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Controllability types of a circular membrane with rotationally symmetric data

by

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Abstract: Various controllability types are demonstrated for a circular membrane with rotationally symmetric initial data and boundary control depending on time only. We prove that the set of initial states, which can be steered to rest in the critical time interval (equal to the diameter of the membrane) by means of L^2−controls is dense in the energy space but contains no eigenmode. We also show that any initial data from a Sobolev space can be transferred to a stationary state. The proof is based on study of exponential families arising in the approach using the method of moments.

Keywords: wave equation, circular membrane, controllability, families of exponentials.

1. Introduction

There are various types of controllability known in control theory for distributed parameter systems (in contrast to systems with finite–dimensional state space). Two of them which are used most often are approximate controllability (the set contains the target space). Two more types are also very important for applications: spectral controllability (or M– controllability) and B–controllability. In the case of M–controllability we are able to reach (from the rest) any eigenmode of the system, i.e., M–controllability is stronger than approximate controllability and, in general, weaker than exact controllability. B–controllability is 'the best' type of controllability — the reachable set coincides with the state space endowed by a natural metrics. In this case we have an isomorphism between the state and the control with minimal norm which drives the system from the rest to this state.

We used to meet different types of controllability for different types of partial differential equations and/or different kinds of controls. For the parabolic type equations with boundary control, M–controllability is intrinsic while for the hyperbolic type equations we have exact controllability or B–controllability for large enough control region and control time (see Russell, 1978, Bardos, Lebeau, and Rauch, 1992) or M–controllability, say, for a rectangular membrane with control on one side of the boundary and time large enough.

In this paper we present an example of a physical system for which we can observe different types of controllability for the same control type. A circular homogeneous membrane is considered and its initial state $(u(\cdot,0), u_t(\cdot,0))$ does not depend on the angle variable (is rotationally symmetric). The problem is to steer the system to the rest at the shortest (critical) time T_* equal to the diameter of the membrane by means of the Neumann type boundary control $f \in L^2(0,T_*)$. In fact, we have a control problem with one spatial variable (the Bessel equation). Using the Fourier method we reduce the problem to a problem of moments with respect to an exponential family.

After a study of the corresponding families we demonstrate that the system is only approximately controllable for the critical time. Particularly, no eigenmode can be steered to the rest in the time interval $[0,T_*]$.

There is another situation if we steer the system not to the rest but to a stationary state — $u(\cdot,T_*) =$ const , $u_t(\cdot,T_*) = 0$. In this case the system turns out to be M–controllable and even UM–controllable: the norm of the control driving the system to a stationary state from an eigenmode is uniformly estimated by the norm of this eigenmode.

Moreover, the system is exactly controllable with respect to initial data from (smoother) Sobolev classes. On the other hand, the system is not B–controllable: not all initial data with finite energy can be steered to a stationary state. In other words, the controllable set could not be described in naturals terms (Sobolev classes or domains of powers of the corresponding operators). B–controllability takes a place if we expand the space of controls to $H^{-1/2}(0,T_*)$ and take the set of rotationally symmetric initial data from $H^{1/2}(\Omega)\times H^{-1/2}(\Omega)$ as the state space, see Avdonin, Ivanov and Russell (2000).

The wave equation in $\Omega\times(0,T)$, where Ω is a unit ball in $\mathbb{R}^n$ (without the rotational symmetry condition on the initial data and control), was considered that the system with the Neumann type boundary control from $L^2(\partial\Omega \times (0,T))$ is exactly controllable for $T > 2$, Graham and Russell (1975), and is not exactly controllable for $T = 2$, Joó (1991a, b). Our aim is to study the 'critical' case of $T = 2$ in detail.

2. Control problems and moment problems

Let $\Omega = \{x = (x_1, x_2) | x_1^2 + x_2^2 < 1\}$ be a unit disk, $\Gamma = \partial\Omega$, and ν be the unit exterior normal vector to Γ. Let us consider the initial boundary value problem

$$\begin{cases} u_{tt} = \Delta u \ \text{ in } \Omega \times (0, T_*), \\ \frac{\partial u}{\partial \nu}\big|_{\Gamma \times (0,T_*)} = f, \\ u\big|_{t=0} = u_0, \quad u_t\big|_{t=0} = u_1 \ \text{ in } \Omega, \end{cases} \tag{1}$$

with control f. In this case the critical time T_* is equal 2.

