Control and Cybernetics

vol. **28** (1999) No. 2

Limiting ellipsoid for reachability set of 2D continuous-discrete linear system with disturbances

by

Ewa Krasoń

Air Force Academy, Dęblin, Poland

Abstract: A model of 2D continuous-discrete linear system is considered whose input disturbances are unknown but bounded by known ellipsoid. The formula for a support function of reachability set of this system is presented. A method of construction of ellipsoid limiting reachability set is proposed and illustrated with an example.

Keywords: 2D continuous-discrete linear systems, reachability set.

1. Introduction

A general 2D continuous-discrete model of linear systems has been investigated in Kaczorek (1994), Kaczorek and Stajniak (1994), Kaczorek (1995), and Stajniak (1995), where solutions, local reachability and controllability as well as minimum energy control problems for particular cases of this model have been considered. In Krasoń (1997) one linear model of 2D continuous-discrete systems with disturbances have been studied. The countepart of reachability set known for models of 1D systems has been introduced. The formulae for the support function and the diameter of reachability set of the system under disturbances limited to ellipsoid has been established. However, in practice, description of reachability set in this form does not allow to apply computer calculation. The idea of ellipsoid approximating reachability set, known for the 1D continuous and discrete systems from Schlaepfer and Schweppe (1972), Schweppe (1973), Chernousko (1981), Honin (1985), Kurzhanski and Valyi (1997), is very useful. A method of construction of ellipsoid limiting the reachability set for 2D continuous-discrete linear control system with ellipsoidal disturbances which will be proposed is founded on the results from Krasoń (1997).

2. The reachability set

Consider the following 2D continuous-discrete linear system

$$\dot{x}(t,k+1) = Ax(t,k) + Cw(t,k) \quad t \in [0,T], \quad k \in [0,N] \tag{1}$$

where $\dot{x}(t,k) = \frac{\partial x(t,k)}{\partial t}$, $x(t,k) \in R^n$ is a state vector, $w(t,k) \in R^q$ stands for the disturbance vector, $A \in R^{n\times n}$ and $C \in R^{n\times q}$ are real matrices.

Boundary conditions for (1) are given by

$$x(t,0) = x_1(t), \quad t \in [0,T] \text{ and } x(0,k) = x_2(k), \quad k \in [0,N] \tag{2}$$

where $x_1(t)$ and $x_2(k)$ are known and $x_1(0) = x_2(0)$.

Assume that the disturbance vectors $w(t,k)$ belong to the ellipsoid

$$W_e = \{w : (w-m)^T Q^{-1}(w-m) \leq 1\} \tag{3}$$

where $m \in R^n$ is its centre, Q is a symmetric positive-definite matrix in $R^{q\times q}$ and ^{T} denotes transposition.

In this paper we treat $w(t,k)$ as the input of the system (1) and we admit the following terminology:

DEFINITION 2.1 *(Krasoń, 1997)*
The reachability set $X^e_{t,k}$ is the set of all possible states at the moment (t,k) of the system (1) with boundary conditions (2) for all possible disturbances from the ellipsoid W_e.

The following two theorems were proved in Krasoń (1997):

THEOREM 2.1 *The reachability set $X^e_{t,k}$ is convex.*

The convex set X in R^n may be described by the support function (Rockafellar, 1972)

$$h(z\,|X) = \max_{x\in X} z^T x, \quad z \in R^n \tag{4}$$

THEOREM 2.2 *The support function of reachability set $X^e_{t,k}$ has the form*

$$h(z\,|X^e_{t,k}) =$$
$$z^T \left[A^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} x_1(s)ds + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) + \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} A^{k-i-1} C m\right] +$$
$$+ \sum_{i=0}^{k-1} (2k-2i-1)^{\frac{1}{2}} \frac{t^{k-i}}{(k-i)!} \left[z^T (A^{k-i-1}C) Q (A^{k-i-1}C)^T z\right]^{\frac{1}{2}}, \quad z \in R^n. \tag{5}$$

Analysis of the formula (5) allows to formulate the following:

THEOREM 2.3 *The reachability set $X^e_{t,k}$ may be presented as the sum*

$$X^e_{t,k} = a_{t,k} + E_{t,0} + E_{t,1} + \ldots + E_{t,k-1}, \tag{6}$$

where $a_{t,k}$ is a vector and $E_{t,i}$ $(i = 0, 1, \ldots, k-1)$ are ellipsoids in R^n.

