Control and Cybernetics

vol. **28** (1999) No. 2

Frequency responses of linear interval plants with delay[1,2]

by

Mikołaj Busłowicz

Białystok Technical University,
Faculty of Electrical Engineering,
ul. Grunwaldzka 11/15, 15-893 Białystok, Poland

Abstract: The paper gives a simple new method for computing envelopes of the magnitude and phase plots of the family of linear time-invariant interval plants with uncertain delay. The undelayed part of the plant is described by the family of interval transfer functions (the nominator and denominator polynomials are interval polynomials). The proposed method requires the knowledge of the Kharitonov polynomials associated with the nominator and denominator polynomials.

Keywords: interval plant, uncertain delay, frequency responses.

1. Introduction

A linear time-invariant interval plant is described by a family of interval transfer functions. The magnitude and phase plots of such a plant are a collection of the magnitude and phase plots corresponding to all transfer functions from the family. Therefore, knowledge of the envelopes of these plots is required in order to effectively carry out the analysis and design of linear time-invariant interval control systems in the frequency domain.

The frequency domain properties of linear interval systems were developed by Bhattacharyya et al. (1995). The problem of computing the frequency response envelopes of families of linear uncertain systems without delay was considered by Levkovich et al. (1995), Garczarczyk (1997), Chen et al. (1998) and Busłowicz (1998).

Bhattacharyya et al. (1995) showed that the Bode magnitude and phase envelopes of the family of the transfer functions are generated by the corresponding extremal set of transfer functions. Levkovich et al. (1995) provided

[1]The original version of this paper was presented at the 9th International Symposium on System Modelling Control which was held in Zakopane, Poland during April, 1998.

[2]The work (Ref. No. PB W/WE/5/98) was supported by the State Committee for Scientific Research of Poland.

an algorithm for computing the magnitude and phase envelopes of interval rational transfer functions. Chen et al. (1998) proposed a numerical method for computing the frequency response template of a class of rational transfer functions whose coefficients are affine functions of interval parameters. Garczarczyk (1997) presented a method for estimating (for all fixed frequencies) the upper and lower boundaries of the family of frequency responses. In this method, the solution of a corresponding linear interval equation is required. A simple computer method for estimating the bounds of the magnitude and phase plots of a family of linear time-invariant systems with uncertain parameters was proposed by Busłowicz (1998). This method is based on the approximation of the value set of the family of transfer functions by rectangle with sides parallel to the axes of the complex plane.

In this paper we give a simple new method for computing the envelopes of the magnitude and phase plots (and also the Bode plots) for the family of linear time-invariant interval plants with an uncertain delay. The proposed method requires knowledge of the Kharitonov polynomials associated with the nominator and denominator polynomials of the family of interval rational transfer functions which describe the undelayed part of the plant. This method is based on the approach similar to the proposed by Levkovich et al. (1995) but it is simpler to apply.

2. Problem formulation

Consider a family of linear time-invariant interval plants with delay described by the family of the strictly proper transfer functions

$$G(s,p,q,\exp(-sh)) = \frac{N(s,p)}{D(s,q)}\exp(-sh), \quad p \in P, q \in Q, h \in H, \tag{1}$$

where $p = [p_0, p_1, \ldots, p_m]^T$ and $q = [q_0, q_1, \ldots, q_{n-1}]^T$ ($m \leq n$) are vectors of uncertain parameters and $H = [h^-, h^+]$ with $0 \leq h^- < h^+$.

The numerator and denominator polynomials are interval polynomials of the form

$$N(s,p) = p_m s^m + p_{m-1}s^{m-1} + \cdots + p_1 s + p_0, \quad p \in P, \tag{2}$$

$$D(s,q) = s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + q_0, \quad q \in Q, \tag{3}$$

where

$$P = \{p : p_k \in [p_k^-, p_k^+], p_k^- \leq p_k^+, k = 0, 1, \ldots, m\} \tag{4}$$

and

$$Q = \{q : q_k \in [q_k^-, q_k^+], q_k^- \leq q_k^+, k = 0, 1, \ldots, n-1\} \tag{5}$$

are the sets of uncertain coefficients p_k $(k = 0, 1, \ldots, m)$ and q_k $(k = 0, 1, \ldots, n-1)$, respectively.

