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Control of degenerate differential systems

by

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Abstract: This work concerns the optimal control of linear degenerate differential equations in Hilbert spaces with convex cost criteria.

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1. Introduction

This paper concerns the controlled system

$$\begin{aligned} &\frac{d}{dt}(My)(t) + Ly(t) = Bu(t),\ t \in (0,T) \\ &(My)(0) = My_0 \end{aligned} \tag{1}$$

in a Hilbert space H.

Here $M : D(M)\subset H \longrightarrow H$ and $L : D(L)\subset H \longrightarrow H$ are linear, closed and densely defined operators, and U (the controller space) is a Hilbert space with the norm denoted $|\cdot|_U$ and scalar product $(\cdot,\cdot)_U$. The norm of H will be denoted by $|\cdot|$ and the scalar product by $(\cdot,\cdot)$. The controller u is taken in $L^2(0,T:U)$ and the solution $y \in L^2(0,T:H)$ is considered in the following weak sense

$$\int_0^T (y(t), M^*\varphi'(t) - L^*\varphi(t))dt + \int_0^T (Bu(t),\varphi(t))dt + (My_0,\varphi(0)) = 0 \tag{2}$$

for all $\varphi \in C^1([0,T];D(M^*)) \cap C([0,T];D(L^*))$ such that $\varphi(T) = 0$.

Here $y_0 \in D(M)$, $\varphi' = \dfrac{d}{dt}\varphi$ and M^*, L^* are the duals of M and L.

This equation was extensively studied in the last years (see, e.g., Carroll, Showalter, 1976, Favini, Yagi, 1999, and the references given there) but there

1995, Sviridyuk, Efremov, 1995). Here we shall study several control problems having (1) as state, and the first one is the convex Bolza control problem

$$\text{Minimize } \int_0^T (g(Cy(t)) + h(u(t)))dt \quad \text{subject to (1)} \tag{3}$$

where the functions g and h satisfy the following condition

(i) $g : Z \longrightarrow R$, $h : U \longrightarrow R$ *are lower semicontinuous, convex and* $C \in L(H, Z)$.

Here Z is a Hilbert space with the norm $|\cdot|_Z$ and scalar product $(\cdot,\cdot)_Z$.

The main difficulty with problem P is that the state equation is singular and so the standard methods of treating convex control problems (see, e.g., Barbu, Precupanu, 1986, Lions, 1968) are not applicable in this situation. This problem will be studied in Sections 2 and 3 with main emphasis on existence and the maximum principle.

In Sections 4 and 5 a linear quadratic control problem will be studied in the framework of strong solutions to the state system (1).

In Section 6 a related problem pertaining the null controllability of the degenerate parabolic equations with internal and boundary control will be studied.

2. Assumptions and formulation of results

We shall denote by $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,T;H) \longrightarrow L^2(0,T;H)$ the linear operator defined by

$$\mathcal{A}y = f \quad \text{iff} \tag{4}$$

$$\int_0^T (y, M^*\varphi' - L^*\varphi)dt + \int_0^T (f,\varphi)dt = 0 \tag{5}$$

for all $\varphi \in C^1([0,T]; D(M^*)) \cap C([0,T]; D(L^*))$ such that $\varphi(T) = 0$. Clearly, $\mathcal{A}$ is closed and densely defined. This means that y is a weak solution to (1) with the right hand side f and the initial value $y_0 = 0$.

The dual operator $\mathcal{A}^*$ is given by

$$\mathcal{A}^*p = f \quad \text{iff} \tag{6}$$

$$\int_0^T (p, (M\psi)' + L\psi)dt = \int_0^T (f,\psi)dt \tag{7}$$

for all $\psi \in C([0,T]; D(L))$, $M\psi \in C^1([0,T]; H)$, $M\psi(0) = 0$.

This means that

$$-M^*\frac{dp}{dt} + L^*p = f \text{ in } (0,T)$$

in the sense of vectorial distributions. It is easily seen that the operator (6) is indeed the dual of $\mathcal{A}$, i.e.,

$$\langle \mathcal{A}y, p\rangle = \langle y, \mathcal{A}^* p\rangle\,, \quad \forall y \in D(\mathcal{A}),\ p \in D(\mathcal{A}^*) \tag{8}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of $L^2(0,T;H)$.