Let us introduce the subspace W^1_{rot} of rotationally symmetric functions in $H^1(\Omega) \times L^2(\Omega)$ (data with finite energy). We assume that the initial data belong to W^1_{rot} and take control f independing on the angle variable; let $f \in L^2(0,2)$.

Denote by $G(T)$ the controllable set, i. e., the set of initial data $(u_0, u_1) \in W^1_{rot}$, which can be steered to zero in the time interval $[0,T]$ by means of controls from $L^2(0,T)$. Let us introduce the following types of controllability of the system (1) (see Avdonin and Ivanov, 1995, Ch. 3, for more details).

DEFINITION 1 *The system (1) is called $W-$ (or approximately) controllable in time T if $G(T)$ is dense in W^1_{rot}.*

DEFINITION 2 *The system (1) is called $M-$ (or spectrally) controllable in time T if for all $n = 1, 2, \ldots$, the pairs $(\pm\varphi_0, \varphi_0)$ and $(\pm i{\omega_n}^{-1}\varphi_n, \varphi_n)$ belong to $G(T)$, i.e., there exist controls $f_n^{\pm}$, $n = 0, 1, \cdots$, steering the system with that initial data to the rest in the time interval $[0,T]$.*

DEFINITION 3 *The system (1) is called $UM-$ controllable in time T if it is $M-$ controllable and controls $f_n^{\pm}$ can be chosen in such a way that their norms are uniformly bounded.*

DEFINITION 4 *The system (1) is called $E-$ (or exactly) controllable in time T with respect to a space $\mathcal{W} \subset W^1_{rot}$ if $G(T)$ contains $\mathcal{W}$.*

DEFINITION 5 *The system (1) is called $B-$ controllable in time T if $G(T)$ coincides with W^1_{rot}.*

The system (1) has stationary states of the form $u(\cdot, T) = \text{ const }, u_t(\cdot, T) = 0$. Therefore it is natural to consider controllability to a stationary state, which we will call *controllability up to a constant.* Denote by $G_0(T)$ the set of initial data $(u_0, u_1) \in W^1_{rot}$, which can be steered to a stationary state in the time

One can introduce the types of controllability up to a constant of the system (1) similar to the types introduced above. If we replace the set $G(T)$ by $G_0(T)$ in the definitions of W–, E–, and B–controllability we obtain the definition of W–, E–, and B–controllability up to a constant .

If we replace zero terminal state by a stationary state in the definitions of M– and UM–controllability we obtain the M– and UM–controllability up to a constant .

Regularity of the solution of the initial boundary value problem (1) is described by the following theorem.

THEOREM 1 *The problem (1) has a unique solution u such that*

$$(u(\cdot,t), u_t(\cdot,t)) \in C\left([0,T], W^1_{rot}\right).$$

The main result of the paper reads as follows.

THEOREM 2 *(i) The system (1) is W–controllable in $T = 2$, but not M–controllable. Moreover, it is impossible to steer to the rest any initial data of the form*

$$(u_0, u_1) = (\pm i{\omega_n}^{-1}\varphi_n, \varphi_n), \quad n = 1, 2, \cdots,. \tag{2}$$

(ii) The system (1) is UM–controllable up to a constant in $T = 2$, i. e., any initial data of the form (2) can be steered to a stationary state by means of controls with uniformly bounded norms.

(iii) The system (1) is not E–controllable up to a constant in $T = 2$ with respect to the space W^1_{rot}.

(iv) The system (1) is E–controllable up to a constant in $T = 2$ with respect to the rotationally symmetric initial data from $H^{3/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$ (for any $\varepsilon > 0$).

(v) If there exists a control driving the system (1) in time $T = 2$ to a given state (or to a given state up to a stationary state), then this control is unique.

