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Control of delay plants via continuous-time GPC principle

by

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Abstract: The continuous-time generalised predictive control (CGPC) is considered in the context of control of continuous-time systems having a transportation delay. It is shown that the basic CGPC design strategy can be given in a form which facilitates a clear discussion of relevant design consequences concerning stability issues. The main results that follow incorporate several solutions to the delay-plant control design problem and a verification of the proposed algorithms in terms of the closed-loop stability.

Keywords: continuous-time systems, system design, delay, predictive control

1. Introduction

CGPC, the continuous-time generalised predictive control strategy (Demircioglu and Gawthrop, 1991, 1992, Demircioglu and Clarke, 1992, Kowalczuk et al., 1996) being a continuous-time restatement of GPC design paradigm (Clarke et al., 1987, Soeterboek, 1992, Sánchez and Rodellar, 1996) and using a similar long horizon quadratic cost function, has proved suitable for practical non-delay plant systems applications using both classical and adaptive control schemes. Therefore it seems to be equally important to accommodate the CGPC strategy to plants incorporating a transportation delay. Of consequence is also the fact that the traditional methods that use Padé approximation of the delay operator (Marshall, 1979, Gawthrop, 1987) have certain fundamental drawbacks in the CGPC control design context, which are connected with the design parameterisation, calculation, and realisation.

In view of the above, the main objective of this presentation is such a development of the CGPC design that shows a new perspective for the predictive principle by covering the case of continuous-time plants with a transportation delay. Specifically, the AF-CGPC-plus-SSR algorithm incorporating anticipation filtering (AF) and tuned with the use of the swiftness method (Kowalczuk and Marcińczyk, 1996, Kowalczuk et al., 1996) is taken into consideration.

2. AF-CGPC design

Let a scalar linear continuous-time plant be described by the following aggregate model (Demircioglu and Gawthrop, 1991, Kowalczuk and Suchomski, 1998a,b) given as

$$Y(s) = \frac{B(s)}{A(s)}U(s) + \frac{C(s)}{A(s)}V(s) \tag{1}$$

where $U(s)$ and $Y(s)$ are the input and the output signals of the plant, $V(s)$ represents a disturbance function, $A(s)$, $B(s)$ and $C(s)$ are polynomials in the Laplace domain. Let $\deg A(s) = N_A$, $\deg B(s) = N_B < N_A$, $\deg B(s) = N_C < N_A$, and $\rho = N_A - N_B$, denote the relative order of the plant. By performing an instrumental polynomial decomposition with the use of the following first Diophantine equation

$$(D1): \quad \frac{s^k C(s)}{A(s)} = \frac{F_k(s)}{A(s)} + E_k(s),\ k = 0, 1, \ldots, \tag{2}$$

the following operator form of the k-th 'derivative' of the plant output

$$Y_k(s) = s^k Y(s) = Y_k^*(s) + E_k^*(s), k = 0, 1, \ldots, \tag{3}$$

can be obtained that has the predictable part of $Y_k(s)$

$$Y_k^*(s) = \frac{E_k(s)B(s)}{C(s)}U(s) + \frac{F_k(s)}{C(s)}Y(s) \tag{4}$$

and the unpredictable-error part of $Y_k(s)$

$$E_k^*(s) = E_k(s)V(s) \tag{5}$$

where $\deg E_k(s) = N_C - N_A + k$ for $k \geq N_A - N_C$, with $E_k(s) = 0$ for $k < N_A - N_C$ and $\deg F_k(s) \leq N_A - 1$ for $k = 0, 1, \ldots$, while $\deg F_k(s) = N_C + k$ for $k < N_A - N_C$. By making another (second) Diophantine decomposition the transfer function $E_k(s)B(s)/C(s)$ is represented by a strictly proper rational part $G_k(s)/C(s)$ and a polynomial part $H_k(s)$

$$(D2): \quad \frac{E_k(s)B(s)}{C(s)} = \frac{G_k(s)}{C(s)} + H_k(s),\ k \geq 0, \tag{6}$$

where $G_k(s) = 0$ and $H_k(s) = 0$ for $k < N_A - N_C$. In a nontrivial case $k \geq N_A - N_C$, one has:

- if $k < \rho$ (possible only for $N_C > N_B$): $H_k(s) = 0$, $G_k(s) = B(s)E_k(s)$, $\deg G_k(s) = N_C - \rho + k$;
- if $k = \rho$: $\deg H_k(s) = 0$, $\deg G_k(s) \leq N_C - 1$;
- if $k > \rho$: $\deg H_k(s) = k - \rho$, $\deg G_k(s) \leq N_C - 1$.

PROPOSITION 2.1 *In order to obtain a realisable transfer function* $F_k(s)/C(s)$ *in the design the postulate* $N_C = N_A - 1$ *will be observed.* ■

The design polynomials are characterised as follows.

LEMMA 2.1 *Properties of design polynomials. For* $N_A \geq 1$ *one has:*

$$\deg E_k(s) = \begin{cases} 0 & \text{if } k = 0, \\ k-1 & \text{if } k \geq 1, \end{cases} \quad \text{with } E_k(s) = 0 \text{ if } k = 0;$$

$$\deg F_k(s) = \begin{cases} = N_A - 1 & \text{if } k = 0, \\ \leq N_A - 1 & \text{if } k \geq 1, \end{cases} \quad \text{with } F_k(s) = C(s) \text{ if } k = 0;$$

$$\deg H_k(s) = k - \rho \text{ if } k \geq \rho, \text{ with } H_k(s) = 0 \text{ if } k < \rho.$$

With $N_A = 1$ *there is*

$$G_k(s) = 0, k \geq 0,$$

while for $N_A \geq 2$ *one has*

$$\deg G_k(s) = \begin{cases} N_B + k - 1 & \text{if } 1 \leq k < \rho, \\ N_A - 2 & \text{if } k \geq \rho, \end{cases} \quad \text{with } G_k(s) = 0 \text{ if } k = 0,$$

and

$$G_k(s) = B(s)E_k(s) \text{ if } 1 \leq k < \rho.$$

■

Different forms of the second Diophantine decomposition (D2) of (6), that governs the way in which $U(s)$ influences $Y_k^*(s)$ of (4), are characterised in the following lemma.

LEMMA 2.2 *Forms of* $Y_k^*(s)$ *for* $\rho \geq 1$.
(A) *Zero solution:* $E_k(s) = H_k(s) = G_k(s) = 0$ *if* $k = 0$.
(B) *Strictly proper rational solutions* ($\rho \geq 2$)*:* $H_k(s) = 0$ *and* $\deg G_k(s) = N_B + k - 1 < N_A - 1$ *if* $1 \leq k < \rho$.
(C) *Proper rational solution:* $\deg H_k(s) = 0$ *and* $\deg G_k(s) \leq N_A - 2$ *if* $k = \rho$.
(D) *Improper rational solutions:* $\deg H_k(s) = k - \rho$ *and* $\deg G_k(s) \leq N_A - 2$ *if* $k > \rho$. ■

Note that $\rho = 0$ is not taken into account, and that for $\rho = 1$ only the solutions (A), (C) or (D) are possible.

The emulator equation of the predictable (4) thus becomes $Y_k^*(s) = Y_k^-(s) + Y_k^+(s)$, $k \geq \rho$, in which $Y_k^-(s)$ denotes an 'observer' part

$$Y_k^-(s) = \frac{G_k(s)}{C(s)} \cdot U(s) + \frac{F_k(s)}{C(s)} \cdot Y(s), \tag{7}$$

with the control signal filtered by strictly proper $G_k(s)/C(s)$ and the plant output filtered by proper transfer function $F_k(s)/C(s)$, and $Y_k^+(s) = H_k(s) \cdot U(s)$ stands for a 'predictor' part, that is based on polynomials $H_k(s) = h_k + h_{k-1}s + \ldots + h_\rho s^{k-\rho}$, $k \geq \rho$, composed of the plant Markov parameters $h_i s$, $i \geq 0$,

$$\frac{B(s)}{A(s)} = \sum_{i=0}^{\infty} h_i s^{-i}. \tag{8}$$

Finally, the required estimate of the kth derivative of the plant output takes the form

$$y_k^*(t) = L^{-1}[Y_k^*(s)] = \begin{cases} y_k^-(t), & \text{if } k < \rho, \\ y_k^-(t) + y_k^+(t) & \text{if } k \geq \rho, \end{cases} \tag{9}$$

where $y_k^-(t) = L^{-1}[Y_k^-(s)]$, $k \geq 0$, and $y_k^+(t) = \sum_{i=0}^{k-\rho} h_{k-i} \cdot u_i(t)$, $k \geq \rho$, with $u_i(t) = d^i u(t)/dt^i$, $i \geq 0$.

