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Second-order necessary conditions of the Kuhn-Tucker type in multiobjective programming problems

by

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Abstract: In this paper, we are concerned with a multiobjective programming problem with inequality constraints. We develop second-order necessary condition of the Kuhn-Tucker type for efficiency and prove that the condition holds under a constraint qualification. Moreover, we give some conditions which ensure that the constraint qualification holds.

Keywords: Multiobjective programming, efficient solutions, second-order necessary conditions, second-order constraint qualification

1. Introduction

In this paper, we deal with second-order necessary conditions for the multiobjective programming problems with inequality constraints. We shall give second-order constraint qualification, and derive the second-order necessary conditions of the Kuhn-Tucker type for a feasible solution to be efficient for the problem. Moreover, we shall give some constraint qualifications which are also sufficient conditions for the second-order constraint qualification. Our work is in the spirit of Kawasaki (1988) but in the context of multiobjective programming problems.

The paper is organized as follow. In Section 2, we formulate the multiobjective programming problems with inequality constraints, and provide some definitions and basic results, which are to be used throughout the paper. In Section 3, following Kawasaki, we define two kinds of second-order approximation sets to the feasible region, and using them, we give second-order necessary conditions of the Kuhn-Tucker type for a feasible solution to be efficient for the multiobjective programming problems. In Section 4, we present various kinds of conditions which guarantee the constraint qualification given in Section 3.

2. Preliminaries

In this section we shall introduce some notations and definitions, which are used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ be in $\mathbb{R}^n$. We shall denote the inner product of x and y by $x \cdot y = \sum_{i=1}^n x_i y_i$.

For any two vectors x and y in $\mathbb{R}^n$, we shall use the following conventions:

$$\begin{array}{lll} x \leqq y & \text{iff} \quad x_i \leqq y_i, & i = 1, \ldots, n, \\ x \leq y & \text{iff} \quad x \leqq y & \text{and} \quad x \neq y, \\ x < y & \text{iff} \quad x_i < y_i, & i = 1, \ldots, n. \end{array}$$

For any two vectors x and y in $\mathbb{R}^2$, we shall use the following conventions:

$$\begin{array}{lll} x \leqq_{lex} y & \text{iff} \quad x_1 < y_1 \text{ or} & x_1 = y_1 \text{ and } x_2 \leqq y_2, \\ x <_{lex} y & \text{iff} \quad x_1 < y_1 \text{ or} & x_1 = y_1 \text{ and } x_2 < y_2. \end{array}$$

the subscript lex is an abbreviation for lexicographic order.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We assume that the functions are twice continuously differentiable on $\mathbb{R}^n$, and for any vector y, we denote the Jacobian (resp. the Hessian) of f and g at $x \in \mathbb{R}^n$ by $\nabla f(x)$ and $\nabla g(x)$ (resp.$\nabla^2 f(x)(y, y)$ and $\nabla^2 g(x)(y, y)$).

We consider the following multiobjective optimization problem:

$$(P) \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t} & x \in A = \{x \in \mathbb{R}^n | g(x) \leqq 0\}, \end{array}$$

Due to the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not obtainable. Thus, for problem (P), the solution is defined in terms of an efficient solution, Yu (1985).

DEFINITION 2.1 *A point $\bar{x} \in A$ is said to be an efficient solution to problem (P), if there is no $x \in A$ such that $f(x) \leq f(\bar{x})$.*

Let $\bar{x} \in A$ be any feasible solution to problem (P), and let E be the subset of indices defined by

$$E = \{\, j \in \{1, 2, \ldots, m\} \mid g_j(\bar{x}) = 0 \,\}. \tag{1}$$

DEFINITION 2.2 *The tangent cone to A at $\bar{x} \in A$ is the set defined by*

$$T_1(A, \bar{x}) = \\ \{\, y \in \mathbb{R}^n \mid \exists x^n \in A, \exists t_n \longrightarrow 0^+ \textit{ such that } x^n = \bar{x} + t_n y + o(t_n) \,\}, \tag{2}$$

where $o(t_n)$ is a vector satisfying $\|o(t_n)\|/t_n \longrightarrow 0$.