Let us rewrite the problem (1) as an initial boundary value problem for the Bessel equation. Set $w_0(r) = u_0(x,y)$, $w_1(r) = u_1(x,y)$, $r := \sqrt{x^2+y^2}$. Then for $w(r,t) = u(x,y,t)$ we have

$$\left\{ \begin{array}{l} w_{tt} = w_{rr} + \frac{1}{r}w_r; \;\; 0 < r < 1, \;\; 0 < t < 2, \\ \frac{\partial w}{\partial r}\big|_{r=1} = f, \;\; 0 < t < 2, \\ w|_{t=0} = w_0, \, w_t|_{t=0} = w_1, \end{array} \right. \tag{3}$$

We will use the following information concerning the 'elliptical part' of this problem. The operator with the boundary condition $\varphi'(1) = 0$ is a selfadjoint operator in the weighted Hilbert space $L^2_r(0,1)$ consisting of functions $\varphi(r)$ such that $\int_0^1 r|\varphi(r)|^2 dr < \infty$. It is known (see, e.g., Bateman and Erdelyi, 1953, Ch. 7; Watson, 1944) that the eigenfrequencies of this operator, ω_n, are nonnegative zeros of the derivative of the zero-order Bessel function $J_0'(z)$ and the eigenfunctions are $J_0(\omega_n r)$ for $n \neq 0$ and constant for $n = 0$. The normalized eigenfunctions are

$$\varphi_n(r) = \frac{\sqrt{2}J_0(\omega_n r)}{|J_0(\omega_n)|}, \quad n = 1, 2, \ldots, \quad \omega_0 = 0, \quad \varphi_0(r) = \sqrt{2}.$$

Let the initial data w_0, w_1 be represented in the form

$$w_0(r) = \sum_{n=0}^{\infty} a_n^0 \varphi_n(r), \quad w_1(r) = \sum_{n=0}^{\infty} a_n^1 \varphi_n(r) \tag{4}$$

and

$$|a_0^0|^2 + |a_0^1|^2 + \sum_{n=1}^{\infty}(|a_n^0 \omega_n|^2 + |a_n^1|^2) < \infty. \tag{5}$$

In terms of the original problem (1), the conditions (4), (5) correspond to $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$.

We will solve the control problem using the method of moments (see Russell, 1978; Avdonin and Ivanov, 1995). For positive integer n we set $\omega_{-n} := -\omega_n$, $\varphi_{-n} := \varphi_n$ and introduce the sequence

$$c_k^0 := i\omega_k a_{|k|}^0 + a_{|k|}^1, \quad k \in \mathbb{Z}. \tag{6}$$

Let us also introduce the exponential family

$$\mathcal{E} := \{e_k\}_{k\in\mathbb{Z}} \cup \{e_0^0\}, \; e_k := \varphi_k(1) e^{i\omega_k t}, \quad e_0^0 := -\varphi_0(1)t;$$

and denote the family $\{e_k\}_{k\in\mathbb{Z}} = \mathcal{E}\backslash\{e_0^0\}$ by $\mathcal{E}_0$.

Two problems of moments may be connected with these exponential families. The first problem is related to the family $\mathcal{E}$:

$$c_k^0 = -(f, e_k)_{L^2(0,T)}, \; k \in \mathbb{Z}; \quad a_0^0 = -(f, e_0^0)_{L^2(0,T)} \tag{7}$$

and the second one — to the family $\mathcal{E}_0$:

$$c_k^0 = -(f, e_k)_{L^2(0,T)}, \; k \in \mathbb{Z}. \tag{8}$$

PROPOSITION 1 *(i) The initial state (4) of the initial boundary value problem (3) can be steered to the zero state in the time interval $[0, T]$ if and only if the moment problem (7) has a solution $f \in L^2(0, T)$.*

(ii) The initial state (4) of the initial boundary value problem (3) can be steered to a stationary state in the time interval $[0, T]$ if and only if the moment