Proof According to Theorem 2.2 we can write

$$h(z\,|X^e_{t,k}) = z^T A^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} x_1(s)ds +$$
$$\sum_{i=0}^{k-1}\left\{ z^T \left[\frac{t^i}{i!}A^i x_2(k-i) + \frac{t^{k-i}}{(k-i)!}A^{k-i-1}Cm\right]\right.$$
$$\left. + \frac{t^{k-i}}{(k-i)!}\left[(2k-2i-1)\, z^T \left(A^{k-i-1}C\right) Q \left(A^{k-i-1}C\right)^T z\right]^{\frac{1}{2}}\right\} \tag{7}$$

From theory of ellipsoids (Schweppe, 1973; Kurzhanski and Valyi, 1997) we know that

$$h(z \mid E) = z^T c + (z^T \Gamma z)^{\frac{1}{2}}, \quad z \in R^n \tag{8}$$

is a support function of the ellipsoid

$$E = \{x \in R^n : (x-c)^T \Gamma^{-1}(x-c) \leq 1\} \tag{9}$$

Hence the sum on the right-hand side of (7) may be treated as the sum of support functions of k ellipsoids

$$E_{t,i} = \{x \in R^n : (x - c_{t,i})^T \Gamma_{t,i}^{-1}(x - c_{t,i}) \leq 1\}, \quad i = 0, 1, \ldots, k-1 \tag{10}$$

with the centres

$$c_{t,i} = \frac{t^i}{i!}A^i x_2(k-i) + \frac{t^{k-i}}{(k-i)!}A^{k-i-1}Cm \tag{11}$$

and the matrices

$$\Gamma_{t,i} = (2k-2i-1)\left[\frac{t^{k-i}}{(k-i)!}\right]^2 (A^{k-i-1}C)Q(A^{k-i-1}C)^T \tag{12}$$

moved by the vector

$$a_{t,k} = A^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} x_1(s)ds. \tag{13}$$

■

EXAMPLE 2.1 We consider the system (1) where:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix},$$

$$x_1(t) = \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}, \quad x_2(k) = \begin{bmatrix} 0 \\ k \\ 1 \end{bmatrix},$$

$$W_e = \left\{ w \in R^2 : (w-m)^T Q^{-1} (w-m) \le 1 \right\},$$

$$m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.05 & 0.04 \\ 0.04 & 0.05 \end{bmatrix}.$$

We assume $t = 0.5$ and $k = 3$.

According to the latest theorem the reachability set $X^e_{0.5;3}$ is the following sum:

$$X^e_{0.5;3} = a_{0.5;3} + E_{0.5;0} + E_{0.5;1} + E_{0.5;2}$$

where $a_{0.5;3} = 0.5A^3 \int_0^{0.5} (0.5-s)^2 x_1(s)\, ds = \begin{bmatrix} 2^{-7} \times 16.(3) \\ 2^{-7} \times 23.(6) \\ 2^{-4} \times 0.(3) \end{bmatrix}$.

$E_{0.5;0}, E_{0.5;1}, E_{0.5;2}$ are the ellipsoids in R^3 with the centres:

$$c_{0.5;0} = x_2(3) + \frac{1}{48} A^2 C m = \begin{bmatrix} 2^{-4} \times 3 \\ 2^{-4} \times 50.(3) \\ 2^{-4} \times 16.(6) \end{bmatrix},$$

$$c_{0.5;1} = \frac{1}{2} A x_2(2) + \frac{1}{8} A C m = \begin{bmatrix} 2^{-3} \times 13 \\ 2^{-3} \times 17 \\ 2^{-3} \times 6 \end{bmatrix},$$

