u_j)\rangle_{(0,T_j)}(\overline{T}/T_j)\\ &= \langle S^*_{T_j}\hat{f}_j, u_j\rangle_{(0,T_j)}(\overline{T}/T_j)\\ &= \langle (S^*_{T_j}\hat{f}_j)(\cdot\, T_j/\overline{T}), \tilde{u}_j\rangle_{(0,\overline{T})}.\end{aligned}$$

Then, assumption (3) and the weak convergence of the sequence $(\tilde{u}_j)_{j\in I\!N}$ imply

$$\begin{aligned}\lim_{j\longrightarrow\infty} \langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} &= \langle (S^*_T\hat{f})(\cdot\, T/\overline{T}), \tilde{u}_*\rangle_{(0,\overline{T})}\\ &= \langle S^*_T\hat{f}, w_*\rangle_{(0,T)}(\overline{T}/T)\\ &= \langle \hat{f}, S_T w_*\rangle_{(0,T)}(\overline{T}/T)\end{aligned}$$

Moreover, since $\|b(\cdot T_j/\overline{T}) - \tilde{b}(\cdot)\|_{(0,\overline{T})} \longrightarrow 0$ $(j \longrightarrow \infty)$ we have

$$\lim_{j\longrightarrow\infty} \langle f, b(\cdot\, T_j/T)\rangle_{(0,\overline{T})} = \langle f, \tilde{b}\rangle_{(0,\overline{T})}.$$

Thus we can conclude that

$$\lim_{j\longrightarrow\infty} \langle f, \tilde{v}_j\rangle_{(0,\overline{T})} = \langle f, (S_T w_* - b)(\cdot\, T/\overline{T})\rangle_{(0,\overline{T})} = \langle f, \tilde{v_*}\rangle_{(0,\overline{T})},$$

so the sequence $(\tilde{v}_j)_{j\in \mathbb{N}}$ converges weakly to $\tilde{v_*}$.

So we obtain the statement

$$\begin{aligned}
\omega(T) + \beta^2 &\leq \|v_*\|^2_{(0,T)} \\
&= (T/\overline{T})\|\tilde{v_*}\|^2_{(0,\overline{T})} \\
&\leq (T/\overline{T}) \liminf_{j\longrightarrow\infty} \|\tilde{v}_j\|^2_{(0,\overline{T})} \\
&= \liminf_{j\longrightarrow\infty} (T_j/\overline{T})\|\tilde{v}_j\|^2_{(0,\overline{T})} \\
&= \liminf_{j\longrightarrow\infty} \|v_j\|^2_{(0,T_j)} \\
&= \liminf_{j\longrightarrow\infty} \omega(T_j) + \beta^2,
\end{aligned}$$

which implies $\omega(T) \leq \liminf_{k\to\infty} \omega(T_k)$, that is, ω is lower semicontinuous in T. □

To show the upper semicontinuity of ω, we use the coefficients of $v_*(T)$ written as a linear combination of the functions $H_i(T)$.

These coefficients form a sequence in l^2 and can be used to express the optimal value $\omega(T)$.

LEMMA 6.3 *Let $T \in [\underline{T}, \overline{T}]$. Then there exist $(\alpha_i(T))_{i\in \mathbb{N}} \in l^2$ such that*

$$\begin{aligned}
v_*(T) &= \sum_{i=1}^{\infty} \alpha_i(T) H_i(T) \text{ and} \\
\omega(T) + \beta^2 &= \sum_{i=1}^{\infty} \alpha_i(T)(c_i - \langle b, H_i(T)\rangle_{(0,T)}).
\end{aligned}$$

Moreover, for all $i \in \mathbb{N}$ the following equation is valid:

$$\sum_{j=1}^{\infty} \alpha_j(T)\langle H_i(T), H_j(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}. \tag{5}$$

Proof. Lemma 5.2 implies that the function $v_*(T)$ is contained in the closure of *span*$\{H_i(T), i \in \mathbb{N}\}$. Hence there exists a sequence $(\alpha_i(T))_{i\in \mathbb{N}}$ such that

$$v_*(T) = \sum \alpha_i(T) H_i(T)$$

Lemma 2.2 implies that the sequence $(\alpha_i(T))_{i\in\mathbb{N}}$ is an element of l^2.

Since $v_*(T) = S_T u_*(T) - b$, we have

$$\begin{aligned}
\omega(T) + \beta^2 &= \|v_*(T)\|^2_{(0,T)} \\
&= \langle \sum_{i=1}^{\infty} \alpha_i(T) H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T) \langle H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T)(c_i - \langle b, H_i(T)\rangle_{(0,T)},
\end{aligned}$$

where the last equality follows from equation (4), which also implies equation (5). □

In the next Lemma, we introduce a maximization problem with value $\omega(T) + \beta^2$, i.e. a dual problem for $P_\infty(T)$.