For each $i = 1, 2, \dots, l$, we shall define the nonempty sets Q^i and Q by

$$Q^i = \{x \in \mathbb{R}^n | g(x) \leqq 0, f_k(x) \leqq f_k(\bar{x}), k = 1, 2, \dots, l, \text{and } k \neq i\},$$
$$Q = \{x \in \mathbb{R}^n | g(x) \leqq 0, f(x) \leqq f(\bar{x})\}.$$

In the case $l = 1$, we set $Q^i = A$.

DEFINITION 2.3 *The linearizing cone to Q at $\bar{x}$ is the set defined by*

$$K_1 = \{ y \in \mathbb{R}^n \mid \nabla f_i(\bar{x}) y \leqq 0,\ i = 1, \dots, l \text{ and } \nabla g_j(\bar{x}) y \leqq 0,\ j \in E \}. \quad (3)$$

3. Second-order necessary conditions

Following Kawasaki (1988), we shall define two kinds of second-order approximation sets to the feasible region. They can be considered as extensions of $T_1(A, \bar{x})$ and K_1 respectively. Using them, we shall give the second-order constraint qualification, under which we derive second-order necessary conditions of the Kuhn-Tucker type for a feasible solution $\bar{x} \in A$ to be an efficient solution to problem (P).

DEFINITION 3.1 *The second-order tangent set to A at $\bar{x} \in A$ is the set defined by*

$$\begin{aligned} T_2(A, \bar{x}) &= \{(y, z) \in \mathbb{R}^{2n} \mid \exists x^n \in A,\ \exists t_n \longrightarrow 0^+ \text{ such that} \\ x^n &= \bar{x} + t_n y + \frac{1}{2} t_n^2 z + o(t_n^2) \}, \end{aligned}$$

where $o(t_n^2)$ is a vector satisfying $\|o(t_n^2)\| / t_n^2 \longrightarrow 0$.

The y-section of $T_2(A, \bar{x})$ is defined by

$$T_2(A, \bar{x})(y) = \{z \in \mathbb{R}^n \mid (y, z) \in T_2(A, \bar{x}) \},$$

DEFINITION 3.2 *The second-order linearizing set to Q at $\bar{x}$ is the set defined by*

$$\begin{aligned} K_2 \equiv \{(y, z) \in \mathbb{R}^{2n} \mid \\ &(\nabla f_i(\bar{x}) y, \nabla f_i(\bar{x}) z + \nabla^2 f_i(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T,\ i = 1, 2, \dots, l, \\ &(\nabla g_j(\bar{x}) y, \nabla g_j(\bar{x}) z + \nabla^2 g_j(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T, j \in E \}. \end{aligned}$$

The y-sections of K_2 is defined by

$$K_2(y) = \{z \in \mathbb{R}^n \mid (y, z) \in K_2\}$$

It is obvious that $K_2(y)$ is closed convex set for each direction $y \in \mathbb{R}^n$.

Before deriving second-order necessary conditions for efficiency, we shall give the following lemma, which shows the relationship between the second-order tangent sets $T_2(Q^i, \bar{x})$ and the second-order linearizing set K_2.

LEMMA 3.1 *Let $\bar{x} \in A$ be any feasible solution to problem (P). Then, we have*

$$\bigcap_{i=1}^{l} \overline{co}[T_2(Q^i, \bar{x})(y)] \subseteq K_2(y)$$

for arbitrary direction y of $\mathbb{R}^n$, where $\overline{co}[T_2(Q^i, \bar{x})(y)]$ denotes the closed convex hull of $T_2(Q^i, \bar{x})(y)$.

Proof. First, we shall show that

$$T_2(Q^i, \bar{x}) \subset K_2^i, \quad i = 1, 2, \ldots, l,$$

where

$$\begin{aligned} K_2^i \equiv \{ &(y, z) \in \mathbb{R}^{2n} \mid \\ &(\nabla f_i(\bar{x})y, \nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T, \quad i = 1, 2, \ldots, l, k \neq i \\ &(\nabla g_j(\bar{x})y, \nabla g_j(\bar{x})z + \nabla^2 g_j(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T, \quad j \in E \ \}. \end{aligned}$$