$$c_{0.5;2} = \frac{1}{8} A^2 x_2(1) + \frac{1}{2} C m = \begin{bmatrix} 1 \\ 2^{-3} \times 6 \\ 2^{-3} \times 9 \end{bmatrix}$$

and the matrices

$$\begin{aligned} \Gamma_{0.5;0} &= \frac{5}{2404} \left(A^2 C\right) Q \left(A^2 C\right)^T \\ &= \begin{bmatrix} 2^{-8} \times 11.25 & 2^{-8} \times 12.75 & 2^{-8} \times 2.5 \\ 2^{-8} \times 12.75 & 2^{-8} \times 15.25 & 2^{-8} \times 2.8(3) \\ 2^{-8} \times 2.5 & 2^{-8} \times 2.8(3) & 2^{-8} \times 0.(5) \end{bmatrix} \end{aligned}$$

$$\Gamma_{0.5;1} = \frac{3}{64} (AC) Q (AC)^T = \begin{bmatrix} 2^{-6} \times 6.75 & 2^{-6} \times 2.79 & 2^{-6} \times 1.98 \\ 2^{-6} \times 2.79 & 2^{-6} \times 1.23 & 2^{-6} \times 0.78 \\ 2^{-6} \times 1.98 & 2^{-6} \times 0.78 & 2^{-6} \times 0.6 \end{bmatrix}$$

$$\Gamma_{0.5;2} = \frac{1}{4} C Q C^T = \begin{bmatrix} 0.0125 & 0.01 & 0.025 \\ 0.01 & 0.0125 & 0.02 \\ 0.025 & 0.02 & 0.05 \end{bmatrix}$$

REMARK 2.1 *Usually, the set $X = \{x : x_1 + x_2; x_1 \in E_1$ and $x_2 \in E_2\}$ is not an ellipsoid. Hence reachability set $X^e_{t,k}$ is not an ellipsoid, neither.*

3. Limiting ellipsoids for reachability set of linear system with disturbances

In practice, efficient determination of the reachability set in the state space of the system (1) is difficult. In many cases, for convenience of computation, finding the ellipsoid covering the set $X_{t,k}^e$ may be useful. In this section the method for constructing the ellipsoids limiting the reachability sets will be described. It is founded on Theorem 2.3.

From formula (7), for fixed t and $k = 1$, we have

$$h(z \mid X_{t,1}^e) = z^T A \int_0^t x_1(s)ds + z^T[x_2(1) + tCm] + t[z^T(CQC^T)z^T]^{\frac{1}{2}}, \; z \in R_n \tag{14}$$

This indicates that the reachability set $X_{t,1}^e$ is the ellipsoid with the centre $m_{t,1} = x_2(1) + tCm$, moved by the vector $a_{t,1} = A\int_0^t x_1(s)ds$, and the matrix $Q_{t,1} = t^2CQC^T$.

When $k \geq 2$ the reachability set $X_{t,k}^e$ is a sum of k ellipsoids $E_{t,i}$ moved by the vector $a_{t,k}$. Then, on the basis of (6), we generate limiting ellipsoids successively:

1) $E_{t,1}^L$ as limiting ellipsoid for $E_{t,0} + E_{t,1}$,

2) $E_{t,2}^L$ as limiting ellipsoid for $E_{t,1}^L + E_{t,2}$,

$\vdots \; \vdots$

$k-1$) $E_{t,k-1}^L$ as limiting ellipsoid for $E_{t,k-2}^L + E_{t,k-1}$.

To this end, the method known from Schlaepfer and Schweppe (1972), Schweppe (1973), and Honin (1985), will be used.