LEMMA 6.4 (DUAL PROBLEM) *For all $T \in [\underline{T}, \overline{T}]$ the following equation holds:*

$$\omega(T) + \beta^2 = \sup_{\alpha\in l^2} -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}$$

$$+2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Proof. For $T \in [\underline{T}, \overline{T}]$, $\alpha \in l^2$, define

$$h(T,\alpha) = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}$$

$$+2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Let $\alpha(T) = (\alpha_i(T))_{i\in\mathbb{N}}$ be as in Lemma 6.3. Then, Lemma 6.3 implies

$$\begin{aligned}
h(T,\alpha(T)) &= -\|v_*(T)\|^2 + 2\sum_{j=1}^{\infty} \alpha_j(T)\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right) \\
&= -(\omega(T) + \beta^2) + 2(\omega(T) + \beta^2) \\
&= \omega(T) + \beta^2. \qquad (6)
\end{aligned}$$

This implies the inequality

$$\omega(T) + \beta^2 \le \sup h(T, \alpha)$$

For $\alpha \in l^2$, $v \in Z(0,T)$ define $\phi(T,v,\alpha)$

$$= \|v\|^2_{(0,T)} + 2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)} - \langle v, H_j(T)\rangle_{(0,T)}\right) \tag{7}$$

Lemma 5.1 implies that $\phi(T,v,\alpha)$ is well-defined.

According to Lemma 5.2, we have

$$\|v_*(T)\|^2_{(0,T)} = \omega(T) + \beta^2$$

and thus equation (5) implies that for all $\alpha \in l^2$

$$\phi(T, v_*(T), \alpha) = \|v_*(T)\|^2_{(0,T)} = \omega(T) + \beta^2.$$

For all $\alpha \in l^2$, the map $\phi(T,\cdot,\alpha)$ is coercive and strictly convex, hence the set

$$M_{\min}(T) = \{v \in Z(0,T) : \phi(T,v,\alpha) = \inf_{w \in Z(0,T)} \phi(T,w,\alpha)\}$$

is nonempty and consists of a single element.

Let $\alpha \in l^2$ be fixed and $M_{\min}(T) = \{w_*\}$. Since the map $\phi(T,\cdot,\alpha) : Z(0,T) \longrightarrow I\!R$ is Fréchet-differentiable, we can derive the equation

$$w_* = \sum_{j=1}^{\infty} \alpha_j H_j(T).$$

Thus, the following equation holds:

$$\begin{aligned}
&\phi(T, w_*, \alpha)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}\\
&+2\sum_{j=1}^{\infty} \alpha_j(T)\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right)\\
&-2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}\\
&= h(T,\alpha).
\end{aligned}$$

Hence for all $\alpha \in l^2$ we have

$$\begin{aligned}
h(T,\alpha) &= \inf_{v \in Z(0,T)} \phi(T,v,\alpha) \\
&\leq \phi(T, v_*(T), \alpha) \\
&= \omega(T) + \beta^2.
\end{aligned} \tag{8}$$

This implies

$$\sup_{\alpha \in l^2} h(T,\alpha) \leq \omega(T) + \beta^2,$$

LEMMA 6.5 (UNIQUENESS OF THE DUAL SOLUTIONS) *For all $T \in [\underline{T}, \overline{T}]$, the point $(\alpha_i(T))_{i\in I\!N} \in l^2$ as defined in Lemma 6.3 is uniquely determined and the unique solution of the dual problem stated in Lemma 6.4.*

Proof. Let $\alpha(T) = (\alpha_i(T))_{i\in I\!N}$ be as in Lemma 6.3. Equation (6) implies that $\alpha(T)$ solves the dual problem.

Lemma 2.2 implies that the function $h(T, \cdot) : l^2 \longrightarrow I\!R$ is strictly concave, hence the dual solution is unique.

Therefore $\alpha(T)$ is uniquely determined. □

Note that for all $T \in [\underline{T}, \overline{T}]$, the dual solution is an element of the space l^2 that is independent of T. This fact is very convenient for our analysis.

LEMMA 6.6 (UNIFORM BOUNDEDNESS OF THE DUAL SOLUTIONS) *Let $T \in [\underline{T}, \overline{T}]$ and $(\alpha_i(T))_{i\in I\!N}$ be as in Lemma 6.3. There exists $r \in I\!R$, such that for all $T \in [\underline{T}, \overline{T}]$*

$$\sum_{i=1}^{\infty} (\alpha_i(T))^2 \leq r.$$

Proof. According to Lemma 2.2, for all $T \in [\underline{T}, \overline{T}]$ we have

$$\begin{aligned}\left(\sum_{i=1}^{\infty} \alpha_i(T)^2\right)^{1/2} &\leq \hat{M} \| \sum_{i=1}^{\infty} \alpha_i(T) H_i(T) \|_{(0,T)} \\ &= \hat{M} \|v_*(T)\|_{(0,T)} \\ &\leq \hat{M} R\end{aligned}$$

with R as defined in the proof of Lemma 6.1. The assertion follows with $r = \hat{M}R$. □

