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Book review:

Generalized Characteristics of First Order PDEs Applications in Optimal Control & Differential Games

by

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The book under review is in principle devoted to one problem: how to extend the method of characteristics (the classical method of constructing the twice differentiable solutions to nonlinear first order PDEs) for the construction of the viscosity solutions to first order PDEs. The notion of characteristics is extended for the following three cases: a) the generalized viscosity solution is nonsmooth while the Hamiltonian may be either smooth or nonsmooth; b) the solution is smooth (classical) but the Hamiltonian is nonsmooth; c) the PDE is the quasilinear second order Euler equation of a variational problem with nonsmooth solution. This is an extension to the characteristics which were found during the study of the singular paths in differential games and optimal control. The regular paths are obtained from the Hamiltonian ODE-systems – the characteristics systems for the Hamilton-Jacobi-Bellman-Isaacs equation, while singular paths are described by similar equations using singular controls. The elimination of singular controls from these equations leads to the notion of singular characteristics which becomes attractive for general nonlinear first order PDEs.

The method of description of regular and singular characteristics, applied in the book, is of differential-geometric type. The classical characteristics define a tangent field on the even-dimensional hypersurface (of codimension one) corresponding to the PDE. The author proposes to use the same characteristics, but defined on the appropriate surface of codimension 3,5,... to construct the nonsmooth - viscosity - solution of the PDE. The integral curves of the characteristics fields on the surface of codimension 3 or more are just the singular characteristics. The regular and singular characteristics are called jointly generalized characteristics and are both used to construct generalized solutions to nonlinear first order PDEs.

The book consists of eight chapters and an appendix:

1. Method of Characteristics in Smooth Problems

3. First Order PDEs in Variational Calculus, Optimal Control and Differential Games
4. Differential Games with Simple Motions on the Manifolds
5. Games of Simple Pursuit and Approach on Two-Dimensional Cone
6. Smooth Solutions of a PDE with Nonsmooth Hamiltonian
7. Shock Waves Related to First Order PDEs
8. Singular Surfaces of Nonsmooth on Multiple Integral Variational Problems

The first chapter contains the classical description of the characteristics method for smooth problems relying on the book of R. Courant. We find in there the construction of twice differentiable solution, the problem of irregular characteristics and the geometrical formulation of the above problem which is particularly convenient for the introduction of the generalized problem (Cauchy problem). Movable boundary problems (arising in differential games) are also considered.

In the second chapter we find the definition of viscosity solution based on the notion introduced by M.G. Crandall, L.C. Evans and P.-L. Lions. This definition is a starting position for introducing the notions of regular and singular points and then of singular curves, singular surfaces and singular manifolds. The latter are the basis for definition and classification of singular characteristics and their equations. This chapter is very important for understanding of the main results of the book and it is really very well written, with many geometrical and intuitive descriptions.

Chapter 3 is a survey of first order PDEs arising in control theory. These are well known equations of Hamilton-Jacobi (in variational calculus), Bellman (in optimal control) and Isaacs (in differential Games). The Hamiltonian, the three fields in mentioned, acquires particular features: in variational calculus it is smooth and convex with respect to the last variable, in optimal control it is convex but not necessarily smooth, in differential games it is not convex and not necessarily smooth. That is why one can meet the focal and equivocal singularities in differential games only.

Chapters four and five are devoted to differential games on the manifolds and surfaces with nonunique geodesic lines connecting the players. This nonuniqueness generates two types of singular surfaces: dispersal and equivocal. For the construction of equivocal surface the theory of Chapter 2 is applied. In Chapter 4 some local properties for the games on a Riemannian manifold with nondegenerate metric are investigated. Based on these results, the complete solution of two games on the conical surface in 3D Euclidean space is constructed.

In Chapter 6 the properties of smooth solutions to first order PDEs with nonsmooth Hamiltonian are investigated. The singular surface considered there is known as the universal one: the solution is smooth, and regular characteristics approach the surface from both sides transversally. It is proved that the solution is at least twice differentiable and subject to some conditions on the singular surface. These conditions generate singular characteristics which correspond to

The generation of the shock waves from the singularity in the initial condition of the nonlinear first order PDE is considered in Chapter seven. Such problems arise in mathematical physics, particularly in conservation laws. Here an independent analysis of singular characteristics in two-dimensional problems is given. Due to low dimensionality one does not need the theory of characteristic field, and the equations of singular characteristics can be derived on the basis of viscosity conditions only. A theorem on the number of shock waves generated by a single nonsmoothness in the initial conditions is proved. It is shown that in a generic position the two extremal waves in the resulting set of waves are the equivocal singular lines, while the intermediate waves are the focal ones. Some possible applications to conservation laws are specified.

Chapter eight demonstrates that the idea of singular characteristics may be useful for the second order PDE as well. A multiple integral variation problem with nonsmooth solutions is considered. At the surface of nonsmoothness two generalized Weierstrass-Erdmann conditions are fulfilled. It is shown that the second Weierstrass-Erdmann condition is the Jacobi bracket of the components of singular Hamiltonian. This allows to state that a system of singular equivocal characteristics describes the propagation of the nonsmoothness in the solution to the quasilinear second order Euler equation.

Each chapter, except for Chapter 9, is followed by a number of exercises. The book may serve as a very good introduction to the classical and nonclassical method of characteristics in solving nonlinear first order PDE. It contains also a good material for a course on the method of singular characteristics, as well as a course on applications of differential equations. Each researcher dealing with first order PDEs or optimal control problems, or game theory, will find this book as very valuable for her or him. The book provides many modern geometrical explanations and fully discussed examples, giving excellent insight the subjects mentioned and inspiring to further study.

The book is very carefully editorially prepared with the known high standard of Birkhäuser Verlag. The only remark concerns the Subject Index – for reader's convenience it should contain more items.

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