This proposition can be proved in a standard way (see, e.g., Russell, 1978; Avdonin and Ivanov, 1995, Ch. 3,5). Using the Fourier method we find the solution of the problem (3) in the form of a series

$$w(x,t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x). \tag{9}$$

If we introduce the coefficients $c_k(t)$

$$c_k(t) := i\omega_k a_{|k|}(t) + \dot{a}_{|k|}(t), \quad k \in \mathbb{Z}, \tag{10}$$

then from (3) we obtain the equalities

$$\begin{aligned} c_k(t) &= c_k^0 e^{i\omega_k t} + \textstyle\int_0^t \varphi_n(1) f(\tau) e^{i\omega_k(\tau - t)}\, d\tau, \\ a_0(t) &= a_0^0 + a_0^1 t + \textstyle\int_0^t \varphi_0(1) f(\tau)(t-\tau)\, d\tau. \end{aligned} \tag{11}$$

which imply the problems of moments (7) and (8).

Solvability of a moment problem depends on 'geometrical' properties of the corresponding exponential families and we introduce an hierarchy of types of the 'linear independence'. Let $\Xi := \{\xi_n\}$ be a family of elements in a Hilbert space $\mathcal{H}$.

DEFINITION 6 *The family Ξ is W– linearly independent (we write $\Xi \in (W)$) if there exists no nonzero sequence $\{a_n\} \in \ell^2$ such that for any element $f \in \mathcal{H}$ satisfying $\sum_n |(f, \xi_n)_{\mathcal{H}}|^2 < \infty$, the series $\sum_n a_n(f, \xi_n)_{\mathcal{H}}$ converges to zero.*

Note that if 'Fourier coefficients' $a = \{(f, \xi_n)_{\mathcal{H}}\}$ belong to ℓ^2 for all $f \in \mathcal{H}$, W– linear independence means: *for any $a = \{a_n\} \in \ell^2$ such that the series $\sum_n a_n \xi_n$ weekly converges to zero in $\mathcal{H}$, we have $a = 0$.*

DEFINITION 7 *The family Ξ is minimal (we write $\Xi \in (M)$) if any element ξ_n does not belong to the closure of the span of the remaining elements:*

$$\xi_n \notin \bigvee_{m \neq n} \xi_m.$$

If the family is minimal, then there exists the unique biorthogonal family $\Xi' = \{\xi_n'\} \in \bigvee \Xi$, such that

$$(\xi_n', \xi_m) = \delta_n^m$$

(δ_n^m is the Kronecker delta).

DEFINITION 8 *The family Ξ is $\star$-uniform minimal (we write $\Xi \in (UM)$) if $\Xi \in (M)$ and norms of biorthogonal elements are uniformly bounded.*

If elements of Ξ are almost normalized, $\|\xi_n\| \asymp 1$, then $\star$-uniform minimality

DEFINITION 9 *The family Ξ is said to be an $\mathcal{L}$-basis (we write $\Xi \in (LB)$) if Ξ forms a Riesz basis in the closure of its span.*

Now we are able to formulate the statement connecting the properties of exponential families with the types of controllability (Avdonin and Ivanov, 1995, Ch. 3).

PROPOSITION 2 *(i) The system (1) is W-controllable (W-controllable up to a constant) if and only if $\mathcal{E} \in (W)$ (correspondingly, $\mathcal{E}_0 \in (W)$).*

(ii) The system (1) is M-controllable (M-controllable up to a constant) if and only if $\mathcal{E} \in (M)$ ($\mathcal{E}_0 \in (M)$).

(iii) The system (1) is UM-controllable (UM-controllable up to a constant) if and only if $\mathcal{E} \in (UM)$ ($\mathcal{E}_0 \in (UM)$).

(iv) The system (1) is B-controllable (B-controllable up to a constant) if and only if $\mathcal{E} \in (LB)$ ($\mathcal{E}_0 \in (LB)$).

3. Exponential families

It is more convenient for us to preserve notations $\mathcal{E}$ and $\mathcal{E}_0$ for families

$$\mathcal{E} := \{e_k\}_{k\in\mathbb{Z}} \cup \{e_0^0\},\ e_k := e^{i\omega_k t},\ \ e_0^0 := t;$$

and $\mathcal{E}_0 := \{e_k\}_{k\in\mathbb{Z}}$. Since factors $\varphi_n(1)$ have the absolute value $\sqrt{2}$, this does not change the properties of the families under consideration.