For any fixed $p \in P$, $q \in Q$ and $h \in H$, the transfer function $G(s, \exp(-sh))$ at $s = j\omega$ can be written in the form

$$G(j\omega, \exp(-j\omega h)) = M(\omega)\exp(j\varphi(\omega)), \tag{6}$$

where

$$M(\omega) = \left|G(j\omega, \exp(-j\omega h))\right|, \tag{7}$$

and

$$\varphi(\omega) = \arg G(j\omega, \exp(-j\omega h)). \tag{8}$$

Let us assume that $\omega \geq 0$ is fixed and then define

$$M^-(\omega) = \min_{p\in P, q\in Q, h\in H} \left|\frac{N(j\omega, p)}{D(j\omega, q)}\exp(-j\omega h)\right|, \tag{9}$$

$$M^+(\omega) = \max_{p\in P, q\in Q, h\in H} \left|\frac{N(j\omega, p)}{D(j\omega, q)}\exp(-j\omega h)\right|, \tag{10}$$

$$\varphi^-(\omega) = \min_{p\in P, q\in Q, h\in H} \arg\left(\frac{N(j\omega, p)}{D(j\omega, q)}\exp(-j\omega h)\right), \tag{11}$$

$$\varphi^+(\omega) = \max_{p\in P, q\in Q, h\in H} \arg\left(\frac{N(j\omega, p)}{D(j\omega, q)}\exp(-j\omega h)\right). \tag{12}$$

From the above it follows that for any fixed $\omega \geq 0$ we have

$$M(\omega) \in [M^-(\omega), M^+(\omega)], \tag{13}$$

and

$$\varphi(\omega) \in [\varphi^-(\omega), \varphi^+(\omega)]. \tag{14}$$

It is easy to see that

$$M^-(\omega) = \frac{\min_{p\in P}|N(j\omega, p)|}{\max_{q\in Q}|D(j\omega, q)|}, \tag{15}$$

$$M^+(\omega) = \frac{\max_{p\in P}|N(j\omega, p)|}{\min_{q\in Q}|D(j\omega, q)|}, \tag{16}$$

and

$$\varphi^-(\omega) = \min_{p\in P}(\arg N(j\omega, p)) - \max_{q\in Q}(\arg D(j\omega, q)) - \omega h^+, \tag{17}$$

$$\varphi^{+}(\omega) = \max_{p \in P}(\arg N(j\omega, p)) - \min_{q \in Q}(\arg D(j\omega, q)) - \omega h^{-}. \tag{18}$$

The aim of this paper is to give a simple method for computing (for any fixed $\omega \geq 0$) the magnitude bounds $M^{-}(\omega)$, $M^{+}(\omega)$ and the phase bounds $\varphi^{-}(\omega)$, $\varphi^{+}(\omega)$. If these bounds are swept over frequency $\omega \geq 0$, we obtain the envelopes of the magnitude and phase plots.

From (15)-(18) it follows that when computing the magnitude and phase bounds for any fixed $\omega \geq 0$ we can consider the interval polynomials (2) and (3) separately. Therefore, we first consider the frequency domain properties of interval polynomials.