For any fixed $i = 1, 2, \ldots, l$, let (y, z) be any element of $T_2(Q^i, \bar{x})$. Then, there exist $x^n \in Q^i$, $t_n \longrightarrow 0^+$ such that

$$x^n = \bar{x} + t_n y + 1/2 t_n^2 z + o(t_n^2).$$

By Taylor's expansion,

$$\begin{aligned} f_k(x^n) &= f_k(\bar{x}) + t_n \nabla f_k(\bar{x}) y \\ &\quad + 1/2 t_n^2 (\nabla f_k(\bar{x}) z + \nabla^2 f_k(\bar{x})(y, y)) + o(t_n^2), \quad k = 1, 2, \ldots, l, k \neq i \\ \Rightarrow 0 &\geqq t_n \nabla f_k(\bar{x}) y \\ &\quad + 1/2 t_n^2 (\nabla f_k(\bar{x}) z + \nabla^2 f_k(\bar{x})(y, y)) + o(t_n^2), \quad k = 1, 2, \ldots, l, k \neq i. \end{aligned}$$

Thus, for n large enough, $\nabla f_k(\bar{x}) y \leqq 0$ and if $\nabla f_k(\bar{x}) y = 0$, we obtain

$$0 \geqq \nabla f_k(\bar{x}) z + \nabla^2 f_k(\bar{x})(y, y) + o(t_n^2)/t_n^2$$

which after letting $n \longrightarrow +\infty$, gives

$$\nabla f_k(\bar{x}) z + \nabla^2 f_k(\bar{x})(y, y) \leqq 0,$$

which implies

$$(\nabla f_k(\bar{x}) y, \nabla f_k(\bar{x}) z + \nabla^2 f_k(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T, \quad k = 1, 2, \ldots, l, k \neq i.$$

Similarly, we have

$$(\nabla g_j(\bar{x}) y, \nabla g_j(\bar{x}) z + \nabla^2 g_j(\bar{x})(y, y))^T \underset{lex}{\leqq} (0, 0)^T, \quad j \in E.$$

Hence,

$$T_2(Q^i,\bar{x}) \subset K_2^i, \quad i = 1,2,\dots,l,$$

and, for any arbitrary direction y of $\mathbb{R}^n$, we have

$$T_2(Q^i,\bar{x})(y) \subset K_2^i(y), \quad i = 1,2,\dots,l.$$

Since, $K_2^i(y)$ is a closed convex set and i is arbitrary, we have

$$\bigcap_{i=1}^{l} \overline{co}[T_2(Q^i,\bar{x})(y)] \subseteq \bigcap_{i=1}^{l} K_2^i(y) \subseteq K_2(y)$$

■

In Lemma 3.1, the converse inclusion does not hold, in general. Therefore, it is reasonable that we assume that

$$K_2(y) \subseteq \bigcap_{i=1}^{l} \overline{co}[T_2(Q^i,\bar{x})(y)] \tag{4}$$

holds for the direction y in order to derive the second-order necessary conditions for a feasible solution $\bar{x} \in A$ to be an efficient solution to problem (P). The condition (4) is called the second-order generalized Guignard constraint qualification for the direction y, and we will refer (4) as to the second-order (GGCQ). Since $K_2(0) = K_1$ and $T_2(A,\bar{x})(0) = T_1(A,\bar{x})$, the above condition contains (GGCQ) of Maeda (1994).

First-order sufficient condition for efficiency is that the following system have no nonzero solution y:

$$\begin{aligned} &\nabla f(\bar{x})y \leqq 0, \\ &\nabla g_E(\bar{x})y \leqq 0, \end{aligned} \tag{5}$$

and the condition of Kuhn-Tucker type for efficiency is equivalent, Maeda (1994), Marusciac (1982), Singh (1987), to the inconsistency of the following system :

$$\begin{aligned} &\nabla f(\bar{x})y \leq 0, \\ &\nabla g_E(\bar{x})y \leqq 0, \end{aligned} \tag{6}$$

The gap between (5) and (6) is caused by the following directions:

$$\begin{aligned} &\nabla f(\bar{x})y = 0, \\ &\nabla g_E(\bar{x})y \leqq 0. \end{aligned} \tag{7}$$

A direction y which satisfies (7) is called a critical direction.