LEMMA 6.7 *Let $u \in Z(0, \overline{T})$. For $T \in [\underline{T}, \overline{T}]$, $i \in I\!N$ define*

$$d_i(T) = \langle u, H_i(T) \rangle_{(0,T)}.$$

Then for all $T \in [\underline{T}, \overline{T}]$, the following equation holds:

$$\lim_{t \longrightarrow T, t\in[\underline{T}, \overline{T}]} \sum_{i=1}^{\infty} (d_i(t) - d_i(T))^2 = 0.$$

Proof. Due to Lemma 5.1, for all $t \in [\underline{T}, \overline{T}]$, we have $(d_i(t))_{i\in I\!N} \in l^2$. The definition of $d_i(T)$ and $H_i(T)$ imply

$$d_i(T) = \langle u, (S_T^*)^{-1} z_i \rangle_{(0,T)} = \langle S_T^{-1} u, z_i \rangle_{(0,T)}.$$

Let $T_1, T_2 \in [\underline{T}, \overline{T}]$, $T_1 < T_2$. Then Lemma 2.1 implies

$$d_i(T_2) - d_i(T_1) = \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_2)} - \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_1)}$$

Analogously to Lemma 5.1 we can prove (by replacing $H_i(T)$ by z_i) that for all $v \in Z(T_1, T_2)$ we have

$$\sum_{i=1}^{\infty} \langle v, z_i \rangle^2_{(T_1,T_2)} \le P^2 \|v\|_{(T_1,T_2)}.$$

This implies

$$\begin{aligned}\sum_{i=1}^{\infty}(d_i(T_2) - d_i(T_1))^2 &= \sum_{i=1}^{\infty} \langle S_{T_2}^{-1} u, z_i \rangle^2_{(T_1,T_2)} \\ &\le \hat{P}^2 \|u\|^2_{(T_1,T_2)}.\end{aligned}$$

On account of

$$\lim_{t \longrightarrow T, t \in [\underline{T}, \overline{T}]} \|u\|_{(t,T)} = 0,$$

the assertion follows. □

LEMMA 6.8 (UPPER SEMICONTINUITY) *The function ω is upper semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_j)_{j \in \mathbb{N}} \in [\underline{T}, \overline{T}]^{\mathbb{N}}$ converging to T be given. Then for all $u \in Z(0, \overline{T})$, the following statement holds:

$$\lim_{j \longrightarrow \infty} \|u\|_{(0,T_j)} = \|u\|_{(0,T)}.$$

Moreover, Lemma 6.7 implies

$$\begin{aligned}\lim_{k \longrightarrow \infty} \sum_{j=1}^{\infty} \left(\langle b, H_j(T_k) \rangle_{(0,T_k)} - \langle b, H_j(T) \rangle_{(0,T)} \right)^2 &= 0 \text{ and} \\ \lim_{k \longrightarrow \infty} \sum_{j=1}^{\infty} \left(\langle u, H_j(T_k) \rangle_{(0,T_k)} - \langle u, H_j(T) \rangle_{(0,T)} \right)^2 &= 0.\end{aligned}$$

Let $(\nu^j)_{j \in \mathbb{N}} \in (l^2)^{\mathbb{N}}$ be a weakly convergent sequence converging to the limit ν^*.

Then for ϕ as defined in (7) we have

$$\begin{aligned}&\lim_{k \longrightarrow \infty} \phi(T_k, u|_{[0,T_k]}, \nu^k) = \lim_{k \longrightarrow \infty} \|u\|^2_{(0,T_k)} \\ &+2 \sum_{j=1}^{\infty} \nu_j^k \left(c_j - \langle b, H_j(T_k) \rangle_{(0,T_k)} - \langle u, H_j(T_k) \rangle_{(0,T_k)} \right) \\ &= \phi(T, u|_{(0,T)}, \nu^*),\end{aligned}$$

i.e. the map

is sequentially weakly continuous. Statement (8) implies

$$h(T,\nu) = \inf_{u\in Z(0,\overline{T})} \phi(T, u|_{(0,T)}, \nu).$$

Hence h is the infimum of sequentially weakly continuous maps. Thus Proposition 1.5.12 in Pedersen (1988) implies that h is sequentially weakly upper semicontinuous, i.e.