2.1. Estimation of future output of the plant

Let $\hat{t}$ be the variable of future time and $\tau \in [0, T]$, $T \in \mathbf{R}_+$, stands for the relative variable of future time: $\hat{t} = t + \tau$. Moreover, let

$$\mathcal{B}_{k_1,k_2}(t; t_1, t_2) = \{\{t^k/k!\}_{k=k_1}^{k_2}, t \in [t_1, t_2], 0 \leq k_1 \leq k_2\} \tag{10}$$

denote the weighted polynomial basis. Assume that the future output can be approximated in $\mathcal{B}_{0,N_y}(\tau; 0, T)$

$$y(\hat{t}) \cong \tilde{y}(\hat{t})|_{\hat{t}=t+\tau} = \sum_{k=0}^{N_y} \frac{\tau^k}{k!} \cdot y_k(t), \tag{11}$$

where $y_i(t) = d^i y(t)/dt^i$, $i \geq 0$, while N_y denotes the plant output prediction order. Seeking for a realisable form $\hat{y}(\hat{t})$ of the predictor $\tilde{y}(\hat{t})$, one can replace the derivatives by their estimates (9)

$$y(\hat{t}) \cong \hat{y}(\hat{t})|_{\hat{t}=t+\tau} = \sum_{k=0}^{N_y} \frac{\tau^k}{k!} \cdot y_k^*(t). \tag{12}$$

PROPOSITION 2.2 *The control sequence can be designed for $N_y \geq \rho$ (see Lemma 2, C and D).* ■

The following matrix representation of $\hat{y}(\hat{t})$ can then be obtained

$$\hat{y}(\hat{t})|_{\hat{t}=t+\tau} = \mathbf{t}^T_{0,N_y}(\tau)\mathbf{H}_{N_y,N_y-\rho}\mathbf{u}_{N_y-\rho}(t) + \mathbf{t}^T_{0,N_y}(\tau)\mathbf{y}_{\bar{N}_y}(t), \tag{13}$$

where

$$\mathbf{t}_{i,j}(\tau) = [\tau^i/i!\ \tau^{i+1}/(i+1)!\ \dots\ \tau^j/j!]^T.\ \mathbf{t}_{i,j}(\tau) \in \mathbf{R}^{j-i},\ 0 \le i \le j, \tag{14}$$

$$\mathbf{H}_{N_1,N_2} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ h_\rho & 0 & \cdots & 0 \\ h_{\rho+1} & h_\rho & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_{N_1} & h_{N_1-1} & \cdots & h_{N_1-N_2} \end{bmatrix}, \mathbf{H}_{N_1,N_2} \in \mathbf{R}^{(N_1+1)\times(N_2+1)},$$

$$N_1 \ge \rho,\ 0 \le N_2 \le N_1 - \rho, \tag{15}$$

$$\mathbf{u}_i(t) = [u(t)\ u_1(t)\ \cdots\ u_i(t)]^T,\ \mathbf{u}_i(t) \in \mathbf{R}^{i+1}, i \ge 0, \tag{16}$$

$$\mathbf{y}_{\bar{i}}(t) = [y(t)\ y_{\bar{1}}(t)\ \cdots\ y_{\bar{i}}(t)]^T,\ \mathbf{y}_{\bar{i}}(t) \in \mathbf{R}^{i+1},\ i \ge 0. \tag{17}$$

2.2. AF-CGPC classical strategy

Let $w(t)$ denote the reference signal sample and $e(t) = w(t) - y(t)$ be the control error sample at time instant t. Assume that the evolution of the reference signal can be anticipated in the future time domain as

$$\bar{w}(\hat{t})|_{\hat{t}=t+\tau} = y(t) + \delta\bar{w}(\hat{t})|_{\hat{t}=t+\tau}, \tag{18}$$

where $\delta\bar{w}(\hat{t})|_{\hat{t}=t+\tau} = e(t)\cdot\beta_r(\tau)$ is an incremental reference composed of a scaling factor $e(t)$ and a normalised reference $\beta_r(\tau)$, which can be represented in the basis $\mathcal{B}_{0,N_y}(\tau; 0, T)$ as a function $\{\beta_r : \tau \to \beta_r(\tau)\}$ via a set of co-ordinates $\mathbf{r}_{N_y} : \beta_r(\tau) = \mathbf{t}^T_{0,N_y}(\tau) \cdot \mathbf{r}_{N_y}$, where the vector $\mathbf{r}_i = [r_0\ \cdots\ r_i]^T$, $\mathbf{r}_i \in \mathbf{R}^{i+1}$, $i \ge 0$, is composed of the initial Markov parameters of a fictitious anticipation filter (AF), employed in the design. The AF mechanism, introduced in order to moderate the command signal (Demircioglu and Gawthrop, 1991, Kowalczuk et al., 1996), can easily be implemented by using a simple first-order filter $F_A(s) = 1/(1+rs)$, $r > 0$, that leads to the following co-ordinates of $\mathbf{r}_{N_y} : r_0 = 0$ and $r_i = (-1)^{i-1}\cdot r^{-i}$, $i = 1, \dots, N_y$. Instead of the future error $e(\hat{t}) = w(\hat{t}) - y(\hat{t})$ one can take the following prediction $\hat{e}(\hat{t}) = \bar{w}(\hat{t}) - \hat{y}(\hat{t})$, in which both the anticipated reference and the predicted output are employed. Denoting an incremental prediction of the output by $\delta\hat{y}(\hat{t})|_{\hat{t}=t+\tau} = \hat{y}(\hat{t})|_{\hat{t}=t+\tau} - y(t)$ one obtains $\hat{e}(\hat{t}) = \delta\bar{w}(\hat{t}) - \delta\hat{y}(\hat{t})$. Now, let us assume that the current and future control $u(\hat{t})$ can be represented in the basis $\mathcal{B}_{0,N_u}(\tau; 0, T)$ as a function $\{\beta_u : \tau \to \beta_u(\tau)\}$ of the incremental variable of future time τ, that is $u(\hat{t})|_{\hat{t}=t+\tau} = \beta_u(\tau) = \beta^t_u(\tau) = \mathbf{t}^T_{0,N_u}(\tau)\mathbf{u}_{N_u}(t)$, where the co-ordinate vector $\mathbf{u}_{N_u}(t)$ is now parameterised by the current time instant t and the control prediction order N_u, which satisfies the design constraint $N_u \leq N_y - \rho$. Hence, the error prediction $\hat{e}(\hat{t})$ takes the form

$$\hat{e}(\hat{t})|_{\hat{t}=t+\tau} =$$
$$e(t) \cdot \mathbf{t}_{0,N_y}^T(\tau)\mathbf{r}_{N_y} + y(t) - \mathbf{t}_{0,N_y}^T(\tau)\mathbf{y}_{\bar{N}_y}(t) - \mathbf{t}_{0,N_y}^T(\tau)\mathbf{H}_{N_y,N_u}\mathbf{u}_{N_u}(t), \quad (19)$$

that, by virtue of (17), can be simplified to

$$\hat{e}(\hat{t})|_{\hat{t}=t+\tau} = \mathbf{t}_{0,N_y}^T(\tau)\left(e(t) \cdot \mathbf{r}_{N_y} - \mathbf{y}_{\bar{\bar{N}}_y}(t)\right) - \mathbf{t}_{0,N_y}^T(\tau)\mathbf{H}_{N_y,N_u}\mathbf{u}_{N_u}(t), \quad (20)$$

where $\mathbf{y}_{\bar{\bar{i}}}(t) = [0 \; y_{\bar{1}} \; \cdots \; y_{\bar{i}}(t)]^T$, $\mathbf{y}_{\bar{\bar{i}}}(t) \in \mathbf{R}^{i+1}$, $i \geq 1$. Having determined the predicted control error $\hat{e}(\hat{t})$ as a function of control $u(\hat{t})$, for $\hat{t} = t + \tau$, one can seek an optimal shape of this control.