The assumptions of Section 1 will be in effect everywhere in the sequel. In addition the following hypotheses will be used:

(ii) *There is $K \in L(Z,H)$ such that the operator $\mathcal{A}_K = \mathcal{A} + KC$ has closed range in $L^2(0,T;H)$ and its kernel $N(\mathcal{A}_K)$ is finite dimensional.*

(iii) *There is $F \in L(H,U)$ such that $\mathcal{A}_F = \mathcal{A} + BF$ has closed range in $L^2(0,T;H)$ and the kernel $N(\mathcal{A}_F^*)$ of its adjoint is finite dimensional.*

(Here we have denoted again by KC (respectively BF) the realization of KC (respectively BF) in $L^2(0,T;H)$.)

(iv) $g(z) \geq \omega_0|z|_Z^2 + C_1,\ \forall z \in Z,\ h(u) \geq \omega_1|u|_U^2 + C_2,\ \forall u \in U$

where $\omega_0, \omega_1 > 0$.

(v) *There are $\alpha_1, \beta_1 \geq 0$, and $\alpha_2, \beta_2 \in R$ such that*

$g(z) \leq \alpha_1|z|_Z^2 + \alpha_2,\ \forall z \in Z,\ h(u) \leq \beta_1|u|_U^2 + \beta_2,\ \forall u \in U.$

Now we are ready to formulate the main results of this section.

THEOREM **2.1.** *Assume that hypotheses* (i),(ii),(iv) *are satisfied and that $y_0 \in D(L)$. Then problem* (P) *has at least one optimal pair (y^*, u^*). If $Z = H$ and $C = I$ then for each $y_0 \in H$ there exists an optimal pair under assumptions* (i), (iv) *only.*

THEOREM **2.2.** *Under assumptions* (i),(iii),(v) *the pair $(y^*, u^*) \in L^2(0,T;H)\times \times L^2(0,T;U)$ is optimal in problem* (P) *if and only if there are $\eta, p \in L^2(0,T;H)$ such that*

$$\mathcal{A}^* p + \eta = 0 \tag{9}$$

$$\eta(t) \in C^*\partial g(y^*(t)), \text{a.e. } t \in (0,T) \tag{10}$$

$$u^*(t) \in \partial h^*(B^* p(t)), \text{a.e. } t \in (0,T). \tag{11}$$

Here $\partial g : Z \longrightarrow 2^Z$ is the subdifferential of g and $\partial h^* : U \longrightarrow 2^U$ is the subdifferential of the conjugate function h^* of h (see, e.g., Barbu, Precupanu, 1986).

The system (1), (9), (10), (11) is the Euler–Lagrange optimality system for problem (3).

Example 1. Consider the optimal control problem

$$\textit{Minimize} \int_Q (g(x,y(x,t)) + h(x,u(x,t)))dx\,dt \ \textit{subject to} \tag{12}$$

$$(x,t) \in Q = \Omega \times (0,T) \tag{14}$$

$$(dy)(x,0) = d(x)y_0(x),\ x \in \Omega;\ y = 0 \text{ in } \Sigma = \partial\Omega \times (0,T)$$

Here Ω is a bounded and open subset of R^n with smooth boundary, $d \in L^1(\Omega)$, $d \geq 0$ a.e. in Ω, $a \in L^\infty(\Omega)$ and $m \in L^\infty(\Omega)$. Equation (13) occurs in the description of certain diffusion processes (see Carroll, Showalter, 1976, and the references given there) as well as in the theory of the Markov stochastic processes (the Wentzell problem). Equation (13) is of the form (1) where $H = U = L^2(\Omega)$, $L = -\Delta$, $D(L) = H_0^1(\Omega) \cap H^2(\Omega)$, $(My)(x) = d(x)y(x)$, $D(M) = \{y \in L^2(\Omega);\ dy \in L^2(\Omega)\}$, $(Bu)(x) = m(x)u(x)$, a.e. $x \in \Omega$. It is readily seen that if $a \geq 0$, a.e. in Ω then the corresponding operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(Q) \longrightarrow L^2(Q)$ defined by (4), (5) has closed range in $L^2(Q)$. Indeed if $\mathcal{A}y_n = f_n$ then in (5) we take φ to be the solution to the boundary value problem (see Lemma 6.1 below)

$$d\varphi_t + \Delta\varphi = y_n \ \text{ in } Q$$

$$d\varphi(x,0) = 0;\ \varphi = 0 \text{ in } \Sigma.$$

This yields

$$\int_\Omega y_n^2(x,t)dx \leq C\int_Q |f_n(x,t)||y_n(x,t)|dx\,dt, \quad \forall n.$$