For the sake of simplicity, we will use the following notation:

$$\begin{aligned} F_i(y,z) &= (\nabla f_i(\bar{x})y, \nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y))^T, \\ G_j(y,z) &= (\nabla g_j(\bar{x})y, \nabla g_j(\bar{x})z + \nabla^2 g_j(\bar{x})(y,y))^T. \end{aligned}$$

Now, we are in a position to state the primal form of our second-order necessary conditions.

THEOREM 3.1 *Let $\bar{x} \in A$ be an efficient solution to problem (P). Assume that the second-order (GGCQ) holds for any critical direction. Then, the following system has no solution (y, z):*

$$\begin{array}{ll} F_i(y,z) <_{lex} 0, & i = 1,2,\ldots,l, \\ F_i(y,z) \leqq_{lex} 0, & \textit{for at least one } i, \\ G_j(y,z) \leqq_{lex} 0, & \forall j \in E. \end{array} \tag{8}$$

Proof. Suppose on the contrary that there exists (y, z) such that (8) hold. Thus, we have $z \in K_2(y)$. Without loss of generality, we may assume that

$$\begin{array}{l} F_1(y,z) <_{lex} 0, \\ F_i(y,z) \leqq_{lex} 0, \quad i = 2,\ldots,l, \end{array}$$

by assumption, we have

$$z \in \overline{co}[T_2(Q^1, \bar{x})(y)].$$

Hence, there exists a sequence $\{z^n\}$ of $co[T_2(Q^1, \bar{x})(y)]$ converging to z. Each z^m can be written as a convex combination of some elements of $T_2(Q^1, \bar{x})(y)$, say $z^{m1}, \ldots, z^{ms}$. For each $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, s$, since $z^{mk} \in T_2(Q^1, \bar{x})(y)$, by definition, there exist $x_n^{mk} \in Q^1$ and $t_n^{mk} \longrightarrow +0$ such that

$$x_n^{mk} = \bar{x} + t_n^{mk} y + \frac{1}{2}(t_n^{mk})^2 z^{mk} + o((t_n^{mk})^2).$$

Then, for all n, we have

$$\begin{aligned} f_i(\bar{x}) \geqq f_i(x_n^{mk}) &= f_i(\bar{x}) + t_n^{mk} \nabla f_i(\bar{x}) y \\ &\quad + \tfrac{1}{2}(t_n^{mk})^2 (\nabla f_i(\bar{x}) z^{mk} + \nabla^2 f_i(\bar{x})(y,y)) + o((t_n^{mk})^2), \\ 0 \geqq g_j(x_n^{mk}) &= g_j(\bar{x}) + t_n^{mk} \nabla g_j(\bar{x}) y \\ &\quad + \tfrac{1}{2}(t_n^{mk})^2 (\nabla g_j(\bar{x}) z^{mk} + \nabla^2 g_j(\bar{x})(y,y)) + o((t_n^{mk})^2). \end{aligned}$$

Then, we have

$$\begin{array}{ll} F_i(y,z) \leqq_{lex} 0, & \text{for } i = 2,3,\ldots,l, \\ G_j(y,z) \leqq_{lex} 0, & \text{for } j \in E. \end{array} \tag{9}$$

On the other hand, since $\bar{x} \in A$ is an efficient solution to problem (P), for all n, we have

$$\begin{aligned} & f_1(x_n^{mk}) \geqq f_1 \bar{x}) \\ f_1(\bar{x}) \leqq f_1(x_n^{mk}) &= f_1(\bar{x}) + t_n^{mk} \nabla f_1(\bar{x}) y \\ &\quad + \tfrac{1}{2}(t_n^{mk})^2 (\nabla f_1(\bar{x}) z^{mk} + \nabla^2 f_1(\bar{x})(y,y)) + o((t_n^{mk})^2). \end{aligned}$$

Then, we have

$$0 \underset{lex}{\leqq} F_1(y, z^{mk}). \tag{10}$$

From (9)-(10), it follows that

$$\begin{aligned} &F_1(y, z^{mk}) \geqq_{lex} 0, \\ &F_i(y, z^{mk}) \leqq_{lex} 0, \quad i = 2, 3, \dots, l, \\ &G_j(y, z^{mk}) \leqq_{lex} 0, \quad j \in E. \end{aligned}$$

From the linearity and the continuity of F_i and G_j with respect to z, we have

$$\begin{aligned} &F_1(y, z) \geqq_{lex} 0, \\ &F_i(y, z) \leqq_{lex} 0, \quad i = 2, 3, \dots, l, \\ &G_j(y, z) \leqq_{lex} 0, \quad j \in E. \end{aligned}$$

This is a contradiction. ■

In particular, the system (8) has no solution of the form (0,z).