Consider the following quadratic cost index

$$J(\mathbf{u}_{N_u}(t)) = \int_{T_1}^{T_2} \hat{e}^2(t+\tau)d\tau + \lambda \int_{T_3}^{T_4} u^2(t+\tau)d\tau, \quad (21)$$

where $\lambda \geq 0$ denotes a control weighting factor, and the pairs (T_1, T_2) and (T_3, T_4), with $T_i \leq T_j$ in each pair (T_i, T_j), determine the error and control signal observation horizons for the predicted error and control effort, respectively. Minimisation of (21)

$$\mathbf{u}_{N_u}^*(t) = \arg \min_{\mathbf{u}_{N_u}(t) \in \mathbf{R}^{N_u+1}} J(\mathbf{u}_{N_u}(t)), \quad (22)$$

yields the solution

$$\mathbf{u}_{N_u}^*(t) = \mathbf{K}_{N_u,N_y}(e(t) \cdot \mathbf{r}_{N_y} - \mathbf{y}_{\bar{\bar{N}}_y}(t)), \quad (23)$$

where

$$\mathbf{K}_{N_u,N_y} = \mathbf{T}_{N_u,N_y}^{-1}\mathbf{H}_{N_y,N_u}^T\mathbf{T}_{N_y}, \; \mathbf{K}_{N_u,N_y} \in \mathbf{R}^{(N_u+1)\times(N_y+1)}, \quad (24)$$

$$\mathbf{T}_{N_u,N_y} = \mathbf{H}_{N_y,N_u}^T\mathbf{T}_{N_y}\mathbf{H}_{N_y,N_u} + \lambda\mathbf{T}_{N_u}, \; \mathbf{T}_{N_u,N_y} \in \mathbf{R}^{(N_u+1)\times(N_u+1)}, \quad (25)$$

$$\mathbf{T}_{N_y} = \mathbf{T}_{0,N_y}^{0,N_y}(T_1, T_2), \; \mathbf{T}_{N_y} \in \mathbf{R}^{(N_y+1)\times(N_y+1)}, \quad (26)$$

$$\mathbf{T}_{N_u} = \mathbf{T}_{0,N_u}^{0,N_u}(T_3, T_4), \; \mathbf{T}_{N_u} \in \mathbf{R}^{(N_u+1)\times(N_u+1)}, \quad (27)$$

$$\mathbf{T}_{k,l}^{m,n}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \mathbf{t}_{k,l}(\tau)\mathbf{t}_{m,n}^T(\tau)d\tau, \; \mathbf{T}_{k,l}^{m,n}(\tau_1, \tau_2) \in \mathbf{R}^{(l-k+1)\times(n-m+1)},$$
$$0 \leq k \leq l, \; 0 \leq m \leq n. \quad (28)$$

As it has been shown in Kowalczuk and Suchomski (1997) for any $\lambda \geq 0$ the solution (23) exists.

2.3. Implementation of control law

The first co-ordinate of $\mathbf{u}^*_{N_u}(t)$ determines the optimal control input $u^*(t)$ at the time instant t

$$u^*(t) = \mathbf{k}^T_{N_y}(e(t)\cdot \mathbf{r}_{N_y} - \mathbf{y}^{=}_{N_y}(t)), \tag{29}$$

where $\mathbf{k}^T_{N_y}$, $\mathbf{k} \in \mathbf{R}^{N_y+1}$, is the first row of $\mathbf{K}_{N_u,N_y}$. The resulting closed-loop control law takes in the Laplace domain the following form

$$U^*(s) = g(W(s) - Y(s)) - M(s)\cdot U^*(s) - N(s)\cdot Y(s), \tag{30}$$

where $g = \mathbf{k}^T_{N_y}\mathbf{r}_{N_y}$ is the controller scalar gain, while $M(s)$ is a strictly proper transfer function

$$M(s) = \frac{G^{=}_{N_C}(s)}{C(s)}, \tag{31}$$

and $N(s)$ is a proper one

$$N(s) = \frac{F^{=}_{N_A}(s)}{C(s)}, \tag{32}$$

The numerator polynomials $F^{=}_{N_A}(s)$ and $G^{=}_{N_C}(s)$ are defined as

$$F^{=}_{N_A}(s) = \mathbf{k}^T_{N_y}\mathbf{F}^{=}_{N_y,N_A}\mathbf{s}_{N_A-1},\ \deg F^{=}_{N_A}(s) = N_A - 1, \tag{33}$$

$$G^{=}_{N_C}(s) = \mathbf{k}^T_{N_y}\mathbf{G}^{=}_{N_y,N_C}\mathbf{s}_{N_C-1},\ \deg G^{=}_{N_C}(s) = N_C - 1 = N_A - 2, \tag{34}$$

where $\mathbf{s}_i = [s^0\ s^1\ \cdots\ s^i]^T$, $i \geq 0$, while elements of matrices

$$\mathbf{F}^{=}_{N_y,N_A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ f_{1,0} & f_{1,1} & \cdots & f_{1,N_A-1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{N_y,0} & f_{N_y,1} & \cdots & f_{N_y,N_A-1} \end{bmatrix},\ \mathbf{F}^{=}_{N_y,N_A} \in \mathbf{R}^{(N_y+1)\times N_A}, \tag{35}$$

$$\mathbf{G}^{=}_{N_y,N_C} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ g_{1,0} & g_{1,1} & \cdots & g_{1,N_C-1} \\ \cdots & \cdots & \cdots & \cdots \\ g_{N_y,0} & g_{N_y,1} & \cdots & g_{N_y,N_C-1} \end{bmatrix},\ \mathbf{G}^{=}_{N_y,N_C} \in \mathbf{R}^{(N_y+1)\times N_C}, \tag{36}$$

are composed of the coefficients of the polynomials of (2) and (6)

$$F_k(s) = f_{k,0} + \cdots + f_{k,N_A-1}s^{N_A-1},\ 1 \leq k \leq N_y, \tag{37}$$

$$G_k(s) = g_{k,0} + \cdots + g_{k,N_C-1}s^{N_C-1},\ 0 \leq k \leq N_y. \tag{38}$$

To facilitate further discussion let us define the following three polynomials

$$E^{=}_{N_y}(s) = \mathbf{k}^T_{N_y}\mathbf{E}^{=}_{N_y,N_y}\mathbf{s}_{N_y-1},\ \deg E^{=}_{N_y}(s) = N_y - 1, \tag{39}$$

$$H^{=}_{\overline{N}_u}(s) = \mathbf{k}^T_{N_y}\mathbf{H}_{N_y,N_u}\mathbf{s}_{N_u},\ \deg H^{=}_{\overline{N}_u}(s) = N_u, \tag{40}$$

$$L^{=}_{\overline{N}_A}(s) = \mathbf{k}^T_{N_y}\mathbf{L}^{=}_{\overline{N}_y,N_A}\mathbf{s}_{N_A-1},\ \deg L^{=}_{\overline{N}_A}(s) = N_A - 1, \tag{41}$$

where the matrix

$$\mathbf{E}^{=}_{\overline{N}_y,N_y} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ e_0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e_{N_y-1} & e_{N_y-2} & \cdots & e_0 \end{bmatrix},\ \mathbf{E}^{=}_{\overline{N}_y,N_y} \in \mathbf{R}^{(N_y+1)\times N_y}, \tag{42}$$

is composed of the coefficients of the quotient polynomials of (2)

$$E_k(s) = \begin{cases} 0 & k = 0, \\ e_{k-1} + e_{k-2}s + \cdots + e_0 s^{k-1} & 1 \le k \le N_y, \end{cases} \tag{43}$$

and the matrix

$$\mathbf{L}^{=}_{\overline{N}_y,N_A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ l_{1,0} & l_{1,1} & \cdots & l_{1,N_A-1} \\ \cdots & \cdots & \cdots & \cdots \\ l_{N_y,0} & l_{N_y,1} & \cdots & l_{N_y,N_A-1} \end{bmatrix},\ \mathbf{L}^{=}_{\overline{N}_y,N_A} \in \mathbf{R}^{(N_y+1)\times N_A}, \tag{44}$$

is composed of the coefficients of the residual polynomials

$$L_k(s) = l_{k,0} + l_{k,1}s + \cdots + l_{k,N_A-1}s^{N_A-1},\ 1 \le k \le N_y, \tag{45}$$

defined by the third Diophantine decomposition

$$(D3): \quad \frac{s^k B(s)}{A(s)} = \frac{L_k(s)}{A(s)} + H_k(s),\ k \ge 0. \tag{46}$$

From the above it follows that $\deg L_k(s) \le N_A - 1$ for $k \ge 0$, while $\deg L_k(s) = N_B + k$ for $k < \rho$, and $L_k(s) = B(s)$ for $k = 0$. Moreover, it can easily be checked that

$$A(s)G^{=}_{\overline{N}_C}(s) + B(s)F^{=}_{\overline{N}_A}(s) = C(s)L^{=}_{\overline{N}_A}(s). \tag{47}$$