THEOREM 3 *Families $\mathcal{E}$ and $\mathcal{E}_0$ possess the following properties.*

(i) The family $\mathcal{E}$ is complete in $L^2(0,2)$ and $\mathcal{E} \notin (M)$.

(ii) The family $\mathcal{E}_0$ is complete and minimal in $L^2(0,2)$, and $\mathcal{E}_0 \notin (LB)$.

(iii) The family $\mathcal{E}_0$ may be presented as a union of two $\mathcal{L}$-basis families.

(iv) $\mathcal{E} \in (W)$.

(v) $\mathcal{E}_0 \in (UM)$.

PROOF of Theorem 3. Let us recall the definition of the generating function (GF) of an exponential family. This notion plays an important role in the theory of nonharmonic Fourier series and was introduced for the first time in Levin (1961) (see also Avdonin and Ivanov, 1995, Khrushchev, Nikol'skii and Pavlov, 1981). The GF of the exponential family $\{e^{i\nu_k t}\}$ is an entire function of the Cartwright class with zero ν_k's. All entire functions of this class with the same zeros are distinguished by the factor e^{a+bz} and the GF of the family $\mathcal{E}$

$$F(z) = zJ_0'(z)$$

is selected by the condition that its indicator diagram is the segment $[i,-i]$ of the imaginary axis. The function $F(z)$ is an odd function and in a sector $|\arg z| < \pi$ the following asymptotics is well known (Bateman and Erdelyi, 1953, Ch. 7):

We will also use the asymptotics of $J_0''(z)$,

$$J_0''(z) = -\sqrt{\frac{2}{\pi z}}\cos(z - \pi/4) + \mathcal{O}(z^{-3/2}), \tag{13}$$

which is obtained from the asymptotics of J_0' by differentiation. Positive zeros, ω_n, of $J_0'(z)$ satisfy the relation

$$\omega_n = \left(n + \frac{1}{4}\right)\pi + O\left(\frac{1}{n}\right), \quad n = 1, 2, \ldots. \tag{14}$$

From (12) we have

$$\int_{-\infty}^{+\infty} \frac{|F(x)|^2}{1+x^2} = \infty,$$

and so $\mathcal{E}$ is not minimal in $L^2(0,2)$, Paley and Wiener (1934), Sedletskii (1982). (Note that the GF of $\mathcal{E}$ has one double zero at the point $z = 0$. The theory of exponential families such that their generating functions have multiple zeros is presented in Sedletskii (1982) and Avdonin and Ivanov (1995), Sec. II.4. In our case all results we need can be obtained basing on the theory of generating functions with simple zeros).

(ii) The GF of the family $\mathcal{E}_0$ is

$$F_0(z) = J_0'(z)$$

and hence (see (12))

$$\int_{-\infty}^{+\infty} \frac{|F_0(x)|^2}{1+x^2} < \infty,$$

that proves minimality of $\mathcal{E}_0$ in $L^2(0,2)$. Since the GF F_0 does not belong to $L^2(0,2)$, we have completeness of $\mathcal{E}$, Levinson (1940).

On the other hand, (12) implies that $|F_0|^2$ does not satisfy the Muckenhoupt condition on straight lines parallel to the real axis (it is a well know fact, complete proof can be found in Avdonin and Ivanov, 1995, Sec. II.3.4). Therefore, the family $\mathcal{E}_0$ does not form a Riesz basis in $L^2(0,2)$, Khrushchev, Nikol'skii and Pavlov (1981), Avdonin and Ivanov (1995).