From the above theorem we get the first-order necessary conditions for efficiency which were already given in Maeda (1994).

Now, we shall state the dual form of Theorem 3.1

THEOREM 3.2 *Let $\bar{x}$ satisfy the assumptions of Theorem 3.1. Then, for each critical direction y, there exist multipliers $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ such that*

$$\begin{aligned} &\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ &\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) + \right)(y, y) \geqq 0, \\ &\lambda > 0, \quad \mu \geqq 0, \quad \mu_j = 0 \quad \forall j \notin E(y), \\ &E(y) = \{\ j \in \{1, \dots, m\} \mid g_j(\bar{x}) = 0, \quad \nabla g_j(\bar{x}) y = 0\ \}. \end{aligned} \tag{11}$$

Proof. Let y be a critical direction. Then, the system:

$$\begin{aligned} &\nabla f(\bar{x}) z + \nabla^2 f(\bar{x})(y, y) \leq 0, \\ &\nabla g_{E(y)}(\bar{x}) z + \nabla^2 g_{E(y)}(\bar{x})(y, y) \leqq 0, \end{aligned} \tag{12}$$

has no solution z, which is equivalent to

$$\begin{aligned} &\nabla f(\bar{x}) z + \nabla^2 f(\bar{x})(y, y)\xi && \leq 0, \\ &\nabla g_{E(y)}(\bar{x}) z + \nabla^2 g_{E(y)}(\bar{x})(y, y)\xi && \leqq 0, \\ &-\xi && < 0, \end{aligned}$$

having no solution $z \in \mathbb{R}^n$, $\xi \in R$. By Slater's theorem of the alternative, Mangasarian (1969), there exist multipliers λ and μ such that either (13) or

(14) holds:

$$
\begin{aligned}
&\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\
&\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) > 0, \\
&\lambda \geq 0, \quad \mu \geqq 0, \quad \mu_j = 0 \ \ \forall j \notin E(y);
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\
&\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) > 0, \\
&\lambda > 0, \quad \mu \geqq 0, \quad \mu_j = 0 \ \ \forall j \notin E(y).
\end{aligned} \tag{14}
$$

Let us assume that (14) does not hold. This is equivalent to the inconsistency of the system

$$
\begin{aligned}
&\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\
&\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) - s.1 = 0, \\
&\lambda > 0, \quad s \geqq 0, \quad \mu \geqq 0, \quad \mu_j = 0 \ \ \forall j \notin E(y).
\end{aligned}
$$

By Tucker's theorem of the alternative, Mangasarian (1969), there exist z and $t \geqq 0$ satisfying

$$
\begin{aligned}
&\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(y,y)t \leq 0, \\
&\nabla g_{E(y)}(\bar{x})z + \nabla^2 g_{E(y)}(\bar{x})(y,y)t \leqq 0.
\end{aligned}
$$

Since (12) has no solution, we have $t = 0$; hence,

$$
\nabla f(\bar{x})z \leq 0, \quad \nabla g_{E(y)}(\bar{x})z \leqq 0.
$$

On the other hand,

$$
\begin{aligned}
&\nabla f(\bar{x})y = 0, \\
&\nabla g_{E(y)}(\bar{x})y = 0, \qquad \nabla g_{E \setminus E(y)}(\bar{x})y < 0,
\end{aligned}
$$

because y is critical. Thus,

$$
\begin{aligned}
&\nabla f(\bar{x})(y + \epsilon z) \leq 0, \\
&\nabla g_E(\bar{x})(y + \epsilon z) \leqq 0,
\end{aligned}
$$

for any sufficiently small $\epsilon > 0$, which contradicts the first-order necessary conditions for efficiency. This completes the proof. ■