From (6) and (46) we obtain the following representations of $F^{=}_{\overline{N}_A}(s)$, $G^{=}_{\overline{N}_C}(s)$ and $L^{=}_{\overline{N}_A}(s)$

$$F^{=}_{\overline{N}_A}(s) = \mathbf{k}^T_{N_y}\mathbf{s}^{=}_{\overline{N}_y} \cdot C(s) - E^{=}_{\overline{N}_y}(s)A(s), \tag{48}$$

$$G^{=}_{\overline{N}_C}(s) = E_{N_y}(s)B(s) - H^{=}_{\overline{N}_y-\rho}(s)C(s), \tag{49}$$

$$L^{=}_{\overline{N}_A}(s) = \mathbf{k}^T_{N_y}\mathbf{s}^{=}_{\overline{N}_y} \cdot B(s) - H^{=}_{\overline{N}_y-\rho}(s)A(s), \tag{50}$$

where $\mathbf{s}^{=}_{\overline{i}} = [0\ s^1\ \cdots\ s^i]^T$, $i \ge 0$. With $\lambda = 0$ certain nontrivial properties of $G^{=}_{\overline{N}_C}(s)$ and $L^{=}_{\overline{N}_A}(s)$ can be proved. Namely, from (15) and (24) it follows that

$$\mathbf{k}^T_{N_y}\mathbf{H}_{N_y,N_y-\rho} = \begin{cases} [1\ 0\ \cdots\ 0 \vdots \mathbf{k}^T_{N_y}\mathbf{H}^{N_u+1}_{N_y,N_y-\rho}] & \text{if } N_u < N_y - \rho, \\ [1\ 0\ \cdots\ 0] & \text{if } N_u = N_y - \rho, \end{cases} \tag{51}$$

where

$$\mathbf{H}_{N_y,N_y-\rho}^{N_u+1} = \begin{bmatrix} 0_{N_u+1,N_y-\rho-N_u} \\ \mathbf{H}_{N_y-N_u-1,N_y-\rho-N_u-1} \end{bmatrix}. \tag{52}$$

As a result $H_{N_u}^{=}(s) = 1$ if $N_u = N_y - \rho$. Thus for $\lambda = 0$ and $N_u = N_y - \rho$ one obtains

$$G_{N_C}^{=}(s) = E_{N_y}^{=}(s)B(s) - C(s), \tag{53}$$

$$L_{N_A}^{=}(s) = \mathbf{k}_{N_y}^T \mathbf{s}_{N_y}^{=} \cdot B(s) - A(s). \tag{54}$$

3. AF-CGPC design for delay plants

The following plant with a transportation delay in the control channel will be considered

$$Y(s) = \frac{B(s)}{A(s)} \cdot U(s)e^{-sT_0} + \frac{C(s)}{A(s)} \cdot V(s),\ T_0 \geq 0. \tag{55}$$

3.1. Delay-predictive solution: AF-DpCGPC

The future k-th derivative, $k \geq 0$, can be denoted by $Y_{k,T_0}^*(s) = Y_k^*(s)e^{sT_0}$. From the development presented in Section 2 it results that

$$Y_{k,T_0}^*(s) = \frac{G_k(s)}{C(s)} \cdot U(s) + \frac{F_k(s)}{C(s)} \cdot Y_{0,T_0}^*(s) + H_k(s) \cdot U(s). \tag{56}$$

From (55) the future output $Y_{0,T_0}^*(s)$ can be computed as

$$Y_{0,T_0}^*(s) = Y(s)e^{sT_0} = \frac{B(s)}{A(s)} \cdot U(s) + \frac{C(s)}{A(s)} \cdot V(s)e^{sT_0}, \tag{57}$$

or approximately by the following 'foreseeable' emulation

$$\hat{Y}_{0,T_0}^*(s) = \frac{B(s)}{A(s)} \cdot U(s) + \frac{C(s)}{A(s)} \cdot V(s). \tag{58}$$

Taking again (55) into account, one arrives at

$$\hat{Y}_{0,T_0}^*(s) = Y(s) + \hat{Y}_{T_0}^{\delta}(s), \tag{59}$$

where the delay-correcting part

$$\hat{Y}_{T_0}^{\delta}(s) = \frac{B(s)}{A(s)}(1 - e^{-sT_0}) \cdot U(s). \tag{60}$$

Consequently, the realisable form of $Y_{k,T_0}^*(s)$, representing the prediction of future output derivatives with negligence of future disturbances, can be shown in the following form

$$\hat{Y}_{k,T_0}^*(s) = \hat{Y}_{k,T_0}^{-}(s) + Y_k^{+}(s), \tag{61}$$

having the 'observer' and 'predictor' parts as follows

$$\hat{Y}^{-}_{k,T_0}(s) = \left[\frac{G_k(s)}{C(s)} + \frac{F_k(s)}{C(s)}\frac{B(s)}{A(s)}(1 - e^{-sT_0})\right] \cdot U(s) + \frac{F_k(s)}{C(s)} \cdot Y(s), \quad (62)$$

$$Y^{+}_{k}(s) = H_k(s) \cdot U(s). \quad (63)$$

With the advanced-future-time variable $\hat{t} = t + T_0 + \tau$, the future output trajectory can be represented in $\mathcal{B}_{0,N_y}(\tau; 0, T)$ by the following equation (see (13))

$$\hat{y}(\hat{t})|_{\hat{t}=t+T_0+\tau} = \mathbf{t}^{T}_{0,N_y}(\tau)\mathbf{H}_{N_y,N_u}\mathbf{u}_{N_u}(t) + \mathbf{t}^{T}_{0,N_y}(\tau)\hat{\mathbf{y}}^{-}_{N_y,T_0}(t), \quad (64)$$

where

$$\hat{\mathbf{y}}^{-}_{N_y,T_0}(t) = [\hat{y}^{-}_{0,T_0}(t)\ \hat{y}^{-}_{1,T_0}(t)\ \cdots\ \hat{y}^{-}_{N_y,T_0}(t)]^T,\ \hat{\mathbf{y}}^{-}_{N_y,T_0}(t) \in \mathbf{R}^{N_y+1}, \quad (65)$$

$$\hat{y}^{-}_{i,T_0}(t) = L^{-1}[\hat{Y}^{-}_{i,T_0}(s)],\ i \geq 0. \quad (66)$$

From (62) it also results that $\hat{y}^{-}_{0,T_0}(t) = y(t) + \hat{y}^{\delta}_{T_0}(t)$, where $\hat{y}^{\delta}_{T_0}(t) = L^{-1}[\hat{Y}^{\delta}_{T_0}(s)]$. Now let

$$\delta\hat{y}(\hat{t})|_{\hat{t}=t+T_0+\tau} = \hat{y}(\hat{t})|_{\hat{t}=t+T_0+\tau} - \hat{y}^{-}_{0,T_0}(t) \quad (67)$$

be the advanced-future incremental prediction of the output. With the conjecture that during the subsequent period T_0 the future setpoint $w(\hat{t})$ will not change, the anticipated reference in τ can be figured similarly to (18)

$$\bar{w}(\hat{t})|_{\hat{t}=t+T_0+\tau} = \hat{y}^{-}_{0,T_0}(t) + \delta\bar{w}(\hat{t})|_{\hat{t}=t+T_0+\tau} \quad (68)$$

where $\delta\bar{w}(\hat{t})$ is the future incremental reference represented in $\mathcal{B}_{0,N_y}(\tau; 0, T)$ as

$$\delta\bar{w}(\hat{t})|_{\hat{t}=t+T_0+\tau} = (w(t) - \hat{y}^{-}_{0,T_0}(t)) \cdot \beta_r(\tau). \quad (69)$$