(iii) We present $\mathcal{E}$ as a union

$$\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+, \quad \mathcal{E}_- := \{e_k\}_{k<0}, \ \mathcal{E}_+ := \{e_k\}_{k\geq 0} \cup \{e_0^0\}.$$

The family

$$\mathcal{E}_+ \cup \{e^{i\pi(n+1/4)t}\}_{n<-1}. \tag{15}$$

forms a Riesz basis in $L^2(0,2)$. Indeed, the family forms an orthogonal basis. The family

$$\{e_k\}_{k\geq 0} \cup \{e^{i\pi(k+1/4)t}\}_{k<0} \tag{16}$$

is a ('small') perturbation of this basis (see (14)) and in view of Avdonin and Ivanov (1995), (Ch.2, Sec. 4.2, statements 2.4.13, 2.4.14), forms a Riesz basis. Hence, the GF of this family, say $\Phi(z)$, satisfies the Muckenhoupt condition on straight lines parallel to the real axis (it is easy to check that $\Phi(z) \asymp 1$). Replacing in the family (16) the exponential $e^{-i3\pi t/4}$ with $k = -1$ by the function t we obtain the family with the GF

$$\Phi_0(z) := \Phi(z)\frac{z}{z+3/4},$$

which satisfies the condition $|\Phi_0(z)| \asymp |\Phi(z)|$ on lines parallel to the real axis. Therefore, the Muckenhoupt condition is also valid for $\Phi_0(z)$ and the corresponding family (15) forms a Riesz basis. Hence the family $\mathcal{E}_+$ forms an $\mathcal{L}$–basis in $L^2(0,2)$ as a part of the basis family.

Similarly, $\mathcal{E}_-$ is a part of the basis family $\mathcal{E}_- \cup \{e^{i\pi(n-1/4)t}\}_{n\geq 0}$.

(iv) Now we are able to prove W-linear independence of $\mathcal{E}$. Suppose that for some sequence $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2$

$$\sum_{k\in\mathbb{Z}} c_k e_k + c_0^0 e_0^0 = 0 \text{ weekly in } L^2. \tag{17}$$

If $c_0^0 = 0$, then all c_k vanish. Indeed, $\mathcal{E}_0 \in (W)$ and (17) with $c_0^0 = 0$ implies $c_k = 0$ for all k. We suppose now that $c_0^0 \neq 0$ and obtain a contradiction.

To use the theory of analytic functions, we introduce the family

$$\tilde{\mathcal{E}} := \{\tilde{e}_k\}_{k\in\mathbb{Z}} \cup \{\tilde{e}_0^0\},\ \tilde{e}_k := e^{i(\omega_k+i)t},\quad \tilde{e}_0^0 := e^{-t}t,$$

which is the image of $\mathcal{E}$ under the isomorphic in $L^2(0,2)$ map $\psi(t) \mapsto e^{-t}\psi(t)$.

Formula (17) implies

$$\sum_{k\in\mathbb{Z}} c_k \tilde{e}_k + c_0^0 \tilde{e}_0^0 = 0 \text{ weekly in } L^2. \tag{18}$$

Denote the family biorthogonal to $\tilde{\mathcal{E}}_0 := \tilde{\mathcal{E}}\backslash\{\tilde{e}_0^0\}$ by $\tilde{\Theta} := \{\tilde{\theta}_k\}_{k\in\mathbb{Z}}$. Multiplying the series (18) in $\mathrm{L}^2(0,2)$ by $\tilde{\theta}_k$ we obtain

$$c_0^0(\tilde{e}_0^0, \tilde{\theta}_k)_{L^2} + c_k = 0,$$

and so

$$\{(\tilde{e}_0^0, \tilde{\theta}_k)_{L^2}\}_{k\in\mathbb{Z}} \in \ell^2. \tag{19}$$

The GF of the family $\tilde{\mathcal{E}}_0$ is where the normalizing factor e^{iz} is taken in order that the function $\tilde{F}_0/(z+i)$ belong to the Hardy space H^2_+ for the upper half plane. It is known (see, e. g. Avdonin and Ivanov, 1995, Ch. 2) that the elements $\tilde{\theta}_k$ may be presented in the form

$$\tilde{\theta}_k = -\mathcal{F}\frac{1}{\sqrt{2\pi}}\frac{\tilde{F}_0(z)}{(z+\omega_k-i)\tilde{F}_0'(-\omega_k+i)}, \tag{20}$$

where $\mathcal{F}$ is the Fourier transform. The inverse transform maps the 'exponential' $\tilde{e}^0_0|_{(0,2)}$ to

$$\mathcal{F}^{-1}\tilde{e}^0_0 = \frac{1}{\sqrt{2\pi}}P\frac{1}{(z+i)^2},$$

where P is the orthoprojector from the Hardy space H^2_+ onto the subspace $H^2_+ \ominus e^{2iz}H^2_+$.