$$\limsup_{j\longrightarrow\infty} h(T_j, \nu^j) \le h(T, \nu^*).$$

For $t \in [\underline{T}, \overline{T}]$, let $\alpha(t) = (\alpha_i(t))_{i\in\mathbb{N}}$. According to Lemma 6.6 there exists $r \in \mathbb{R}$ such that for all k we have

$$\sum_{i=1}^{\infty} (\alpha_i(T_k))^2 \le r.$$

Hence there exists a subsequence $(t_j)_{j\in\mathbb{N}}$ of $(T_j)_{j\in\mathbb{N}}$ for which

$$\limsup_{k\longrightarrow\infty} h(T_k, \alpha(T_k)) = \lim_{k\longrightarrow\infty} h(t_k, \alpha(t_k))$$

and such that the sequence $(\alpha(t_k))_{k\in\mathbb{N}} \in (l^2)^{\mathbb{N}}$ converges weakly to a point $\alpha^* \in l^2$. Then, due to Lemma 6.4 we have

$$\begin{aligned}
\limsup_{k\longrightarrow\infty} \omega(T_k) + \beta^2 &= \limsup_{k\longrightarrow\infty} h(T_k, \alpha(T_k)) \\
&= \lim_{k\longrightarrow\infty} h(t_k, \alpha(t_k)) \\
&\le h(T, \alpha^*) \\
&\le \omega(T) + \beta^2.
\end{aligned}$$

Hence $\limsup_{k\longrightarrow\infty} \omega(T_k) \le \omega(T)$, i.e. ω is upper semicontinuous on $[\underline{T}, \overline{T}]$. □

Now we state the main result of this section.

THEOREM 6.1 (CONTINUITY) *The function ω is continuous on the interval $[\underline{T}, \overline{T}]$.*

Proof. Lemma 6.2 and Lemma 6.8 together yield the assertion. □

LEMMA 6.9 *If $T^* > \underline{T}$, then $\omega(T^*) = 0$.*

Proof. Assumption (A0) implies $T^* \le \overline{T}$.

By Lemma 5.2 the set $U(T, \beta, c)$ is nonempty if and only if

$$\omega(T) = \|S_T u_*(T) - b\|^2_{(0,T)} - \beta^2 = \|v_*(T)\|_{(0,T)} - \beta^2 \le 0.$$

Hence the definition of T^* implies

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : \omega(T) \le 0\}. \tag{9}$$

7. Continuity of the value function for the discretized problem

LEMMA 7.1 *For all $N \in I\!N$, the function ω_N is continuous on the interval $[\underline{T}, \overline{T}]$.*

Proof. The assertion follows analogously to Theorem 6.1, by replacing the infinite series by finite sums and the infinite systems of moment equations by the corresponding finite systems. The dual solutions of problem $P_N(T)$ are elements of $I\!R^N$. □

8. Uniform convergence of the value functions for the discretized problems

In this section we present the result that is announced in the title of the present paper, a theorem about uniform convergence of the optimal value functions for the discretized problems. This theorem shows that if the discretization level is large enough, the discretized problem yields an arbitrarily good approximation for the optimal value function ω, uniformly on the whole interval $[\underline{T}, \overline{T}]$.

THEOREM 8.1 (UNIFORM CONVERGENCE) *The sequence $(\omega_N)_{N \in I\!N}$ converges uniformly and monotone to ω on $[\underline{T}, \overline{T}]$.*

Proof. The definitions of $P_\infty(T)$ and $P_N(T)$ imply that for all $N \in I\!N$ the following inequality holds:

$$\omega_N(T) \leq \omega_{N+1}(T) \leq \omega(T).$$

Hence for all $T \in [\underline{T}, \overline{T}]$, the sequence $(\omega_N(T))_{N \in I\!N}$ is convergent and

$$\lim_{N \longrightarrow \infty} \omega_N(T) \leq \omega(T).$$

The proof of Lemma 5.2 implies that

$$\begin{aligned} \lim_{N \longrightarrow \infty} \omega_N(T) - \beta^2 &= \liminf_{N \longrightarrow \infty} \|v_N(T)\|^2_{(0,T)} \\ &\geq \|v_*(T)\|^2_{(0,T)} \\ &= \omega(T) - \beta^2, \end{aligned}$$

where we have used the fact that the function $\| \cdot \|_{(0,T)}$ is sequentially weakly lower semicontinuous. Hence for all $T \in [\underline{T}, \overline{T}]$, we have

$$\lim_{N \longrightarrow \infty} \omega_N(T) = \omega(T).$$

Thus the sequence of functions $(\omega_N)_{N \in I\!N}$ converges pointwise to the function ω. By Lemma 7.1, for all $N \in I\!N$ the functions ω_N are continuous. By Theorem 6.1, the limit function ω is also continuous. Hence Dini's Theorem (see Pedersen, 1988) implies the uniform convergence. □.

THEOREM 8.2 *For all $N \in \mathbb{N}$, the optimal value functions ω_N of the discretized problems are continuous. The value function ω of the original problem is also continuous.*

The sequence $(\omega_N)_{N\in\mathbb{N}}$ converges uniformly and monotone to ω on $[\underline{T},\overline{T}]$.