The prediction of control error can now be given as

$$\begin{aligned}&\hat{e}(\hat{t}) = \bar{w}(\hat{t}) - \hat{y}(\hat{t}) = \\ &\mathbf{t}^{T}_{0,N_y}(\tau)[(e(t) - \hat{y}^{\delta}_{T_0}(t)) \cdot \mathbf{r}_{N_y} - \hat{\mathbf{y}}^{=}_{N_y,T_0}(t)] - \mathbf{t}^{T}_{0,N_y}(\tau)\mathbf{H}_{N_y,N_u}\mathbf{u}_{N_u}(t), \quad (70)\end{aligned}$$

where $\hat{\mathbf{y}}^{=}_{N_y,T_0}(t) = [0\ \hat{y}^{-}_{1,T_0}\ \cdots\ \hat{y}^{-}_{N_y,T_0}(t)]^T$, $\hat{\mathbf{y}}^{=}_{N_y,T_0}(t) \in \mathbf{R}^{N_y+1}$. With the modified cost function

$$J_{T_0}(\mathbf{u}_{N_u}(t)) = \int_{T_1}^{T_2} \hat{e}^2(t + T_0 + \tau)d\tau + \lambda \int_{T_3}^{T_4} u^2(t + \tau)d\tau, \quad (71)$$

and $T_4 \leq T_2$, one gains the following optimal control input

$$u^*(t) = \mathbf{k}^{T}_{N_y}[(e(t) - \hat{y}^{\delta}_{T_0}(t)) \cdot \mathbf{r}_{N_y} - \hat{\mathbf{y}}^{=}_{N_y,T_0}(t)], \quad (72)$$

where $\mathbf{k}^{T}_{N_y}$ is the first row of $\mathbf{K}_{N_u,N_y}$ from (24). The respective closed-loop control law has in the Laplace domain the form of (30) with the input-observer transfer function incorporating a system delay factor

$$M(s) = M_{T_0}(s) = \frac{G^{=}_{N_C}(s)}{C(s)} + \left(g + \frac{F^{=}_{N_A}(s)}{C(s)}\right)\frac{B(s)}{A(s)}(1 - e^{-sT_0}), \quad (73)$$

and the proper transfer function $N(s)$ of (32) having no delay ingredient.

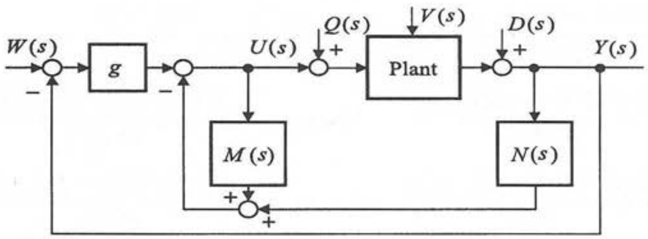

Figure 1. Closed-loop control system

3.2. Stiff solution: AF-DsCGPC

For the plant delay T_0 of a very small value the output prediction $Y^*_{0,T_0}(s)$ can be anticipated in the simplest form of $\hat{Y}^*_{0,T_0}(s)$, gained by assuming $\hat{Y}^\delta_{T_0}(s) = 0$ in (59)

$$Y^*_{0,T_0}(s) \cong \bar{Y}^*_{0,T_0}(s) = \hat{Y}^*_{0,T_0}(s)|_{\hat{Y}^\delta_{T_0}(s)=0} = Y(s). \tag{74}$$

Hence the error-prediction formula (see (70)) acquires the form of

$$\hat{e}(\hat{t})|_{\hat{t}=t+T_0+\tau} = \mathbf{t}^T_{0,N_y}(\tau)(e(t)\cdot\mathbf{r}_{N_y} - \mathbf{y}^{=}_{N_y}(t)) - \mathbf{t}^T_{0,N_y}(\tau)\mathbf{H}_{N_y,N_u}\mathbf{u}_{N_u}(t), \tag{75}$$

which, used in the cost function (71), results in the closed-loop control law of (30).

4. Properties of the CGPC Design

4.1. Internal stability analysis

In order to investigate the internal stability conditions of the resulting closed-loop control system, consider the structure shown in Fig. 1 with four input signals $(W(s), V(s), D(s), Q(s))$ and two output signals $(Y(s), U(s))$ discerned. Thus, the following matrix transfer function can be considered

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} T_{wy}(s) & T_{vy}(s) & T_{dy}(s) & T_{qy}(s) \\ T_{wu}(s) & T_{vu}(s) & T_{du}(s) & T_{qu}(s) \end{bmatrix} \begin{bmatrix} W(s) \\ V(s) \\ D(s) \\ Q(s) \end{bmatrix}. \tag{76}$$

4.1.1. Basic AF-CGPC design

For the non-delay plant of (1) one has

$$T_{wy}(s) = \frac{gB(s)}{P_0(s)}, \quad T_{wu}(s) = \frac{gA(s)}{P_0(s)}, \tag{77}$$

$$T_{vy}(s) = \frac{C(s) + G^{=}_{\overline{N}_C}(s)}{P_0(s)}, \quad T_{vu}(s) = \frac{-(gC(s) + F^{=}_{\overline{N}_A}(s))}{P_0(s)}, \tag{78}$$

$$T_{dy}(s) = \frac{A(s)(C(s) + G^{=}_{\overline{N}_C}(s))}{P(s)}, \quad T_{du}(s) = \frac{-A(s)(gC(s) + F^{=}_{\overline{N}_A}(s))}{P(s)}, \tag{79}$$

$$T_{qy}(s) = \frac{B(s)(C(s) + G^{=}_{\overline{N}_C}(s))}{P(s)}, \quad T_{qu}(s) = \frac{-B(s)(gC(s) + F^{=}_{\overline{N}_A}(s))}{P(s)}, \tag{80}$$

where $P(s)$ denotes the characteristic polynomial

$$P(s) = C(s)P_0(s), \tag{81}$$

and

$$P_0(s) = A(s) + gB(s) + L^{=}_{\overline{N}_A}(s), \ \deg P_0(s) = N_A. \tag{82}$$

COROLLARY 4.1 *The closed-loop control system is internally stable if and only if the polynomials $C(s)$ and $P_0(s)$ are Hurwitz.* ■

4.1.2. AF-DpCGPC control system

According to the delay plant model of (55), one should take into account the following transfer functions of (76)

$$T_{wy}(s) = \frac{gB(s)e^{-sT_0}}{P_0(s)}, \quad T_{wu}(s) = \frac{gA(s)}{P_0(s)}, \tag{83}$$

$$T_{vy}(s) = \frac{R(s)}{A(s)P_0(s)}, \quad T_{vu}(s) = \frac{-(gC(s) + F^{=}_{\overline{N}_A}(s))}{P_0(s)}, \tag{84}$$

$$T_{dy}(s) = \frac{R(s)}{P(s)}, \quad T_{du}(s) = \frac{-A(s)(gC(s) + F^{=}_{\overline{N}_A}(s))}{P(s)}, \tag{85}$$

$$T_{qy}(s) = \frac{B(s)R(s)e^{-sT_0}}{P_{T_0}(s)}, \quad T_{qu}(s) = \frac{-B(s)(gC(s) + F^{=}_{\overline{N}_A}(s))e^{-sT_0}}{P(s)}, \tag{86}$$

where the numerator component function is defined as

$$R(s) = A(s)(C(s) + G^{=}_{\overline{N}_C}(s)) + B(s)(gC(s) + F^{=}_{\overline{N}_A}(s))(1 - e^{-sT_0}), \tag{87}$$

and the closed-loop characteristic polynomial is given by

$$P_{T_0}(s) = A(s)P(s). \tag{88}$$

A crucial conjecture concluded from the above is stated in the form of the following corollary.

COROLLARY 4.2 *For Hurwitz polynomial $P(s)$, the AF-DpCGPC control system is internally stable if and only if the plant jest BIBO-stable.* ■

As it can be seen from the above the AF-DpCGPC controller working as a time-delay compensator, like the classical Smith predictor (Smith, 1959, Laughlin et al., 1987, Palmor, 1996), can be used for stable plants only.

4.1.3. AF-DsCGPC design

The above mentioned restriction on the plant stability is not valid for the AF-DsCGPC design. The transfer functions (76) can, in this case, be given by the following expressions

$$T_{wy}(s) = \frac{gB(s)C(s)e^{-sT_0}}{\bar{P}_{T_0}(s)}, \quad T_{wu}(s) = \frac{gA(s)C(s)}{\bar{P}_{T_0}(s)}, \tag{89}$$

$$T_{vy}(s) = \frac{C(s)(C(s) + G^{=}_{\overline{N}_C}(s))}{\bar{P}_{T_0}(s)}, \quad T_{vu}(s) = \frac{-C(s)(gC(s) + F^{=}_{\overline{N}_A}(s))}{\bar{P}_{T_0}(s)}, \tag{90}$$

$$T_{dy}(s) = \frac{A(s)(C(s) + G^{=}_{\overline{N}_C}(s))}{\bar{P}_{T_0}(s)}, \quad T_{du}(s) = \frac{-A(s)(gC(s) + F^{=}_{\overline{N}_A}(s))}{\bar{P}_{T_0}(s)}, \tag{91}$$

$$\begin{aligned} T_{qy}(s) &= \frac{B(s)(C(s) + G^{=}_{\overline{N}_C}(s))e^{-sT_0}}{\bar{P}_{T_0}(s)}, \\ T_{qu}(s) &= \frac{-B(s)(gC(s) + F^{=}_{\overline{N}_A}(s))e^{-sT_0}}{\bar{P}_{T_0}(s)}, \end{aligned} \tag{92}$$

where $\bar{P}_{T_0}(s)$ denotes the generalised characteristic polynomial

$$\bar{P}_{T_0}(s) = A(s)(C(s) + G^{=}_{\overline{N}_C}(s)) + B(s)(gC(s) + F^{=}_{\overline{N}_A}(s))e^{-sT_0} \tag{93}$$

which has the following asymptotic ($T_0 \to 0$) property $\bar{P}_{T_0}(s) \to P(s)$. Apparently, the closed-loop control system is internally stable if and only if all roots of $\bar{P}_{T_0}(s)$ are in the open LHP.