Then, using the unitary property of the Fourier transform, we have

$$\begin{aligned}&(\tilde{e}^0_0,\tilde{\theta}_k)_{L^2(0,2)} = (\mathcal{F}^{-1}\tilde{e}^0_0,\mathcal{F}^{-1}\tilde{\theta}_k)_{H^2_+}\\ &= \left(\frac{1}{\sqrt{2\pi}}P\frac{1}{(z+i)^2}, -\frac{1}{\sqrt{2\pi}}\frac{\tilde{F}_0(z)}{(z+\omega_k-i)\tilde{F}_0'(-\omega_k+i)}\right)_{H^2_+}.\end{aligned}$$

Since $\theta_k \in L^2(0,2)$, we can omit the orthoprojector P and with the help of the Cauchy formula, we derive for the complex conjugated inner products that

$$\begin{aligned}&(\tilde{\theta}_k,\tilde{e}^0_0)_{L^2(0,2)} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{F}_0(z)}{(z-i)^2(z+\omega_k-i)\tilde{F}_0'(-\omega_k+i)}\,dz\\ &= \frac{\text{const}}{\omega_k J_0''(-\omega_k)}.\end{aligned}$$

Taking into account that ω_k are zeroes of $J'(z)$ we have from the asymptotics (12), (13), and (14) that for large $|k|$

$$\left|\frac{1}{\omega_k J_0''(-\omega_k)}\right| \asymp 1/\sqrt{|k|}$$

which contradicts (19). Thus, (17) is possible only for zero sequence $\{c_k\}_{k\in\mathbb{Z}}$. In view of the statement (iii) this proves W–linear independence of $\mathcal{E}$.

(v) It is clear that we can estimate elements $\tilde{\theta}_k$ (biorthogonal to the family $\tilde{\mathcal{E}}_0$) instead the elements biorthogonal to $\mathcal{E}_0$, since these families are connected by the factor e^t. From (20) we have

$$\|\tilde{\theta}_k\|^2_{L^2(0,2)} = \|\mathcal{F}^{-1}\tilde{\theta}_k\|^2_{H^2_+} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|J_0'(x-i)|^2}{|x+\omega_k-i|^2|J_0''(-\omega_k)|^2}\,dx.$$

The integrand is estimated by

$$\frac{1}{|x-i||x+\omega_k-i|^2|J_0''(-\omega_k)|^2} \prec \frac{|k|}{(x^2+1)^{1/2}(|x+k|+1)^2}.$$

Standard estimation of the integral leads to the inequality

$$\|\tilde{\theta}_k\|^2_{L^2(0,2)} \prec 1.$$

The theorem is proved. ∎

REMARK 1 *The theorem tells, in particular, that $\mathcal{E}$ forms both a Bessel and a Hilbert family, i.e., it is a frame (see Seip, 1995).*

REMARK 2 *The control problem under consideration gives us the unique (to our knowledge) example of the exponential family ($\mathcal{E}_0$) which is not a Riesz basis but is uniformly minimal and complete.*

4. Proof of the main theorem

PROOF of Theorem 1. First, we prove that for each t the state $(u(\cdot,t), u_t(\cdot,t))$ is in W^1_{rot}. In view of the presentation of the solution (9), formulas (10) and (11), we see that this is true, if $\{c_k(t)\} \in \ell^2$. Since $\mathcal{E}$ is a union of two $\mathcal{L}$–basis families (Theorem 3 (iii)), the moment problem (7) gives this inclusion. Continuity in t can be proved in the standard way (see, e.g., Avdonin and Ivanov, 1995, Ch. 3). The theorem is proved. ∎

PROOF of Theorem 2. (i) W–controllability of the system (1) follows from Proposition 1 (i), 2 (i), and W–linear independence of $\mathcal{E}$ (Theorem 3 (iv)). Propositions 1 (i), 2 (ii) and non-minimality of $\mathcal{E}$ (Theorem 3 (i)) imply lack of M–controllability.