1) Limiting ellipsoid $E_{t,1}^L$ has the centre

$$m_{t,1} = c_{t,0} + c_{t,1} = x_2(k) + tAx_2(k-1) + \left(\frac{t^k}{k!}A^{k-1} + \frac{t^{k-1}}{(k-1)!}A^{k-2}\right)Cm \tag{15}$$

and the matrix

$$Q_{t,1} = (p_{t,1}^{-1} + 1)\Gamma_{t,0} + (p_{t,1} + 1)\Gamma_{t,1}, \quad p_{t,1} \geq 0. \tag{16}$$

If $p_{t,1}$ is the unique positive root of the equation

$$d_{t,1}p_{t,1}^3 + b_{t,1}p_{t,1}^2 - b_{t,1}p_{t,1} - c_{t,1} = 0, \tag{17}$$

where $d_{t,1} = tr(\Gamma_{t,1}\Gamma_{t,1})$; $b_{t,1} = tr(\Gamma_{t,0}\Gamma_{t,1})$; $c_{t,1} = tr(\Gamma_{t,0}\Gamma_{t,0})$ then the matrix $Q_{t,1}$ has the minimal norm ($\|Q\| = [tr(QQ^T)]^{\frac{1}{2}}$) in the class (16).

2) Limiting ellipsoid $E_{t,2}^L$ has the centre

$$m_{t,2} = m_{t,1} + c_{t,2} = x_2(k) + tAx_2(k-1) + \frac{t^2}{2}A^2x_2(k-2) + \\ + \left[\frac{t^k}{k!}A^{k-1} + \frac{t^{k-1}}{(k-1)!}A^{k-2} + \frac{t^{k-2}}{(k-2)!}A^{k-3}\right]Cm \tag{18}$$

and the matrix

$$Q_{t,2} = (p_{t,2}^{-1} + 1)Q_{t,1} + (p_{t,2} + 1)\Gamma_{t,2} \tag{19}$$

where $p_{t,2}$ is the unique positive root of the equation

$$d_{t,2}p_{t,2}^3 + b_{t,2}p_{t,2}^2 - b_{t,2}p_{t,2} - c_{t,2} = 0, \tag{20}$$

$$d_{t,2} = tr(\Gamma_{t,2}\Gamma_{t,2});\ b_{t,2} = tr(Q_{t,1}\Gamma_{t,2});\ c_{t,2} = tr(Q_{t,1}Q_{t,1}).$$

$\vdots\ \vdots$

$k-1$) Limiting ellipsoid $E^L_{t,k-1}$ is determined by parameters

$$m_{t,k-1} = \sum_{i=0}^{k-1}\left[\frac{t^i}{i!}A^i x_2(k-i) + \frac{t^{k-i}}{(k-i)!}A^{k-i-1}Cm\right] \tag{21}$$

$$Q_{t,k-1} = \left(p^{-1}_{t,k-1}+1\right)Q_{t,k-2} + (p_{t,k-1}+1)\Gamma_{t,k-1}, \tag{22}$$

where $p_{t,k-1}$ is the unique positive root of the equation

$$d_{t,k-1}p^3_{t,k-1} + b_{t,k-1}p^2_{t,k-1} - b_{t,k-1}p_{t,k-1} - c_{t,k-1} = 0, \tag{23}$$

$d_{t,k-1} = tr(\Gamma_{t,k-1}\Gamma_{t,k-1})$; $b_{t,k-1} = tr(Q_{t,k-2}\Gamma_{t,k-1})$; $c_{t,k-1} = tr(Q_{t,k-2}Q_{t,k-2})$.

Finally

$$E^x_{t,k} = E^L_{t,k-1} + a_{t,k} \tag{24}$$

is the ellipsoid limiting the reachability set $X^e_{t,k}$. The support function of this ellipsoid, according to (8), has the form

$$h(z\,|E^x_{t,k}) = z^T(a_{t,k} + m_{t,k-1}) + (z^T Q_{t,k-1}z)^{\frac{1}{2}}, \quad z \in R^n \tag{25}$$

where the vectors $a_{t,k}$, $m_{t,k-1}$ and the matrix $Q_{t,k-1}$ are described by the formulae (13), (21) and (22), respectively.