4.2. Properties of the CGPC Design

4.2.1.

For the predictive design and for the stiff solution the transfer functions of (76) are asymptotically ($T_0 \to 0$) given by (77)-(80).

4.2.2.

For $A(0) = 0$ from (50) it results that $L^{=}_{\overline{N}_A}(0) = 0$. Thus the following corollary, which is true for all the considered GPC-based strategies (AF-CGPC, AF-DpCGPC and AF-DsCGPC), can be stated.

COROLLARY 4.3 *For the CGPC control of a plant with a pole in the origin (a zero root of $A(s)$), the closed loop transfer function $T_{wy}(s)$ is of unity DC gain*

$$T_{wy}(0) = \frac{gB(0)}{A(0) + gB(0) + L^{=}_{\overline{N}_A}(0)}\bigg|_{A(0)=0} = \frac{gB(0)}{gB(0)} = 1. \tag{94}$$

■

On the other hand, with $\lambda = 0$, from (50) and (51) one has $L_{\overline{\overline{N}}_A}(0) = -A(0)$ and what follows.

COROLLARY 4.4 *If there is no pole in the origin (i.e. if $A(0) \neq 0$) the zero nominal steady-state error for the positional tracking (in a stable closed-loop system) is guaranteed by using $\lambda = 0$.* ■

4.2.3.

Let us consider the following special design settings

$$\lambda = 0 \text{ and } N_u = N_y - \rho. \tag{95}$$

In this case, by virtue of (54) and (82), one obtains

$$P_0(s) = \bar{P}_0(s)B(s), \tag{96}$$

where

$$\bar{P}_0(s) = g + \mathbf{k}_{N_y}^T \mathbf{s}_{\overline{\overline{N}}_y} = \mathbf{k}_{N_y}^T(\mathbf{r}_{N_y} + \mathbf{s}_{\overline{\overline{N}}_y}),\ \deg \bar{P}_0(s) = \rho. \tag{97}$$

This formula motivates the AF-DpCGPC design: with the first-order anticipation filter $1/(1 + rs)$, $r > 0$, one has $\bar{P}_0(s)|_{s=-r^{-1}} = 0$, which means that the AF-DpCGPC procedure can be regarded as a partial pole placement approach (Demircioglu and Gawthrop, 1991). Therefore, by the use of (95), the transfer functions of (76) acquire their reduced-order forms

$$T_{wy}(s) = \frac{gB(s)e^{-sT_0}}{\bar{P}_0(s)}, \quad T_{wu}(s) = \frac{gA(s)}{B(s)\bar{P}_0(s)}, \tag{98}$$

$$T_{vy}(s) = \frac{\bar{R}(s)}{A(s)\bar{P}_0(s)}, \quad T_{vu}(s) = \frac{-(gC(s) + F_{\overline{\overline{N}}_A}(s))}{B(s)\bar{P}_0(s)}, \tag{99}$$

$$T_{dy}(s) = \frac{\bar{R}(s)}{C(s)\bar{P}_0(s)}, \quad T_{du}(s) = \frac{-A(s)(gC(s) + F_{\overline{\overline{N}}_A}(s))}{B(s)C(s)\bar{P}_0(s)}, \tag{100}$$

$$T_{qy}(s) = \frac{B(s)\bar{R}(s)e^{-sT_0}}{A(s)C(s)\bar{P}_0(s)}, \quad T_{qu}(s) = \frac{-(gC(s) + F_{\overline{\overline{N}}_A}(s))e^{-sT_0}}{C(s)\bar{P}_0(s)}, \tag{101}$$

where

$$\bar{R}(s) = A(s)E_{\overline{\overline{N}}_y}(s) + (gC(s) + F_{\overline{\overline{N}}_A}(s))(1 - e^{-sT_0}) \tag{102}$$

from which one can draw the following conclusion.

COROLLARY 4.5 *With $\lambda = 0$ and $N_u = N_y - \rho$ the application of the AF-DpCGPC design is restricted to stable and minimum phase plant models, whereas the use of the CGPC design with $T_0 = 0$ is confined to minimum phase models.* ■

Polynomial:	$A(s)$	$B(s)$	$C(s)$
Input: $W(s)$	-	+	-
Input: $V(s)$	+	+	-
Input: $D(s)$	-	+	+
Input: $Q(s)$	+	-	+

Table 1. Internal stability paths: + indicates the impact of the model polynomial

4.2.4.

In order to obtain a pair of input signals sufficient for determining the internal stability conditions let us consider the Table. 1.

Note that the pair $(W(s), V(s))$ chosen in a 'natural' way is not sufficient for the internal stability analysis. Instead, the appropriate pairs are: $(W(s), Q(s))$, $(V(s), D(s))$, $(V(s), Q(s))$ and $(D(s), Q(s))$.

4.3. Markov parameters of Padé approximants

It is instructive to observe that a first order system described as

$$\frac{B(s)}{A(s)} = \frac{b_0}{s-p}, p \in \mathbf{R}, \tag{103}$$

has the following Markov parameters:

$$h_0 = 0,\ h_i = b_0 p^{i-1},\ i \geq 1. \tag{104}$$

It is thus clear that for $|p| > 1$ the sequence $\{h_i\}_0^\infty$ is composed of the Markov parameters with a monotonically increased modulus, and an alternately changing sign for $p < -1$. This result is in contradiction with the conditions (Kowalczuk and Suchomski, 1997) of convergence of the Markovian linear-system representations that is established by a limited series of Markov parameters. In the case of complex poles, this increase need not be monotonic. For instance, a second order system with two complex conjugate poles (p, p^*)

$$\frac{B(s)}{A(s)} = \frac{b_0 + b_1 s}{(s-p)(s-p^*)},\ p, p^* \in \mathcal{C}, \mathrm{Im}\ p \neq 0, \tag{105}$$

has the following Markov parameters:

$$h_0 = 0,\ h_i = 2|\alpha| p^{i-1} \cos\phi_i,\ i \geq 1,\ \alpha = \frac{b_0 + b_1 p}{p - p^*},\ \alpha \in \mathcal{C},$$
$$\phi_i = \arg\alpha + (i-1) \cdot \arg p,\ i \geq 1. \tag{106}$$

Consequently, with $|p| > 1$ a similar divergence effect as in the case of simple poles can be observed. The (m, n) – Padé approximant $P_{m,n}(s, T_0)$ to the delay operator e^{-sT_0}

$$P_{m,n}(s, T_0) = \frac{N_m(s, T_0)}{D_n(s, T_0)}, \tag{107}$$

where $N_m(s, T_0)$ with $\deg N_m(s, T_0) = m$ and $D_n(s, T_0)$ with $\deg D_n(s, T_0) = n$, are the polynomials in s, is defined by (Baker, 1975)

$$e^{-sT_0} - P_{m,n}(s, T_0) = \mathcal{O}(s^{m+n+1}). \tag{108}$$

Thus $P_{m,n}(s, T_0)$ matches the power series of e^{-sT_0} at orders 1 through $m + n$. The first three 'symmetrical' ($m = n$) Padé approximants for the delay are (Baker, 1975, Baker et al., 1981)

$$\begin{aligned} P_{1,1}(s, T_0) &= \frac{1 - sT_0/2}{1 + sT_0/2}, \; P_{2,2}(s, T_0) = \frac{1 - sT_0/2 + s^2T_0^2/12}{1 + sT_0/2 + s^2T_0^2/12}, \\ P_{3,3}(s, T_0) &= \frac{1 - sT_0/2 + s^2T_0^2/10 - s^3T_0^3/120}{1 + sT_0/2 + s^2T_0^2/10 + s^3T_0^3/120}. \end{aligned} \tag{109}$$