3. Frequency domain properties of interval polynomials

Consider an interval constant degree polynomial

$$w(s, a) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \quad a \in A, \tag{19}$$

where

$$A = \{a : a_i \in [a_i^{-}, a_i^{+}], a_i^{-} \leq a_i^{+}, i = 0, 1, \ldots, n, a_n^{-} > 0\}. \tag{20}$$

Denote by $w_k(s)$ $(k = 1, \ldots, 4)$ the Kharitonov polynomials associated with the interval polynomial (19). These polynomials are of the form (see Busłowicz, 1997; Bhattacharyya et al., 1995, for example)

$$w_1(s) = a_0^{+} + a_1^{+} s + a_2^{-} s^2 + a_3^{-} s^3 + a_4^{+} s^4 + \ldots \tag{21}$$

$$w_2(s) = a_0^{-} + a_1^{-} s + a_2^{+} s^2 + a_3^{+} s^3 + a_4^{-} s^4 + \ldots \tag{22}$$

$$w_3(s) = a_0^{-} + a_1^{+} s + a_2^{+} s^2 + a_3^{-} s^3 + a_4^{-} s^4 + \ldots \tag{23}$$

$$w_4(s) = a_0^{+} + a_1^{-} s + a_2^{-} s^2 + a_3^{+} s^3 + a_4^{+} s^4 + \ldots \tag{24}$$

For any fixed $\omega \geq 0$ the value set $w(j\omega, A) = \{w(j\omega, a) : a \in A\}$, associated with the interval polynomial (19), is a rectangle with the vertices $w_k(j\omega)$ $(k = 1, \ldots, 4)$ and with sides parallel to the axes of the complex plane (Busłowicz, 1997; Bhattacharyya et al., 1995, for example).

Let us denote

$$u_k(\omega) = \Re w_k(j\omega), \quad v_k(\omega) = \Im w_k(j\omega), \quad k = 1, \ldots, 4, \tag{25}$$

where the Kharitonov polynomials $w_k(s)$ $(k = 1, \ldots, 4)$ have the forms given by (21)-(24).

Because the value set $w(j\omega, A)$ is a rectangle with sides parallel to the axes of the complex plane, for any fixed $\omega \geq 0$ we have:

1. the maximum and minimum arguments of $w(j\omega, a)$ occur on one of the vertices of the rectangle $w(j\omega, A)$, that is
$$\max_{a\in A} \arg(w(j\omega, a)) = \max\{\arg(w_k(j\omega)), \quad k = 1, \dots, 4\}, \tag{26}$$
$$\min_{a\in A} \arg(w(j\omega, a)) = \min\{\arg(w_k(j\omega)), \quad k = 1, \dots, 4\}, \tag{27}$$
2. the maximum absolute value of $w(j\omega, a)$ always occurs at a vertex of the rectangle $w(j\omega, A)$, that is
$$\max_{a\in A} |w(j\omega, a)| = \max\{|w_k(j\omega)|, \quad k = 1, \dots, 4\}, \tag{28}$$
3. the minimum absolute value of $w(j\omega, a)$ can occur on an edge of the rectangle $w(j\omega, A)$. This value, from the geometrical point of view, is the distance $d(\omega)$ of the rectangle $w(j\omega, A)$ from the origin of the complex plane. This distance can be computed by the following algorithm:
   - if $u_1(\omega)u_2(\omega) \leq 0$ and $v_1(\omega)v_2(\omega) \leq 0$ (that is $0 \in w(j\omega, A)$), then $d(\omega) = \min_{a\in A} |w(j\omega, a)| = 0$,
   - if $u_1(\omega)u_2(\omega) < 0$ and $v_1(\omega)v_2(\omega) > 0$ (that is $w(j\omega, A)$ crosses the imaginary axis), then $d(\omega) = \min(|v_1(\omega)|, |v_2(\omega)|)$,
   - if $u_1(\omega)u_2(\omega) > 0$ and $v_1(\omega)v_2(\omega) < 0$ (that is $w(j\omega, A)$ crosses the real axis), then $d(\omega) = \min(|u_1(\omega)|, |u_2(\omega)|)$,
   - if $u_1(\omega)u_2(\omega) > 0$ and $v_1(\omega)v_2(\omega) > 0$ (that is $w(j\omega, A)$ does not cross the axes of the complex plane), then $d(\omega) = \min\{|w_k(j\omega)|, \quad k = 1, \dots, 4\}$.