REMARK 3.1 *From (21) it follows that location of the centre of ellipsoid $E^x_{t,k}$ depends on boundary conditions (2), centre m of ellipsoid W_e, matrices A and C of the system (1) and, obviously, values of t and k. We know from the theory of ellipsoids that directions and lengths of the axes of a limiting ellipsoid are determined by eigen-vectors and eigen-values of the matrix $Q_{t,k-1}$ which is constructed on the basis of matrices Q, A and C.*

We will denote

$$r = d(E^x_{t,k}) - d(X^e_{t,k}) \tag{26}$$

where

$$d(X) = \max_{\|z\|\le 1}[h(z\,|X) + h(-z\,|X)] \tag{27}$$

is a diameter of the set $X \in R^n$ defined in Kurzhanski (1977) as a maximum length of the projection of the set X onto the straight line αz, $\alpha \in R$, for $\|z\| \le 1$.

We can admit the value of r as a certain estimation measure of reachability set $X^e_{t,k}$ by limiting ellipsoid $E^x_{t,k}$. Diameter $d(E^x_{t,k})$ is the longest axis of the ellipsoid $E^x_{t,k}$.

EXAMPLE 3.1 We consider the system (1) where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_2(k) = \begin{bmatrix} k \\ 1 \end{bmatrix},$$

$$W_e = \{w \in R^2 : w^T w \leq 0.01\}, \quad m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

For $t = 1$ and $k = 2$, according to (7), the support function of reachability set has the form

$$\begin{aligned} h(z\,|X^e_{1,2}) &= z^T \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left\{ z^T \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{\sqrt{3}}{20} \left(z^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} z \right)^{\frac{1}{2}} \right\} \\ &+ \left\{ z^T \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{10} \left(z^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z \right)^{\frac{1}{2}} \right\} \end{aligned}$$

On account of Theorem 2.3 we treat it as a sum of the support functions of two ellipsoids:

$E_{1,0}$ with the centre $c_{1,0} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the matrix $\Gamma_{1,0} = \begin{bmatrix} 0.0075 & 0 \\ 0 & 0.03 \end{bmatrix}$,

$E_{1,1}$ with the centre $c_{1,1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the matrix $\Gamma_{1,1} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$,

after having been moved by the vector $a_{1,2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

From (15) and (24) it follows that limiting ellipsoid $E^x_{1,1}$ has a centre $m_{1,1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. To obtain the matrix $Q_{1,1}$ for this ellipsoid we solve the equation

$$0.0002p^3 + 0.000375p^2 - 0.000375p - 0.00095625 = 0$$

where $0.0002 = tr\,(\Gamma_{1,1}\Gamma_{1,1})$; $0.000375 = tr\,(\Gamma_{1,0}\Gamma_{1,1})$; $0.00095625 = tr\,(\Gamma_{1,0}\Gamma_{1,0})$.

The unique positive root of the last equation is $p = 1.5$. Using (16) we may write

$$Q_{1,1} = \frac{5}{3} \begin{bmatrix} 0.0075 & 0 \\ 0 & 0.03 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} = \begin{bmatrix} 0.0375 & 0 \\ 0 & 0.075 \end{bmatrix}$$

The greater eigenvalue of this matrix is equal 0.075. The longer axis and simultaneously the diameter of the limiting ellipsoid has the value $d(E^x_{1,1}) = 2\sqrt{0.075} = 0.5476$. From Krasoń (1997) we have in the same example $d(X^e_{1,2}) = 0.5464$. According to (26), $r = 0.0012$.

4. Limiting ellipsoid for reachability set of linear control system with disturbances

Consider a 2D continuous-discrete linear control system described by equation

$$\dot{x}(t,k+1) = Ax(t,k) + Bu(t,k) + Cw(t,k), \quad t \in [0,T], \quad k \in [0,N] \tag{28}$$

where $u(t,k) \in R^p$ is a control vector, $w(t,k) \in R^q$ denotes a disturbance vector and $A \in R^{n\times n}$, $B \in R^{n\times p}$, $C \in R^{n\times q}$ are real matrices. The boundary conditions for the system (28) have the form (2). Moreover, assume that $u(t,k) \in U \subset R^p$.