The above approximate Padé delay operators can be shown with the use of Markov parameters

$$P_{n,n}(s, T_0) = \sum_{i=0}^{\infty} h_{n,i}(T_0)s^{-i} \tag{110}$$

which can be computed using, for instance, the following recursive rules

$$n = 1 : h_{1,0} = -1, \; h_{1,1} = \frac{4}{T_0}, \; h_{1,i} = -\frac{2}{T_0}h_{1,i-1}, \; i \geq 2; \tag{111}$$

$$\begin{aligned} &n = 2 : h_{2,0} = 1, \; h_{2,1} = -\frac{12}{T_0}, \; h_{2,2} = \frac{72}{T_0^2}, \\ &h_{2,i} = -\frac{6}{T_0^2}(2h_{2,i-2} + T_0h_{2,i-1}), \; i \geq 3; \end{aligned} \tag{112}$$

$$\begin{aligned} &n = 3 : h_{3,0} = -1, \; h_{3,1} = \frac{24}{T_0}, \; h_{3,2} = -\frac{288}{T_0^2}, \; h_{3,3} = \frac{2256}{T_0^3}, \\ &h_{3,i} = -\frac{12}{T_0^3}(10h_{3,i-3} + 5T_0h_{3,i-2} + T_0^2h_{3,i-1}), \; i \geq 4. \end{aligned} \tag{113}$$

Note that with $\sigma = sT_0$ and $|s| = 1$ a sufficient value of T_0 can be (in terms of the above mentioned convergence conditions) defined as $T_{0,n} = |\sigma_{n,\max}| = \max\{|\sigma| : D_n(\sigma, 1) = 0, \sigma \in \mathcal{C}\}$, $n = 1, 2, \ldots$. On the other hand, $T_{0,n}$ can be

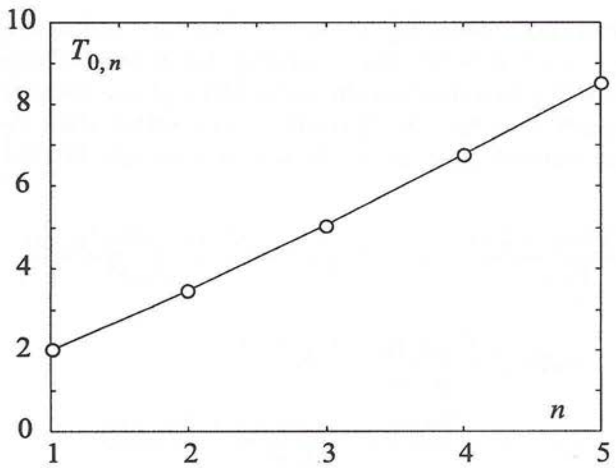

Figure 2. Minimum value of delay suitable for the Padé approximants of order n.

interpreted as a minimum time delay ($T_0 \geq T_{0,n}$) guaranteeing that the worst-case root ($s_{n,\max} = \sigma_{n,\max}/T_{0,n}$) and thus all the poles of the n-th order Padé approximant (107) are within the unit circle ($T_0 : \max\{|s| : D_n(s,T_0) = 0,\ s \in \mathcal{C}\} \leq 1$), which means that in terms of previously given definitions $T_0 \geq T_{0,n}$. The values of $T_{0,n}$'s are shown in Fig. 2 (where $T_{0,1} = 2$, $T_{0,2} = 2\sqrt{3}, \ldots$) to produce an 'almost' linear relationship toward n within the range of $1 \leq n \leq 5$, satisfactory from a practical viewpoint.

4.4. Padé approximants solution: AF-DaCGPC

With the nth order Padé approximant $P_{n,n}(s,T_0)$ the following model of the controlled plant (55) is obtained

$$Y(s) = \frac{B(s)N_n(s,T_0)}{A(s)D_n(s,T_0)} \cdot U(s) + \frac{C(s)D_n(s,T_0)}{A(s)D_n(s,T_0)} \cdot V(s), \tag{114}$$

which has the disturbance channel unchanged. Performing the standard CGPC design procedure for the above model we arrive at the control law of (30) having the strictly proper transfer function

$$M(s) = M_{T_0}(s) = \frac{G^{=}_{\overline{N}_C}(s)}{C(s)D_n(s,T_0)}, \ \deg G^{=}_{\overline{N}_C}(s) = N_A + n - 2, \tag{115}$$

and the low-order proper transfer function $N(s) = F^{=}_{\overline{N}_A}(s)/C(s)$, $\deg F^{=}_{\overline{N}_A}(s) = N_A - 1$, given by (32). In this case, instead of (47), one has the following relation

$$A(s)G^{=}_{\overline{N}_C}(s) + B(s)N_n(s,T_0)F^{=}_{\overline{N}_A}(s) = C(s)L^{=}_{\overline{N}_A}(s), \tag{116}$$

with the instrumental polynomial $L^{=}_{\overline{N}_A}(s)$ of (41) that, this time, is of high order $\deg L^{=}_{\overline{N}_A}(s) = N_A + n - 1$. It is also worth noticing that here the design is based on Markov parameters which describe the high-order rational transfer function of the control channel $B(s)N_n(s,T_0)/(A(s)D_n(s,T_0))$ rather than the original lower order transfer function $B(s)/A(s)$. The resulting transfer functions of (76) are the following:

$$T_{wy}(s) = \frac{gB(s)C(s)D_n(s,T_0)e^{-sT_0}}{P_n(s)}, \quad T_{wu}(s) = \frac{gA(s)C(s)D_n(s,T_0)}{P_n(s)}, \tag{117}$$

$$T_{vy}(s) = \frac{C(s)(C(s)D_n(s,T_0) + G^{=}_{\overline{N}_C}(s))}{P_n(s)}, \quad T_{vu}(s) = \frac{-C(s)D_n(s,T_0)(gC(s) + F^{=}_{\overline{N}_A}(s))}{P_n(s)}, \tag{118}$$

$$T_{dy}(s) = \frac{A(s)(C(s)D_n(s,T_0) + G^{=}_{\overline{N}_C}(s))}{P_n(s)}, \quad T_{du}(s) = \frac{-A(s)D_n(s,T_0)(gC(s) + F^{=}_{\overline{N}_A}(s))}{P_n(s)}, \tag{119}$$

$$T_{qy}(s) = \frac{B(s)(C(s)D_n(s,T_0) + G^{=}_{\overline{N}_C}(s))e^{-sT_0}}{P_n(s)}, \quad T_{qu}(s) = \frac{-B(s)D_n(s,T_0)(gC(s) + F^{=}_{\overline{N}_A}(s))e^{-sT_0}}{P_n(s)}, \tag{120}$$

where

$$P_n(s) = A(s)(C(s)D_n(s,T_0) + G^{=}_{\overline{N}_C}(s)) + B(s)D_n(s,T_0)(gC(s) + F^{=}_{\overline{N}_A}(s))e^{-sT_0}. \tag{121}$$

Since $\bar{P}_{T_0}(s)$ of (93) fits the above given $P_n(s)$ for $n = 0$, it is clear that the stiff design DsCGPC can be interpreted in terms of the DaCGPC with the zero-order Padé approximant $P_{0,0}(s,T_0)$.

5. Illustrative examples

Two numerical examples are considered.

5.1.

Let us start from a stable and non minimum-phase plant model

$$Y(s) = \frac{2-s}{(1+s)^2}e^{-0.5s} \cdot U(s) + \frac{1+0.5s}{(1+s)^2} \cdot V(s).$$

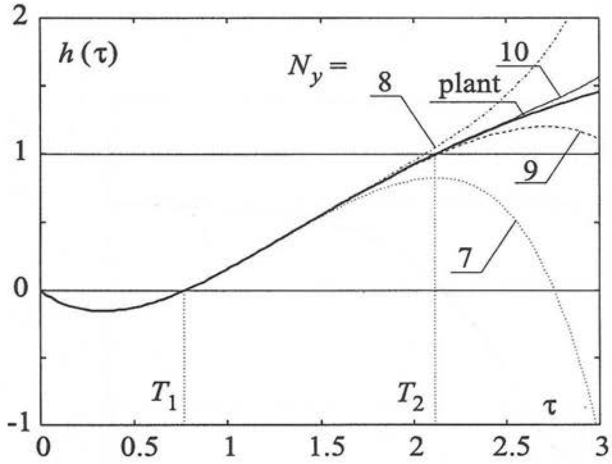

Figure 3. Example 5.1: SSR tuning.