Hence $\{y_n\}$ is bounded in $L^2(Q)$ and this clearly implies that $R(\mathcal{A})$ is closed. Thus if one assumes that $\exists\lambda \in R$ such that

$$a(x) + \lambda m(x) \geq 0, \quad \text{a.e. } x \in \Omega$$

then assumption (iii) is satisfied with

$$(Fy)(x) = -\lambda y(x), \quad \text{a.e. } x \in \Omega,\ y \in L^2(\Omega).$$

The functions $g : \Omega \times R \longrightarrow R$, $h : \Omega \times R \longrightarrow R$ are convex and continuous in y and u, measurable in x, and satisfy the conditions

$$\begin{aligned} &\omega_0 y^2 + C_1 \leq g(x,y) \leq \alpha_1 y^2 + \alpha_2, \quad \text{a.e.} \quad x \in \Omega, y \in R \\ &\omega_1 u^2 + C_2 \leq h(x,u) \leq \beta_1 u^2 + \beta_2, \quad \text{a.e.} \quad x \in \Omega, u \in R \end{aligned} \tag{15}$$

where $\omega_0, \omega_1, \alpha_1, \beta_1 > 0$.

Then, assumptions (i), (ii), (iv) are satisfied. We may apply Theorems 2.1, 2.2 to conclude that problem (12) has at least one solution. Moreover, every optimal pair (y^*, u^*) is characterized by the Euler–Lagrange system

$$d(x)p_t(x,t) + \Delta p(x,t) - a(x)p(x,t) \in \partial_y g(x, y^*(x,t)) \text{ in } Q$$

Example 2. Consider the control system governed by the degenerate wave equation

$$\begin{aligned} (d(x)y_t(x,t))_t - \Delta y(x,t) = u(x,t), &\qquad (x,t) \in Q \\ y(x,0) = y_0(x),\ (dy_t)(x,0) = d(x)y_1(x), &\qquad x \in \Omega \\ y = 0 &\qquad \text{in } \Sigma \end{aligned} \tag{16}$$

where $d \in L^\infty(\Omega)$, $d \geq 0$, a.e. in Ω.

We may rewrite (16) as

$$\begin{aligned} &\frac{d}{dt} M \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + L \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad t \in (0,T) \\ &M \begin{pmatrix} y \\ z \end{pmatrix}(0) = M \begin{pmatrix} y_0 \\ z_1 \end{pmatrix} \end{aligned} \tag{17}$$

in $H = H_0^1(\Omega) \times L^2(\Omega)$, $U = L^2(\Omega)$ where

$$M = \left\| \begin{matrix} 1 & 0 \\ 0 & d \end{matrix} \right\|, \ L = \left\| \begin{matrix} 0 & -1 \\ -\Delta & 0 \end{matrix} \right\|, \ B = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|.$$

Assumption (iii) is satisfied with

$$F \begin{pmatrix} y \\ z \end{pmatrix} = -z, \ \ \forall \begin{pmatrix} y \\ z \end{pmatrix} \in H.$$

Indeed by the equation

$$(\mathcal{A} + BF) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

we see that

$$\int_\Omega |\nabla y(x,t)|^2 dx + \int_Q z^2 dx\, dt \leq C \int_Q (|\nabla f||\nabla y| + |zg|) dx\, dt$$

which clearly implies that $R(\mathcal{A} + BF)$ is closed in $L^2(0,T; H_0^1(\Omega) \times L^2(\Omega))$.

Hence, Theorems 2.1 and 2.2 are applicable for the cost functional

$$\int_Q (g(x, y(x,t), y_t(x,t)) + h(x, u(x,t))) dx\, dt$$

where g and h satisfy conditions of the form (14).