REMARK 8.1 For $N \in \mathbb{N}$, $T \in [\underline{T},\overline{T}]$ define $\Omega(N,T) = \omega_N(T)$ and let $\Omega(\infty,T) = \omega(T)$. Then Theorem 8.2 implies that for all sequences $(N_k)_{k\in\mathbb{N}}$ with $N_k \in \mathbb{N}\cup\{\infty\}$, $(T_k)_{k\in\mathbb{N}}$ where $T_k \in [\underline{T},\overline{T}]$ with

$$\lim_{k\longrightarrow\infty}(N_k,T_k) = (M,S) \in (\mathbb{N}\cup\{\infty\})\times[\underline{T},\overline{T}]$$

the statement

$$\lim_{k\longrightarrow\infty}\Omega(N_k,T_k) = \Omega(M,S), \tag{10}$$

holds, that is the function Ω is continuous on $(\mathbb{N}\cup\{\infty\})\times[\underline{T},\overline{T}]$. For $M \in \mathbb{N}$, (10) is equivalent to the continuity of ω_M. If $N_k = \infty$ for all $k \in \mathbb{N}$, (10) is equivalent to the continuity of ω. Using the compactness of $[\underline{T},\overline{T}]$, we can also deduce from (10) the equation

$$\lim_{k\longrightarrow\infty}\max_{T\in[\underline{T},\overline{T}]}|\Omega(N_k,T)-\Omega(\infty,T)| = 0,$$

i.e. the uniform convergence of the sequence $(\omega_N)_{N\in\mathbb{N}}$ to ω.

Hence except for the statement about monotone convergence, Theorem 8.2 is equivalent to the statement that the function Ω is continuous on $(\mathbb{N}\cup\{\infty\})\times[\underline{T},\overline{T}]$. Note, however, that in the proof of Theorem 8.1 Dini's Theorem can only be applied due to the fact that for fixed $T \in [\underline{T},\overline{T}]$, the sequence of numbers $(\omega_N(T))_{N\in\mathbb{N}}$ is increasing. Moreover, in the proof of the continuity of ω, we have used the fact that for fixed $T \in [\underline{T},\overline{T}]$, $P_\infty(T)$ is a convex problem.

Continuity results of the type of Theorem 8.2 are well known in different settings, for example Theorem 5.5.1 from Rolewicz (1987). This theorem basically states that with feasible sets that give a continuous set-valued map of the parameter, the corresponding optimal value function is continuous.

To show that the feasible set map is continuous, both lower and upper semicontinuity of the set-valued map has to be shown. This approach requires at least as much work as to show that the optimal value function is both upper and lower semicontinuous, as we have done.

The purpose of this paper is to examine the behaviour of optimal value functions that occur if the method of moments is used so that the moment equations appear as constraints. This problem is important since the method of moments is suitable for a numerical treatment of problems of time-optimal control and in this approach the optimal value functions that we consider occur

To compute T^* numerically, we consider the sequence $(T_N^*)_{N\in I\!N}$ defined as follows. For $N \in I\!N$, let

$$T_N^* = \inf\{T \in [\underline{T}, \overline{T}] : \omega_N(T) \leq 0\}.$$

Since $\omega_N \leq \omega_{N+1} \leq \omega$, for all $N \in I\!N$ we have $T_N^* \leq T_{N+1}^* \leq T^*$. Hence $\lim_{N\longrightarrow\infty} T_N^* \leq T^*$.

Analogously to Lemma 6.9 we can prove the following: If $T_N^* > \underline{T}$, then the equation $\omega_N(T_N^*) = 0$ holds. Hence if there exists $N_0 \in I\!N$ such that $T_{N_0}^* > \underline{T}$, Theorem 8.1 yields

$$\omega(\lim_{N\longrightarrow\infty} T_N^*) = \lim_{N\longrightarrow\infty} \omega_N(T_N^*) = 0;$$

since $\lim_{N\longrightarrow\infty} T_N^* \leq T^*$, by (9) this yields

$$\lim_{N\longrightarrow\infty} T_N^* = T^*.$$

This implies the following Lemma.

LEMMA 8.1 *If $T^* > \underline{T}$ the sequence $(T_N^*)_{N\in I\!N}$ converges monotonically to T^* and for N large enough, we have $\omega_N(T_N^*) = 0$.*

For the problem of time-minimal control of an Euler–Bernoulli beam, Lemma 8.1 has been stated in Krabs (1996).

9. Lipschitz and Hölder conditions

In this section, we consider the standard minimum norm problem

$$Q_\infty(T): \ \min \|u\|^2_{(0,T)} \text{ s.t.}$$

$$\langle u, z_j\rangle_{(0,T)} = c_j \ \ (j \in I\!N)$$

for $T \in [\underline{T}, \overline{T}]$ with optimal value $\varphi(T)$:

$$\varphi(T) = \min\{\|u\|^2_{(0,T)} : u \in Z(0,T),\ \langle u, z_j\rangle_{(0,T)} = c_j \ \ (j \in I\!N)\}.$$

We give an assumption that ensures that the optimal value function satisfies a certain Hölder condition with exponent $1/2$. We also present an assumption that implies a certain Lipschitz condition. Our assumptions are regularity conditions for the solutions of problem $Q_\infty(T)$.