By using $N_u = 0$, $\lambda = 0$ and $r = 0.75$ with $T_1 = 0.763$, $T_2 = 2.125$ and $N_y = 10$ of the SSR rules (see Fig. 3) the DpCGPC control law of (30) characterised by $g = 2.1445$, $F_{\overline{N}_A}^{=}(s) = -1.5963 - 0.5241s$ and $G_{\overline{N}_C}^{=}(s) = 2.1927$ results in the closed-loop behaviour shown in Fig. 4.

5.2.

The second example deals with double-integrators with various time delays

$$Y(s) \doteq \frac{1}{s^2} \cdot U(s)e^{-sT_0} + \frac{1+s}{s^2} \cdot V(s),\ T_0 = 0.2, 0.5, 1, 2$$

and the SSR tuning CGPC settings $T_1 = 0$, $T_2 = 1.4142$, $N_y = 2$, $N_u = 0$, $\lambda = 0$, $r = 1$, lead to the stiff DsCGPC controller of (30) with $g = 0.7678$, $M(s) = 1.7678/(1+s)$, $N(s) = 1.7678s/(1+s)$, and the step responses given in Fig. 5.

Following the SSR-tuning rule with the 1st-order Padé approximant used for different delays, we obtain the different primary CGPC design parameters:

$$T_0 = 0.2 : N_y = 45, T_1 = 0.256, T_2 = 1.614;$$

$$T_0 = 0.5 : N_y = 22, T_1 = 0.639, T_2 = 1.914;$$

$$T_0 = 1.0 : N_y = 15, T_1 = 1.278, T_2 = 2.417;$$

$$T_0 = 2.0 : N_y = 11, T_1 = 2.556, T_2 = 3.458.$$

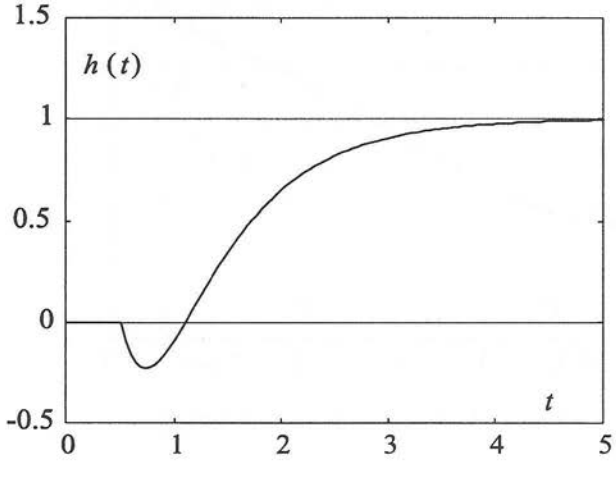

Figure 4. Example 5.1: closed-loop step response.

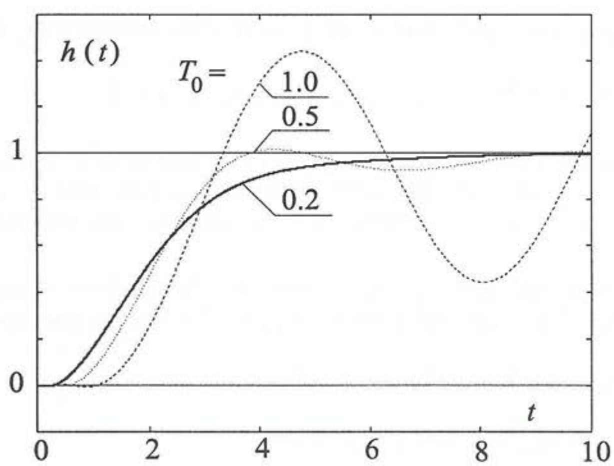

Figure 5. Example 5.2: closed-loop step response.

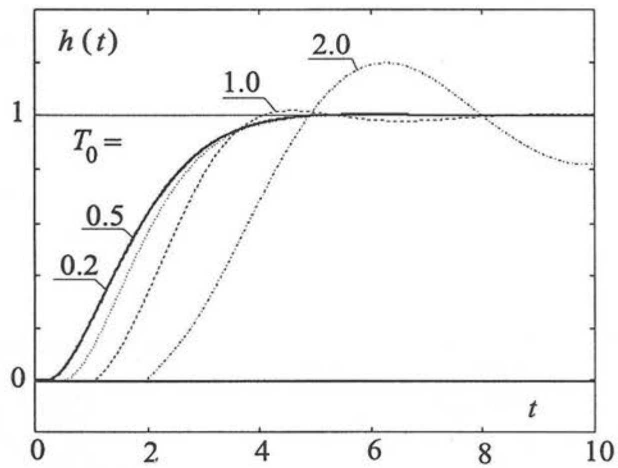

Figure 6. Example 5.2: the DaCGPC control of delay plants, based on the 1st-order Padé approximants.

Consequently, with $N_u = 0$, $\lambda = 0$ and $r = 1$ the DaCGPC controller obtains different form:

$$T_0 = 0.2 : g = 1.1748, M_{T_0}(s) = (2.4882 + 0.1768s)/(1 + 1.1s + 0.1s^2),$$

$$N(s) = 1.7678s/(1 + s);$$

$$T_0 = 0.5 : g = 1.2994, M_{T_0}(s) = (3.6906 + 0.4420s)/(1 + 1.25s + 0.25s^2),$$

$$N(s) = 2.5981s/(1 + s);$$

$$T_0 = 1.0 : g = 1.4240, M_{T_0}(s) = (6.0117 + 0.9006s)/(1 + 1.5s + 0.5s^2),$$

$$N(s) = 3.4074s/(1 + s);$$

$$T_0 = 2.0 : g = 1.5115, M_{T_0}(s) = (11.9964 + 1.9834s)/(1 + 2s + 1s^2),$$

$$N(s) = 5.0065s/(1 + s).$$

The resulting closed-loop behaviours are depicted in Fig. 6. The corresponding Markov representations (Kowalczuk and Suchomski, 1997) of the plant model are given in Fig. 7.

Good performance of both the predictive and stiff controllers for small T_0 is evident. On the other hand, a similar effect achieved with the Padé approximant approach, can be justified by the fact that even for small delays there is, within the limited time interval, a 'suitable' Markov representation that can be determined in the SSR-tuning operation.

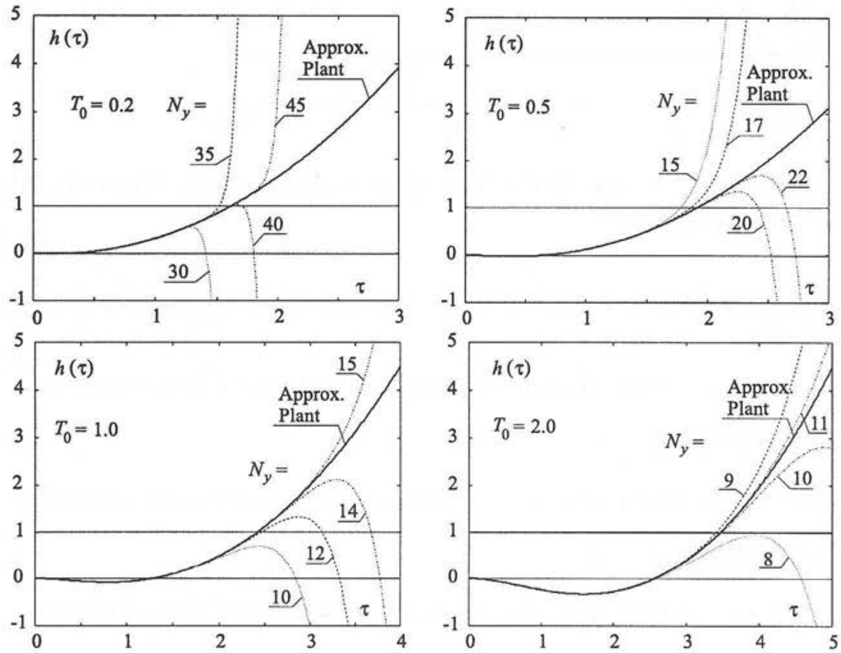

Figure 7. Example 5.2: SSR selection of the CGPC tuning parameters for model with 1st-order Padé approximation.

6. Conclusions

Applicability of the continuous-time generalised predictive control has been extended to the dynamical plants that includes a pure transportation delay. Three treatments of the delay-related problem referred to as the delay-predictive solution, the stiff solution, and the delay-Padé-approximant solution have been proposed. Certain theoretical properties have been discussed, including the issues of stability and realisability of the CGPC systems. Consequently, limitations of the traditional methods based on Padé approximation of the delay operator has been explained that exhibit via the necessity of choosing large observation horizons. Finally, a sample of numerical results illustrating efficacy of the methodology applied to different plant models, has been provided.

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