Example 3. (The Sobolev equation)

$$\begin{aligned} &\frac{d}{dt}(I + \Delta)y - \Delta y = (Bu)(x,t) \quad \text{in} \quad Q \\ &((I + \Delta)y)(x,0) = (I + \Delta)y_0(x), \qquad x \in \Omega \end{aligned}$$

Here $B \in L(U, L^2(\Omega))$ where U is a real Hilbert space (the controller space). We are in the general situation presented above where $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $My = (I + \Delta)y$, $D(M) = H_0^1(\Omega) \cap H^2(\Omega)$. It is readily seen that in this case the corresponding operator $\mathcal{A}$ has closed range in $L^2(0,T; L^2(\Omega))$ and $N(\mathcal{A}^*) = \{0\} = N(\mathcal{A})$. (We refer the reader to Carroll, Showalter, 1976, Favini, Yagi, 1999, and Sviridyuk, 1995, for physical examples and a treatment of such an equation.)

3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. We may equivalently write problem (3) as

$$\min \Big\{ \int_0^T (g(C(y_0 + z)) + h(u))dt;\ \mathcal{A}z = Bu + Ly_0,$$
$$u \in L^2(0,T;H),\ z \in L^2(0,T;H) \Big\}. \tag{19}$$

Let $\{z_n, u_n\}$ be a minimizing sequence for (19), i.e.,

$$d \leq \int_0^T (g(C(y_0 + z_n)) + h(u_n))dt \leq d + \frac{1}{n} \tag{20}$$

where d is the infimum in (19). We have

$$\mathcal{A}_K z_n = Bu_n + KCz_n + Ly_0$$

where K and $\mathcal{A}_K$ are defined as in assumption (ii).

We may write $z_n = z_n^1 + z_n^2$ where $z_n^2 \in N(\mathcal{A}_K)$ and $z_n^1 \in R(\mathcal{A}_K^*)$. (By assumption (ii), $L^2(0,T;H) = N(\mathcal{A}_K) \oplus R(\mathcal{A}_K^*)$.) Since $\mathcal{A}_K^{-1} \in L(R(\mathcal{A}_K), R(\mathcal{A}_K^*))$ it follows by assumption (iii) that

$$|z_n^1| \leq C_3,\ \forall n. \tag{21}$$

Moreover, $z_n^2 = z_n^3 + z_n^4$ where $z_n^3 \in R(\widetilde{C}^*)$ and $z_n^4 \in N(\widetilde{C})$. We have denoted by $\widetilde{C}$ the realization of C in the space $N(\mathcal{A}_K) \subset L^2(0,T;H)$. Since $N(\mathcal{A}_K)$ is finite dimensional we have

$$N(\mathcal{A}_K) = N(\widetilde{C}) \oplus R(\widetilde{C}^*),$$

and so

$$|z_n^3| \leq C_4,\quad \forall n$$

because $\{Cz_n^2 = \widetilde{C}z_n^3\}$ is bounded in $L^2(0,T;H)$ by the assumptions of Theorem 2.1. We have also

$$\mathcal{A}_K(z_n^1 + z_n^3) = Bu_n + KC(z_n^1 + z_n^3) + Ly_0.$$

Let

$$u_n \longrightarrow u^* \text{ weakly in } L^2(0,T;U).$$

Since $\mathcal{A}_K$ is closed we have

$$\mathcal{A}(z^*) = Bu^* + Ly_0$$

while by (20) we see that

$$\int_0^T (g(C(y_0+z^*) + h(u^*))dt = d$$

because the convex integrand is weakly lower semicontinuous.

Hence $\{y^* = z^* + y_0, u^*\}$ is optimal. Assume now that $Z = H$ and $C = I$. Let (y_n, u_n) be a minimizing sequence for problem (3). We have as above

$$d \le \int_0^T (g(y_n) + h(u_n))dt \le d + \frac{1}{n}$$

and since the map $u \longrightarrow y$ (y is a weak solution to (1)) has closed graph in $(L^2(0,T;U)\times L^2(0,T;H))_w$ we infer by assumption (iv) that

$$(y_n, u_n) \longrightarrow (y^*, u^*) \text{ weakly in } L^2(0,T;H)\times L^2(0,T;U),$$

where (y^*, u^*) is an optimal pair of (3). This completes the proof. ■

Proof of Theorem 2.2.