The last part of the statement (i) is also the consequence of non-minimality of $\mathcal{E}$. Let us suppose that the initial data have the form (2) and we are able to steer this state to the rest. Then there exists a solution f of the moment problem (7) for the sequence $c^0_k = 2\delta^k_{-n}$ (or $c^0_k = 2\delta^k_n$) and $a^0_0 = 0$. Hence f is orthogonal to all remaining elements e_k, $k \neq n$ (or $k \neq -n$) and e^0_0. It is impossible, since the exponential family $\mathcal{E}$ preserves completeness when we remove an arbitrary element. For $\mathcal{E}_0 = \mathcal{E}\backslash\{t\}$ this was proved above, and the proof is valid for all exponentials.

(ii) This fact follows from the uniform minimality of $\mathcal{E}_0$ (Theorem 3 (v)).

(iii) In view of Theorem 1, in our case E–controllability coincides with B–controllability, which does not take place by Theorem 3 (ii) and Propositions 2 (iv).

(iv) Let $(u_0, u_1) \in H^{3/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$. Then for coefficients c^0_k (connected with coefficients in (4) by (6)) we have

$$\sum |c^0_k|^2 |\omega_k|^{1+2\varepsilon} < \infty.$$

Denote the family biorthogonal to $\mathcal{E}_0$ by $\Theta := \{\theta_k\}_{k\in\mathbb{Z}}$. A formal solution of the moment problem (8) is given by

$$f(t) = \sum c^0_k \theta_k(t)$$

and if this series converges weakly, then it presents the 'real' solution of the moment problem.

In view of the uniform minimality of $\mathcal{E}_0$, this series converges even in L^2–norm:

$$\|\sum c_k^0\theta_k\| \leq \sum |c_k^0|\|\theta_k\| \prec \sum |c_k^0| = \sum |c_k^0 k^{1/2+\varepsilon}||k^{-1/2-\varepsilon}| < \infty.$$

(v) The uniqueness of the control follows (see the moment problems) from the completeness of both families $\mathcal{E}$ and $\mathcal{E}_0$ (Theorem 3 (i),(ii)).

The theorem is proved. ∎

REMARK 3 *Note that in the proof of E-controllability we used very rough estimates and the 'Sobolev' orders $3/2+\varepsilon$, $1/2+\varepsilon$ of the exact controllable space are not sharp.*

REMARK 4 *We see that the reachable (up to a constant) set of the system does not coincide with – and is dense in the rotationally symmetric subspace of $H^1(\Omega)\times L^2(\Omega)$, and contains the rotationally symmetric subspace of $H^{3/2+\varepsilon}(\Omega)\times H^{1/2+\varepsilon}(\Omega)$. In some sense, the control space L^2 is not intrinsic for the system. In Avdonin, Ivanov and Russell (2000) it has been proved that the system is B–controllable for the control space $H^{-1/2}(0,2)$ and the rotationally symmetric initial data from $H^{1/2}(\Omega)\times H^{-1/2}(\Omega)$. It means that the map*

state (=initial data) $\mapsto$ control

is a bounded one–to–one correspondence for these spaces. Formally, it is easy to find such map for any control space endowing the controllable space by the norm of the control, which steers the system from the given state to the rest. The point is that if a system is not B–controllable such approach may lead to norms, which have not a 'natural' description (see, e.g., Haraux, 1988, Lebeau, 1992). In contrast to such cases, the control systems with basis of exponential families (B–controllable systems) have proper descriptions, Avdonin, Ivanov, and Russell (2000).

REMARK 5 *It is interesting to compare our problem with the problem for a regular string. Let us consider Neumann boundary conditions and L^2-control acting at one end point during the critical time — double optical length of the string. In this case (for a regular string) it is known that*

(i) the reachable set is a subspace of the energy space of codimension 1, in particular, the system is not approximately controllable,

(ii) the system is B–controllable up to a constant (up to a stationary state).

REMARK 6 *The situation changes drastically if we take control time T longer than the critical $T_* = 2$. Then our system ((1) or (3)) is B–controllable and the set of controls driving the system from the given state to the rest has infinite dimension.*

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