4. Computation of magnitude and phase bounds

Let us denote by $N_k(s)$ $(k = 1, \dots, 4)$ and by $D_i(s)$ $(i = 1, \dots, 4)$ the Kharitonov polynomials associated with the interval polynomials $N(s,p)$, $p \in P$, and $D(s,q)$, $q \in Q$, respectively. These polynomials are of the form given by (21)-(24) for the interval polynomials $N(s,p)$ and $D(s,q)$, respectively.

For any fixed $\omega \geq 0$ the value set $N(j\omega, P) = \{N(j\omega, p) : p \in P\}$ associated with the interval polynomial (2) is a rectangle with the vertices $N_k(j\omega)$ $(k = 1, \dots, 4)$ and with sides parallel to the axes of the complex plane. Similarly, the value set $D(j\omega, Q) = \{D(j\omega, q) : q \in Q\}$ associated with the interval polynomial (3) is a rectangle with the vertices $D_i(j\omega)$ $(i = 1, \dots, 4)$ and with sides parallel to the axes of the complex plane.

Hence, from (26) and (27) we have

$$\max_{p\in P}(\arg N(j\omega, p)) = \max\{\arg N_k(j\omega), k = 1, \dots, 4\}, \tag{29}$$

$$\min_{p\in P}(\arg N(j\omega, p)) = \min\{\arg N_k(j\omega), k = 1, \dots, 4\}, \tag{30}$$

$$\max_{q\in Q}(\arg D(j\omega, q)) = \max\{\arg D_i(j\omega), i = 1, \dots, 4\}, \tag{31}$$

$$\min_{q\in Q}(\arg D(j\omega,q)) = \min\{\arg D_i(j\omega), i=1,\ldots,4\}. \tag{32}$$

From (17), (18) and (29)-(32) it follows that for any fixed $\omega \geq 0$ the phase bounds $\varphi^-(\omega)$ and $\varphi^+(\omega)$ can be computed from the formulae

$$\begin{aligned}\varphi^-(\omega) &= \min\{\arg N_k(j\omega),\ k=1,\ldots,4\}\\ &- \max\{\arg D_i(j\omega),\ i=1,\ldots,4\} - \omega h^+,\end{aligned} \tag{33}$$

$$\begin{aligned}\varphi^+(\omega) &= \max\{\arg N_k(j\omega),\ k=1,\ldots,4\}\\ &- \min\{\arg D_i(j\omega),\ i=1,\ldots,4\} - \omega h^-.\end{aligned} \tag{34}$$

Let us introduce the following notation

$$U_{Nk}(\omega) = \Re N_k(j\omega), \quad V_{Nk}(\omega) = \Im N_k(j\omega), \quad k=1,\ldots,4, \tag{35}$$

$$U_{Di}(\omega) = \Re D_i(j\omega), \quad V_{Di}(\omega) = \Im D_i(j\omega), \quad i=1,\ldots,4. \tag{36}$$

The formula (15) can be written in the form

$$M^-(\omega) = N_{\min}(\omega)/D_{\max}(\omega), \tag{37}$$

where

$$N_{\min}(\omega) = \min_{p\in P}|N(j\omega,p)|, \quad D_{\max}(\omega) = \max_{q\in Q}|D(j\omega,q)|. \tag{38}$$

From the above and Section 3 it follows that for any fixed $\omega \geq 0$, $M^-(\omega)$ is computed from (37), where

$$D_{\max}(\omega) = \max\{|D_i(j\omega)|, \quad i=1,\ldots,4\} \tag{39}$$

and $N_{\min}(\omega)$ is computed by the following algorithm:

- if $U_{N1}(\omega)U_{N2}(\omega) \leq 0$ and $V_{N1}(\omega)V_{N2}(\omega) \leq 0$, then $N_{\min}(\omega) = 0$,
- if $U_{N1}(\omega)U_{N2}(\omega) < 0$ and $V_{N1}(\omega)V_{N2}(\omega) > 0$, then $N_{\min}(\omega) = \min(|V_{N1}(\omega)|, |V_{N2}(\omega)|)$,
- if $U_{N1}(\omega)U_{N2}(\omega) > 0$ and $V_{N1}(\omega)V_{N2}(\omega) < 0$, then $N_{\min}(\omega) = \min(|U_{N1}(\omega)|, |U_{N2}(\omega)|)$,
- if $U_{N1}(\omega)U_{N2}(\omega) > 0$ and $V_{N1}(\omega)V_{N2}(\omega) > 0$, then $N_{\min}(\omega) = \min\{|N_k(j\omega)|,\ k=1,\ldots,4\}$.

The formula (16) can be written in the form

$$M^+(\omega) = N_{\max}(\omega)/D_{\min}(\omega), \tag{40}$$

where

$$N_{\max}(\omega) = \max_{p\in P}|N(j\omega,p)|, \quad D_{\min}(\omega) = \min_{q\in Q}|D(j\omega,q)|. \tag{41}$$

From Section 3 it follows that for any fixed $\omega \geq 0$, $M^+(\omega)$ is computed from (40), where

$$N_{\max}(\omega) = \max\{|N_k(j\omega)|,\ k=1,\ldots,4\} \tag{42}$$

and $D_{\min}(\omega)$ is computed by the following algorithm:

- if $U_{D1}(\omega)U_{D2}(\omega) \leq 0$ and $V_{D1}(\omega)V_{D2}(\omega) \leq 0$, then $D_{\min}(\omega) = 0$,
- if $U_{D1}(\omega)U_{D2}(\omega) < 0$ and $V_{D1}(\omega)V_{D2}(\omega) > 0$, then $D_{\min}(\omega) = \min(|V_{D1}(\omega)|, |V_{D2}(\omega)|)$,
- if $U_{D1}(\omega)U_{D2}(\omega) > 0$ and $V_{D1}(\omega)V_{D2}(\omega) < 0$, then $D_{\min}(\omega) = \min(|U_{D1}(\omega)|, |U_{D2}(\omega)|)$,
- if $U_{D1}(\omega)U_{D2}(\omega) > 0$ and $V_{D1}(\omega)V_{D2}(\omega) > 0$, then $D_{\min}(\omega) = \min\{|D_i(j\omega)|,\ i = 1, \dots, 4\}$.

On computing the magnitude bounds $M^-(\omega)$, $M^+(\omega)$ and the phase bounds $\varphi^-(\omega)$, $\varphi^+(\omega)$ for all frequencies $\omega \in [\omega_{\min},\ \omega_{\max}]$ with the step $\Delta\omega$ by the proposed method we obtain the magnitude and phase envelopes of the family of the magnitude and phase plots.

5. Illustrative example

Consider an interval plant, the transfer function of which has the form

$$G(s, p, q, \exp(-sh)) = \frac{N(s,p)}{D(s,q)} \exp(-sh), \quad h \in H = [0.1, 0.3], \tag{43}$$

where

$$N(s,p) = p_1 s + p_0, \quad p_1 \in [10, 20], \quad p_0 \in [20, 60], \tag{44}$$

$$D(s,q) = s^2 + q_1 s + q_0, \quad q_1 \in [30, 80], \quad q_0 \in [0, 20]. \tag{45}$$

The Kharitonov polynomials associated with the interval polynomials (44) and (45), respectively, have the following forms

$$N_1(s) = 20s + 60, \quad N_2(s) = 10s + 20,$$

$$N_3(s) = 20s + 20, \quad N_4(s) = 10s + 60,$$

$$D_1(s) = s^2 + 80s + 20, \quad D_2(s) = s^2 + 30s,$$

$$D_3(s) = s^2 + 80s, \quad D_4(s) = s^2 + 30s + 20.$$

On computing the magnitude and phase envelopes by the proposed method, we obtain the Bode envelopes (i.e. plots of $20 \log M^-(\omega)$ and $20 \log M^+(\omega)$ versus $\log \omega$ and plots of $\varphi^-(\omega)$ and $\varphi^+(\omega)$ versus $\log \omega$) shown in Figure 1. These envelopes were computed using the programs of the MATLAB package.