EXAMPLE 3.1 *Consider the following problem:*

$$\begin{aligned} min \quad & (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1^2 - x_2^2, x_2 - x_1), \\ s.t \quad & g_1(x_1,x_2) = x_1 + x_2 \leqq 0, \\ & g_2(x_1,x_2) = x_1 - x_2 \leqq 0. \end{aligned}$$

The $\bar{x} = (x_1, x_2)^T = (0,0)^T$ *is an efficient solution and*

$$\begin{aligned} Q^1 &= \{(x_1,x_2)^T \in I\!R^2 | x_1 = x_2,\ x_1 \leqq 0\}, \\ Q^2 &= \{(x_1,x_2)^T \in I\!R^2 | x_1^2 = x_2^2,\ x_1 \leqq 0\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} K_2(0) \quad &= \overline{co}[T_2(Q^1,\bar{x})(0)] \cap \overline{co}[T_2(Q^2,\bar{x})(0)] \\ &= \{(z_1,z_2)^T \in I\!R^2 | z_1 = z_2,\ z_1 \leqq 0\}. \end{aligned}$$

Moreover, for each critical direction $y \neq 0$, *we have*

$$\begin{aligned} K_2(y) \quad &= \overline{co}[T_2(Q^1,\bar{x})(y)] \cap \overline{co}[T_2(Q^2,\bar{x})(y)] \\ &= \{(z_1,z_2)^T \in I\!R^2 | z_1 = z_2\}. \end{aligned}$$

Therefore, the (GGCQ) holds at $\bar{x}$ *for any critical direction* y. *Then, for* $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1,1,0,1)$, *we have*

$$\sum_{i=1}^{2} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{2} \mu_j \nabla g_j(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and for each critical direction y,

$$(y_1,y_2)\left(\sum_{i=1}^{2} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{2} \mu_j \nabla^2 g_j(\bar{x}) \right)(y_1,y_2)^T = y_1^2 - y_2^2 = 0.$$

4. Sufficient conditions for the second-order generalized Guignard CQ

In the preceding section, we introduced the second-order constraint qualification (4). In this section, we shall present several conditions which guarantee (4).

THEOREM 4.1 *Let* y *be any critical direction. If any of conditions (a) through (e) holds, then (4) holds.*

(a) *Ben-Tal's Constraint Qualification (BTCQ): For each* $i = 1, 2, \ldots, l$, *the system*

$$\begin{aligned} &\nabla f_k(\bar{x})v + \nabla^2 f_k(\bar{x})(y,y) < 0, \quad k = 1,2,\ldots,l \text{ and } k \neq i, \\ &\nabla g_j(\bar{x})v + \nabla^2 g_j(\bar{x})(y,y) < 0, \quad j \in E(y), \end{aligned}$$

has a solution $v \in I\!R^n$.

(b) *Cottle-Type Constraint Qualification (CCQ): For each* $i = 1, 2, \ldots, l$, *the system*

$$\begin{aligned} &\nabla f_k(\bar{x})v < 0, \quad k = 1,2,\ldots,l \text{ and } k \neq i, \\ &\nabla g_j(\bar{x})v < 0, \quad j \in E, \end{aligned}$$

has a solution $v \in I\!R^n$.

(c) *Slater's Constraint Qualification (SCQ):* f_i, $i = 1, 2, \ldots, l$, *and* g_j, $j = 1, \ldots, m$ *are all convex on* $\mathbb{R}^n$, *and for each* $i = 1, 2, \ldots, l$, *the system*

$$f_k(x) < f_k(\bar{x}), \quad k = 1, 2, \ldots, l \text{ and } k \neq i,$$
$$g_j(x) < 0, \quad j = 1, \ldots, m.$$

has a solution $x \in \mathbb{R}^n$.

(d) *Mangasarian-Fromovitz's Constraint Qualification (MFCQ):* $\nabla f_i(\bar{x})$, $i = 1, 2, \ldots, l$ *are linearly independent and the system*

$$\nabla f_i(\bar{x})v = 0, \quad i = 1, 2, \ldots, l,$$
$$\nabla g_j(\bar{x})v < 0, \quad j \in E.$$

has a solution $v \in \mathbb{R}^n$.