We need some additional notation. Let a sequence $(\lambda_j)_{j\in I\!N}$ of numbers greater than or equal to 1 be given. Assume that there is a number $s > 0$ such that

$$\sum^{\infty} \frac{1}{\ } < \infty$$

For a sequence $(a_j)_{j\in I\!N}$ of real numbers and $r \in I\!R$ let

$$\|c\|_r = \left(\sum_{j=1}^{\infty} |a_j|^2 (\lambda_j)^r\right)^{1/2}.$$

Define the space of sequences

$$l_r^2 = \{(a_j)_{j\in I\!N} : \|a\|_r < \infty\}.$$

For $t \in [\underline{T}, \overline{T}]$, define the linear operator $A(t) : l^2 \longrightarrow l^2$, $A(t)\alpha = \left(\sum_{j=1}^{\infty} \alpha_j \langle z_i, z_j\rangle_{(0,t)}\right)_{i\in I\!N}$.

Up to now we have studied the optimal value function. The following lemma contains a result about the sensitivity of the optimal solutions with respect to the parameter t.

LEMMA 9.1 *Let $c \in l^2$. For $t \in [\underline{T}, \overline{T}]$, let $\eta(t) = A(t)^{-1}c$. As before, assume that A1 and A2 hold. Then for all t_1, $t_2 \in [\underline{T}, \overline{T}]$, the following inequality is valid:*

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \leq M^2 P \|\sum_{j=1}^{\infty} \eta_j(t_1) z_j\|_{(t_1,t_2)}.$$

In particular, this implies

$$\lim_{t_2 \longrightarrow t_1} \|\eta(t_1) - \eta(t_2)\|_{l^2} = 0.$$

Moreover, if the functions z_i are continuous and

$$\max_{t\in[0,\overline{T}]} |z_i(t)| \leq 1, \quad i \in I\!N, \tag{11}$$

and for some $r \geq s$ the sequence c is in $A(t_1)(l_r^2)$, then the following inequalities hold:

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \leq \sqrt{|t_1 - t_2|} M^2 P \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r}\right)^{1/2} \tag{12}$$

$$|\varphi(t_1) - \varphi(t_2)| \leq \sqrt{|t_1 - t_2|} M^2 P \|c\|_{l^2} \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r}\right)^{1/2} \tag{13}$$

Inequality (13) shows that the optimal value function φ satisfies a Hölder con-

Proof. The definition of $\eta(t)$ implies

$$\begin{aligned} & A(t_2)(\eta(t_1)-\eta(t_2)) \\ = \; & A(t_2)\eta(t_1) - c \\ = \; & (A(t_2)-A(t_1))\eta(t_1) \\ = \; & \left(\langle z_i, \sum_{j=1}^{\infty} \eta_j(t_1) z_j \rangle_{(t_1,t_2)} \right)_{i \in I\!N} . \end{aligned}$$

Let $u = \sum_{j=1}^{\infty} \eta_j(t_1) z_j \in L^2[0,\overline{T}]$. Then Lemma 5.1 implies

$$\|A(t_2)(\eta(t_1)-\eta(t_2))\|_{l^2}^2 = \sum_{i=1}^{\infty} \langle z_i, u\rangle_{(t_1,t_2)}^2 \leq P^2 \|u\|_{(t_1,t_2)}^2 .$$

Hence the following inequality holds:

$$\|\eta(t_1)-\eta(t_2)\|_{l^2} \leq M^2 \|A(t_2)(\eta(t_1)-\eta(t_2)\|_{l^2} \leq M^2 P \|u\|_{(t_1,t_2)},$$

and the first assertion follows. Due to (11) we have

$$\begin{aligned} \|u\|_{(t_1,t_2)}^2 &= \left| \int_{t_1}^{t_2} u(s)^2 \, ds \right| \\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\eta_i(t_1)\langle z_i, z_j\rangle_{(t_1,t_2)} \eta_j(t_1)| \\ &\leq \left(\sum_{i=1}^{\infty} |\eta_i(t_1)|\right)\left(\sum_{j=1}^{\infty} |\eta_j(t_1)|\right) |t_1 - t_2| \\ &\leq \left(\sum_{i=1}^{\infty} |\eta_i(t_1)| \lambda_i^{r/2} \lambda_i^{-r/2}\right)^2 |t_1 - t_2| \\ &\leq \|\eta(t_1)\|_{l_r^2}^2 \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r}\right) |t_1 - t_2|, \end{aligned}$$

hence if $\eta(t_1) \in l_r^2$, (12) follows. Now (13) is a consequence of the equation $\varphi(t_1) - \varphi(t_2) = c^T(\eta(t_1) - \eta(t_2))$ and the Cauchy–Schwarz inequality. □

The proof of Lemma 9.1 only works for the standard minimum norm problem $Q_\infty(T)$ and not for problem $P_\infty(T)$.

Note that the dual space of l_r^2 is l_{-r}^2.