DEFINITION 4.1 *The reachability set $X^e_{t,k}(u)$ is the set of all possible states at the moment (t,k) of the system (28) with the boundary conditions (2) for all possible disturbances from ellipsoid W_e, for a fixed $u \in U$.*

Krasoń (1997) proves the following

THEOREM 4.1 *The support function of the reachability set $X^e_{t,k}(u)$ of the system (28) with boundary conditions (2) is given by*

$$\begin{aligned} h\left(z\,|X^e_{t,k}(u)\right) &= z^T A^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} x_1(s)ds + \\ &+ \sum_{i=0}^{k-1} z^T A^{k-i-1} B \int_0^t \frac{(t-s)^{k-i-1}}{(k-i-1)!} u(s,i)ds + \\ &+ \sum_{i=0}^{k-1} \left\{ z^T \left[\frac{t^i}{i!} A^i x_2(k-i) + \frac{t^{k-i}}{(k-i)!} A^{k-i-1} C m \right] + \right. \\ &\left. + \frac{t^{k-i}}{(k-i)!} \left[(2k-2i-1)\, z^T \left(A^{k-i-1}C\right) Q \left(A^{k-i-1}C\right)^T z \right]^{\frac{1}{2}} \right\} \\ z &\in R^n \end{aligned} \tag{29}$$

From the above formula and Theorem 2.3 follows the

THEOREM 4.2 *The reachability set $X^e_{t,k}(u)$ may be presented as the following sum*

$$X^e_{t,k}(u) = a_{t,k}(u) + E_{t,0} + E_{t,1} + \ldots + E_{t,k-1} \tag{30}$$

where:

$$a_{t,k}(u) = a_{t,k} + b_{t,k}(u) \tag{31}$$

$a_{t,k}$ has the form (13),

$$b_{t,k}(u) = \sum_{i=0}^{k-1} A^{k-i-1} B \int_0^t \frac{(t-s)^{k-i-1}}{(k-i-1)!} u(s,i)ds$$

and $E_{t,i}$ $(i = 0,1,\ldots,k-1)$ are ellipsoids in R^n described by (10), (11) and (12).

Using the procedure from Section 3 it is possible to construct ellipsoid $E^e_{t,k}(u)$ covering the reachability set $X^e_{t,k}(u)$. From (24), (30) and (31) it follows that

$$E^x_{t,k}(u) = E^x_{t,k} + b_{t,k}(u) \tag{32}$$

EXAMPLE 4.1 Consider the system (28) where A, C, $x_1(t)$, $x_2(k)$, m and Q have the form from Example 3.1 and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. To simplify, we do not determine the end of control and assume $u(s,i) = u_i$. Then, for $t = 1$ and $k = 2$, from Theorem 4.2 we have

$$b_{1,2}(u) = \sum_{i=0}^{1} \frac{1}{(2-i)!} A^{1-i} B u_i = u_0 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_2.$$

In Example 3.1 the limiting ellipsoid $E^x_{1,2}$ has been constructed for the reachability set $X^e_{1,2}$ of the system (1)

$$E^x_{1,2} = \left\{ x \in R^2 : (x - m_{1,1})^T Q^{-1}_{1,1} (x - m_{1,1}) \leq 1 \right\}$$

where $m_{1,1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $Q_{1,1} = \begin{bmatrix} 0.0375 & 0 \\ 0 & 0.075 \end{bmatrix}$, $Q^{-1}_{1,1} = \begin{bmatrix} 26.(6) & 0 \\ 0 & 13.(3) \end{bmatrix}$.
The limiting ellipsoid $E^x_{1,2}(u)$ has the centre $m_{1,1}(u) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + u_0 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_1$.
Finally

$$\begin{aligned} E^x_{1,2}(u) \; = \; & \left\{ x \in R^2 : \left[x - \left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} + u_0 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_1 \right) \right]^T \right. \\ & \left. \begin{bmatrix} 26.(6) & 0 \\ 0 & 13.(3) \end{bmatrix} \left[x - \left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} + u_0 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_1 \right) \right] \leq 1 \right\} \end{aligned}$$