6. Conclusion

In the paper a new method for computing the envelopes of the magnitude and phase plots of the interval plant with delay is given. The proposed method requires knowledge of the Kharitonov polynomials associated with the nominator and denominator polynomials of the family of interval transfer functions which describe the undelayed part of the plant. Therefore, this method is simple to apply.

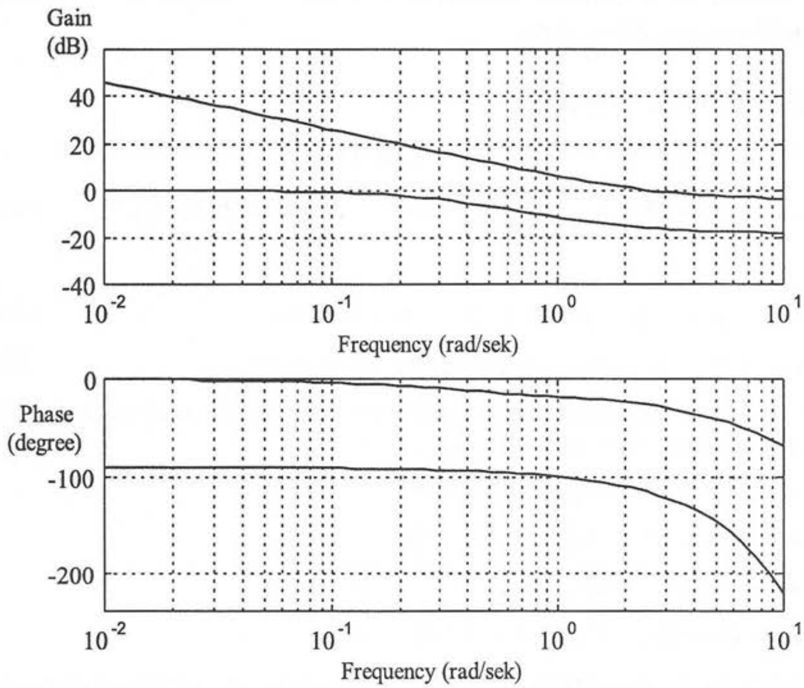

Figure 1. Bode magnitude and phase envelopes.

References

BHATTACHARYYA, S.P., CHAPELLAT, H., KEEL, L.H. (1995) *Robust Control: The Parametric Approach.* New York, Prentice Hall PTR.

BUSŁOWICZ, M. (1997) *Stability of Linear Time-Invariant Systems with Uncertain Parameters.* Białystok, Dział Wydawnictw i Poligrafii Politechniki Białostockiej, Seria Rozprawy Naukowe, 48 (in Polish).

BUSŁOWICZ, M. (1998) Frequency responses of linear systems with uncertain parameters. *Proc. of 21-st Sem. on Fundamentals of Electrotechnics and Circuit Theory,* Gliwice-Ustroń, 435-438 (in Polish).

CHEN, J.-J., HWANG, C. (1998) Computing frequency responses of uncertain systems. *IEEE Trans. Circuits Syst.*, **I**, 45, 304-307.

GARCZARCZYK, Z. (1997) On frequency responses of linear systems with interval parameters. *Proc. of 20-th Sem. on Fundamentals of Electrotechnics and Circuit Theory,* Gliwice-Ustroń, 2, 403-406.

LEVKOVICH, A., ZEHEB, E., COHEN, N. (1995) Frequency response envelopes of a family of uncertain continuous-time systems. *IEEE Trans. Circuits Syst.*, **I**, 42, 156-165.