LEMMA 9.2 *For* $t \in [\underline{T}, \overline{T}]$, $\alpha \in l_r^2$, *let*

$$(D(t)\alpha)_i = \sum^{\infty} z_i(t) z_j(t) \alpha_j$$

Assume that (11) holds. Let $r \geq s$. Then $D(t)$ is a continuous linear map from l_r^2 into l_{-r}^2 and for all $\alpha \in l_r^2$

$$\|D(t)\alpha\|_{l_{-r}^2} \leq \|\alpha\|_{l_r^2} \left(\sum_{i=1}^{\infty} \lambda_i^{-r} \right). \tag{14}$$

Proof. Let $\beta \in l_r^2$. Then

$$\begin{aligned}
|\beta^T D(t)\alpha| &= |\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta_i z_i(t) z_j(t) \alpha_j| \\
&\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\beta_i \alpha_j| \\
&= \left(\sum_{i=1} |\beta_i| \right) \left(\sum_{j=1}^{\infty} |\alpha_j| \right) \\
&= \left(\sum_{i=1}^{\infty} |\beta_i| \lambda_i^{r/2} \lambda_i^{-r/2} \right) \left(\sum_{j=1}^{\infty} |\alpha_j| \lambda_j^{r/2} \lambda_j^{-r/2} \right) \\
&\leq \|\beta\|_{l_r^2} \left(\sum_{i=1}^{\infty} \lambda_i^{-r} \right) \|\alpha\|_{l_r^2},
\end{aligned}$$

where for the last line we have applied the Cauchy–Schwarz inequality twice. Hence the inequality (14) follows. □

LEMMA 9.3 *Assume that (A1) and (11) hold. Assume that the functions z_i are continuously differentiable with*

$$\max_{t \in [0, \overline{T}]} |z_i'(t)| \leq \sqrt{\lambda_i}, \;\; i \in I\!N. \tag{15}$$

Let $r \geq s + 1$. For $t \in [\underline{T}, \overline{T}]$, $\alpha \in l_r^2$, let

$$\overline{A}(t)\alpha = \left(\sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} \alpha_j \right)_{i \in I\!N}.$$

Then $\overline{A}(t)$ is a bounded linear operator from l_r^2 into l_{-r}^2. $\overline{A}(t)$ is Fréchet-differentiable with respect to t, and

$$\left(\overline{A}'(t)\alpha \right)_i = \sum^{\infty} z_i(t) z_j(t) \alpha_j = (D(t)\alpha)_i.$$

Proof. Due to (A1), for $\alpha \in l_r^2$ we have

$$\|\overline{A}(t)\alpha\|_{l^2_{-r}} \leq \|\overline{A}(t)\alpha\|_{l^2} \leq P^2\|\alpha\|_{l^2} \leq P^2\|\alpha\|_{l^2_r}.$$

Let $h \neq 0$ be such that $t + h \in [\underline{T}, \overline{T}]$. The Taylor expansion implies the existence of numbers $\xi_{ij} \in (0, \overline{T})$ such that

$$\left|\frac{1}{h}\langle z_i, z_j\rangle_{(t,t+h)} - z_i(t)z_j(t)\right|$$

$$= \frac{|h|}{2}\left|z_i(\xi_{ij})z_j'(\xi_{ij}) + z_i'(\xi_{ij})z_j(\xi_{ij})\right| \leq \frac{|h|}{2}(\sqrt{\lambda_i} + \sqrt{\lambda_j}).$$

Let $\alpha \in l_r^2$. Define $a_i = \sum_{j=1}^{\infty}(\sqrt{\lambda_i} + \sqrt{\lambda_j})\alpha_j$. Then for all $\beta \in l_r^2$, the following inequality holds:

$$\begin{aligned}\left|\sum_{i=1}^{\infty} a_i\beta_i\right| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\beta_i|(\sqrt{\lambda_i} + \sqrt{\lambda_j})|\alpha_j| \\ &\leq \left(\sum_{i=1}^{\infty}|\beta_i|\sqrt{\lambda_i}\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\right) \\ &\quad + \left(\sum_{i=1}^{\infty}|\beta_i|\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\sqrt{\lambda_j}\right).\end{aligned}$$

For $q \geq s$, we define a positive number C_q by the equation

$$C_q = \left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^q}\right)^{1/2} < \infty. \tag{16}$$

For $\gamma \in l_r^2$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}|\gamma_i|\sqrt{\lambda_i} &= \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{r/2}\lambda_i^{(1-r)/2} \\ &\leq \|\gamma\|_{l_r^2}\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{r-1}}\right) = \|\gamma\|_{l_r^2}C_{r-1}.\end{aligned}$$

Moreover,

$$\sum_{i=1}^{\infty}|\gamma_i| \leq \|\gamma\|_{l_r^2}C_r.$$

Hence for all $\beta \in l_r^2$ we have the inequality

$$\left|\sum^{\infty} a_i\beta_i\right| \leq \|\beta\|_{l^2}\|\alpha\|_{l^2}2C_{r-1}C_r.$$