(e) *Linear Constraint Qualification (LCQ):* f_i, $i = 1, 2, \ldots, l$, *and* g_j, $j \in E$, *are all linear.*

Proof. To show that if any of conditions (a) through (e) holds then (4) holds, it is only sufficient to show that (a) guarantees (4); since (d) yields (b) which in turn yields (a), and (c) yields (b).

Next, we show that (a) guarantees (4). Let z be any element of $K_2(y)$. Then, we have

$$\nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y, y) \leqq 0, \quad i = 1, \ldots, l,$$
$$\nabla g_j(\bar{x})z + \nabla^2 g_j(\bar{x})(y, y) \leqq 0, \quad \forall j \in E(y).$$

First, we shall show that, for each $i = 1, 2, \ldots, l$,

$$z \in T_2(Q^i, \bar{x})(y).$$

By assumption, there exists $v \in \mathbb{R}^n$ such that

$$\nabla f_k(\bar{x})v + \nabla^2 f_k(\bar{x})(y, y) < 0, \quad k = 1, 2, \ldots, l \text{ and } k \neq i,$$
$$\nabla g_j(\bar{x})v + \nabla^2 g_j(\bar{x})(y, y) < 0, \quad j \in E(y),$$

For any positive sequence $\{t_n\}$ converging to 0, we shall define the sequence $\{z^n\}$ converging to z by

$$z^n = z + t_n(v - z).$$

From (15), for each n, we have

$$\nabla f_k(\bar{x})z^n + \nabla^2 f_k(\bar{x})(y, y) < 0, \quad i = 1, \ldots, l \text{ and } k \neq i,$$
$$\nabla g_j(\bar{x})z^n + \nabla^2 g_j(\bar{x})(y, y) < 0, \quad j \in E(y).$$

For each z^n, $n = 1, 2, \ldots$, and any positive sequence $\{\mu_s\}$ converging to 0, we shall define the sequence $\{x^{ns}\}$ converging to $\bar{x}$ by

$$x^{ns} = \bar{x} + \mu_s y + 1/2\mu_s^2 z^n + o(\mu_s^2).$$

Then, for all s sufficiently large, we have

$$\begin{aligned} x^{ns} &= \bar{x} + \mu_s y + 1/2\mu_s^2 z^n + o(\mu_s^2), \\ f_k(x^{ns}) &= f_k(\bar{x} + \mu_s y + 1/2\mu_s^2 z^n + o(\mu_s^2)) \\ &= f_k(\bar{x}) + \mu_s \nabla f_k(\bar{x})y + 1/2\mu_s^2(\nabla f_k(\bar{x})z^n + \nabla^2 f_k(\bar{x})(y, y)) + o(\mu_s^2) \\ &\leq f_k(\bar{x}), \quad k = 1, 2, \ldots, l \text{ and } k \neq i, \end{aligned}$$

and

$$\begin{aligned} g_j(x^{ns}) &= g_j(\bar{x} + \mu_s y + 1/2\mu_s^2 z^n + o(\mu_s^2)) \\ &= g_j(\bar{x}) + \mu_s \nabla g_j(\bar{x}) y + 1/2\mu_s^2(\nabla g_j(\bar{x}) z^n + \nabla^2 g_j(\bar{x})(y,y)) + o(\mu_s^2) \\ &< g_j(\bar{x}) \\ &= 0, \quad j \in E. \end{aligned}$$

For $j \notin E$, from the continuity of g_j, it follows that

$$g_j(x^{ns}) < 0, \quad \text{all } s \text{ sufficiently large.}$$

Hence, we have that

$$x^{ns} \in Q^i, \quad \text{all } s \text{ sufficiently large.}$$

and

$$z^n \in T_2(Q^i, \bar{x})(y).$$

Since $T_2(Q^i, \bar{x})(y)$ is closed, we have

$$z \in T_2(Q^i, \bar{x})(y), \quad i = 1, 2, \ldots, l.$$

Therefore, we have

$$z \in \bigcap_{i=1}^l T_2(Q^i, \bar{x})(y) \subseteq \overline{co}[T_2(Q^i, \bar{x})(y)].$$

It is easily proved that the condition (e) quarantees (4). This completes the proof. □

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