Let (y^*, u^*) be optimal. Consider the approximating control problem

$$\min\left\{\int_0^T (g_\varepsilon(Cy(t)) + h(u(t)))dt + \frac{1}{2}\int_0^T |u(t) - u^*(t)|_U^2 dt + \right.$$
$$\left. +\frac{1}{2}\int_0^T |y(t) - y^*(t)|^2 dt + \frac{1}{2\varepsilon}\int_0^T |v(t)|^2 dt\right\} \tag{22}$$

subject to

$$\frac{d}{dt}My + Ly = Bu + v;\ (My)(0) = My_0 \tag{23}$$

Here g_ε is the regularization of g, i.e.,

$$g_\varepsilon(z) = \inf\left\{\frac{|z-\theta|_Z^2}{2\varepsilon} + g(\theta);\ \theta \in Z,\ \varepsilon > 0\right\}.$$

We recall (see, e.g., Barbu, Precupanu, 1986) that g_ε is convex, Fréchet differ-

By Theorem 2.1 and its proof it follows that (22) has a solution $(y_\varepsilon, u_\varepsilon, v_\varepsilon)$. We have

$$\int_0^T (g_\varepsilon(C(y_\varepsilon+\lambda z)) + h(u_\varepsilon+\lambda u))dt + \frac{1}{2}\int_0^T (|u_\varepsilon+\lambda u-u^*|_U^2 +$$

$$+|y_\varepsilon+\lambda z-y^*|^2)dt + \frac{1}{2\varepsilon}\int_0^T |v_\varepsilon+\lambda(\mathcal{A}z-Bu)|^2 dt \geq$$

$$\geq \int_0^T (g_\varepsilon(Cy_\varepsilon)+h(u_\varepsilon))dt + \frac{1}{2}\int_0^T (|u_\varepsilon-u^*|_U^2 + |y_\varepsilon-y^*|^2)dt + \frac{1}{2\varepsilon}\int_0^T |v_\varepsilon|^2 dt,$$

$$\forall\lambda>0,\ \forall z\in D(\mathcal{A}),$$

because $y_\varepsilon+\lambda z$ is a weak solution to

$$\frac{d}{dt}My+Ly = Bu_\varepsilon+v_\varepsilon+\lambda\mathcal{A}z, \quad (My)(0)=My_0.$$

This yields

$$\int_0^T ((C^*\nabla g_\varepsilon(Cy_\varepsilon), z) + h'(u_\varepsilon, u))dt + \int_0^T (u_\varepsilon-u^*, u)_U dt +$$
$$+\int_0^T (y_\varepsilon-y^*, z)dt + \frac{1}{\varepsilon}\int_0^T (\mathcal{A}z, v_\varepsilon)dt - \frac{1}{\varepsilon}\int_0^T (Bu, v_\varepsilon)dt \tag{24}$$

for all $z\in D(\mathcal{A}), u\in L^2(0,T;U)$. (Here h' is the directional derivative of h.) We set $p_\varepsilon = \frac{1}{\varepsilon}v_\varepsilon$ and take $u=0$ in (3.6). We get

$$\langle\mathcal{A}z, p_\varepsilon\rangle + \int_0^T (C^*\nabla g_\varepsilon(Cy_\varepsilon)+y_\varepsilon-y^*, z)dt = 0,\ \forall z\in D(\mathcal{A}).$$

Hence, $p_\varepsilon\in D(\mathcal{A}^*)$ and

$$-\mathcal{A}^*p_\varepsilon = C^*\nabla g_\varepsilon(Cy_\varepsilon)+y_\varepsilon-y^*. \tag{25}$$

Then by (24) we have

$$\int_0^T (h'(u_\varepsilon, u) - (u, B^*p_\varepsilon+u^*-u_\varepsilon)_U)dt \geq 0$$

for all $u\in L^2(0,T;U)$. This yields

$$B^*p_\varepsilon+u^*-u_\varepsilon \in \partial h(u_\varepsilon), \text{ a.e. in } (0,T). \tag{26}$$

On the other hand, we have

$$\int_0^T (g_\varepsilon(Cy_\varepsilon)+h(u_\varepsilon))dt + \frac{1}{2}\int_0^T (|y_\varepsilon-y^*|^2+|u_\varepsilon-u^*|_V^2)dt + \tag{27}$$

Selecting a subsequence we may assume that

$$\begin{aligned} u_\varepsilon &\longrightarrow \bar{u} \text{ weakly in } L^2(0,T;U) \\ y_\varepsilon &\longrightarrow \bar{y} \text{ weakly in } L^2(0,T;H) \end{aligned} \tag{29}$$

where $(\bar{y},\bar{u})$ satisfy the system (1).