Thus we conclude that

$$\|(a_i)_i\|_{l^2_{-r}} \leq 2\|\alpha\|_{l^2_r} C_{r-1} C_r.$$

Then we obtain the statement

$$\begin{aligned}
& \| \left[\frac{\overline{A}(t+h) - \overline{A}(t)}{h} - D(t) \right] \alpha \|_{l^2_{-r}} \\
= & \| \left(\sum_{j=1}^{\infty} [\frac{1}{h}\langle z_i, z_j \rangle_{(t,t+h)} - z_i(t) z_j(t)] \alpha_j \right)_i \|_{l^2_{-r}} \\
\leq & \| \left(\frac{h}{2} \sum_{j=1}^{\infty} (\sqrt{\lambda_i} + \sqrt{\lambda_j}) \alpha_j \right)_i \|_{l^2_{-r}} \\
= & \frac{|h|}{2} \|(a_i)_i\|_{l^2_{-r}} \\
\leq & |h| \, \|\alpha\|_{l^2_r} C_{r-1} C_r. \qquad (17)
\end{aligned}$$

So for $h \longrightarrow 0$ the assertion that $\overline{A}$ is Fréchet–differentiable in t follows. □

The following theorem contains a sufficient condition for a kind of Lipschitz condition for φ.

THEOREM 9.1 *Let $r \geq s+1$. Assume that (A1), (11) and (15) hold. Let $t \in [\underline{T}, \overline{T})$ be such that $c \in A(t)(l^2_r)$. Then there exists a constant $L(t) > 0$ such that for all $t_2 \in (t, \overline{T}]$, the following inequality is valid:*

$$\varphi(t) \geq \varphi(t_2) \geq \varphi(t) - L(t)\,(t_2 - t).$$

Proof. Let $t_2 \in (t, \overline{T}]$ and $h = t_2 - t > 0$. Let u_* be the solution of $Q_\infty(t)$. Define $\hat{u}(s) := u_*(s)$, if $s \in [0,t]$, $\hat{u}(s) := 0$ if $s \in (t, t_2]$. Then for all $i \in I\!N$ we have

$$\langle \hat{u}, z_i \rangle_{(0,t_2)} = \langle u_*, z_i \rangle_{(0,t)} = c_i,$$

hence

$$\varphi(t_2) \leq \|\hat{u}\|^2_{(0,t+h)} = \|u_*\|^2_{(0,t)} = \varphi(t).$$

Moreover, due to Lemma 6.4 we obtain the statement

$$\begin{aligned}
\frac{\varphi(t+h) - \varphi(t)}{h} = & \frac{1}{h} \Bigg(\sup_{\alpha \in l^2} - \sum_{i,j=1}^{\infty} \alpha_i \alpha_j \langle z_i, z_j \rangle_{(0,t+h)} + 2 \sum_{j=1}^{\infty} \alpha_j c_j \\
& + \sum_{i,j=1}^{\infty} \eta_i(t) \eta_j(t) \langle z_i, z_j \rangle_{(0,t)} - 2 \sum_{j=1}^{\infty} \eta_j(t) c_j \Bigg)
\end{aligned}$$

$$\begin{aligned}
&\geq -\frac{1}{h}\sum_{i,j=1}^{\infty}\eta_i(t)\eta_j(t)\langle z_i, z_j\rangle_{(t,t+h)} \\
&= -\eta(t)^T\frac{A(t+h)-A(t)}{h}\eta(t) \\
&= -\eta(t)^T\left(\frac{A(t+h)-A(t)}{h} - D(t)\right)\eta(t) \\
&\quad -\eta(t)^T D(t)\eta(t) \\
&\geq -\eta(t)^T D(t)\eta(t) - \|\eta(t)\|_{l_r^2}^2 C_{r-1}C_r|h|,
\end{aligned}$$

where the last line follows from (17).

Let $L(t) = \eta(t)^T D(t)\eta(t) + \|\eta(t)\|_{l_r^2}^2 C_{r-1}C_r[\overline{T} - \underline{T}] > 0$. Then

$$\varphi(t+h) \geq \varphi(t) - L(t)\,h,$$

and the assertion follows. □

REMARK 9.1 The fact that φ is decreasing is well-known, but the lower bound for $\varphi(t_2)$ in Theorem 9.1 appears to be new.

Conditions (11) and (15) hold for trigonometric moment problems of the form

$$\begin{aligned}
\int_0^T u(t)\sin(\sqrt{\lambda_i}t)\,dt &= c_{2i-1}, \\
\int_0^T u(t)\cos(\sqrt{\lambda_i}t)\,dt &= c_{2i},\ i \in I\!N
\end{aligned}$$

that appear for example in the characterization of the set of feasible controls for the exact control of hyperbolic partial differential equations (see, for example Krabs, 1982).

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