THEOREM 4.3 *The diameter of the ellipsoid $E^x_{t,k}(u)$ limiting the reachability set $X^e_{t,k}(u)$ of the system (28) with boundary conditions (2) for disturbances from the set W_e is equal to the diameter of the ellipsoid $E^x_{t,k}$ limiting the reachability set $X^e_{t,k}$ of the system (1) with the same boundary conditions and for disturbances of the same type.*

Proof Taking into account definition (27), (32) and the appropriate properties of a support function we can write

$$\begin{aligned} d\left(E^x_{t,k}(u)\right) &= \max_{\|z\| \leq 1} \left\{ \left[z^T b^u_{t,k} + h\left(z \left| E^x_{t,k} \right.\right) \right] + \left[-z^T b^u_{t,k} + h\left(-z \left| E^x_{t,k} \right.\right) \right] \right\} \\ &= d\left(E^x_{t,k}\right) \end{aligned} \tag{33}$$

5. Concluding remarks

A method of construction of ellipsoid limiting the reachability set $X^e_{t,k}$ for 2D continuous-discrete linear system (1) with boundary conditions (2) under disturbances bounded by ellipsoid (3) has been proposed. This algorithm is based on Theorem 2.3 and the method from Honin (1985). The centre of the ellipsoid $E^x_{t,k}$ may be regarded as the estimation of the state vector $x(t,k)$ for the system (1). The error of this estimation does not exceed half the length of the longest axis of limiting ellipsoid $E^x_{t,k}$. In Section 4 the method of construction of ellipsoid $E^x_{t,k}(u)$ covering the reachability set $X^e_{t,k}(u)$ for continuous-discrete linear control system (28) has been described. It may be useful in the search for a control rule which assures achievement of the target set given in the space of states of the system (28) or in minimization of a selected performance index.

References

BARMISH, B.R., LIN, Y.H. (1978) Reachability of targets subject to disturbances bounded by convex sets. *Int. J. Control*, **28**, 5, 771-783.

CHERNOUSKO, F.L. (1981) Ellipsoidal estimates for the attainable set of a controlled systems. *Appl. Math. and Mech.*, **45**, 1, 11-19.

HONIN, E.A. (1985) Guaranteed estimates of the state of linear system by means ellipsoids. *Evolutional systems in estimation problems*, Swierdlowsk, UNC, 104-123.

KACZOREK, T. (1994) Singular 2-D continuous-discrete linear systems. *Bull. Pol. Acad. Sci.*, **42**, 3, 417-422.

KACZOREK, T., STAJNIAK, A. (1994) Local reachability and minimum energy control of 2-D continuous-discrete linear systems. *Proc. 10-th Intern. Conf. Systems Engineering*, 6-8 Sept. Coventry University, UK.

KACZOREK, T. (1995) Controllability and minimum energy control of 2-D continuous-discrete linear systems. *Appl. Math. and Comp. Sci.*, **5**, 1, 5-21.

KRASOŃ, E. (1997) Diameter of reachability set for 2-D continuous-discrete linear system with disturbances. *Appl. Math. and Comp. Sci.*, **7**, 4.

KURZHANSKI, A.B. (1977) *Control and Observation under Uncertainty Conditions.* Nauka, Moscow (in Russian).

KURZHANSKI, A.B., VALYI, I. (1997) *Ellipsoidal Calculus for Estimation and Control.* Boston, Birkhauser Verlag.

RESHETNIAK, I.N. (1989) Sum of ellipsoids in guaranteed estimation problems. *PMM*, **53**, 2, 249-254.

ROCKAFELLAR, R.T. (1972) *Convex Analysis.* Princeton University Press.

SCHLAEPFER, F.M., SCHWEPPE, F.C. (1972) Continuous-time state estimation under disturbances bounded by convex sets. *IEEE Trans. Automat. Contr.*, **AC-17**, 2, 197-205.

SCHWEPPE, F.C. (1973) *Uncertain Dynamic Systems.* Prentice Hall, Englewood Cliffs.

STAJNIAK, A. (1995) *Analysis of 2-D Continuous Discrete Systems* (in Polish). Ph.D.Thesis, Warsaw University of Technology.