Then by (29) we see that (the convex integrand is weakly lower semicontinuous)

$$\begin{aligned} &\int_0^T (g(C\bar{y}) + h(\bar{u}))dt + \\ &+ \limsup_{\varepsilon\longrightarrow 0} \left(\frac{1}{2} \int_0^T (|y_\varepsilon - y^*|^2 + |u_\varepsilon - u^*|_U^2 + \frac{1}{\varepsilon}|v_\varepsilon|^2)dt \right) \\ &\leq \int_0^T (g(Cy^*) + h(u^*))dt. \end{aligned} \tag{30}$$

Since (y^*,u^*) is optimal in (3) we conclude that

$$\begin{aligned} u_\varepsilon &\longrightarrow u^* \text{ strongly in } L^2(0,T;U) \\ y_\varepsilon &\longrightarrow y^* \text{ strongly in } L^2(0,T;H) \\ v_\varepsilon &\longrightarrow 0 \text{ strongly in } L^2(0,T;H). \end{aligned} \tag{31}$$

Next, by (26) and assumption (ii) we see that

$$\int_0^T |B^* p_\varepsilon|_U^2 dt \leq C_4, \ \forall t \in (0,T). \tag{32}$$

and we may rewrite (25) as

$$-(\mathcal{A}_F)^* p_\varepsilon = C^* \nabla g_\varepsilon(y_\varepsilon) + y_\varepsilon - y^* - F^* B^* p_\varepsilon. \tag{33}$$

Since the right hand side of (33) is bounded in $L^2(0,T;H)$, we have

$$p_\varepsilon = p_\varepsilon^1 + p_\varepsilon^2$$

where $p_\varepsilon^2 \in N(\mathcal{A}_F^*)$ and

$$\int_0^T |p_\varepsilon^1|^2 dt \leq C_5, \quad \forall \varepsilon > 0. \tag{34}$$

On the other hand, by (33) we may write

$$p_\varepsilon^2 = p_\varepsilon^3 + p_\varepsilon^4$$

where $B^* p_\varepsilon^4 = 0$ and $|p_\varepsilon^3|_{L^2(0,T;H)} \leq C_6$.

(The restriction of B^* to $N(\mathcal{A}_F^*)$ has closed range.) We have therefore and $\{q_\varepsilon = p_\varepsilon^1 + p_\varepsilon^2\}$ is bounded in $L^2(0,T;H)$. Then we may pass to the limit in (26), (35) to get (9) as claimed.

The sufficiency of the system (9), (10), (11) for optimality follows in a standard way (see, e.g., Barbu, Precupanu, 1986, Chap. IV) from the definition of ∂g, ∂h^* and the duality equality (8).

4. Strong solutions to degenerate differential equations

In this section we outline some results on degenerate differential equations and their solutions in a special but important case.

Here H denotes a (complex) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We are given two closed linear operators L, M from H into itself, with domain $\mathcal{D}(L)$ and $\mathcal{D}(M)$, respectively, such that

$$\mathcal{D}(L) \subseteq \mathcal{D}(M),\ 0 \in \rho(L). \tag{36}$$

Given $f \in L^2(0,\tau;H)$, $u_0 \in \mathcal{D}(L)$, when $\tau > 0$ is fixed, we define a solution u to the initial value problem

$$\frac{d}{dt}(Mu)(t) + Lu(t) = f(t),\ 0 < t < \tau, \tag{37}$$

$$(Mu)(0) = Mu_0, \tag{38}$$

as an element u of $L^2(0,\tau;\mathcal{D}(L))$, such that $Mu \in H^1(0,\tau;H)$, the equation (37) holds almost everywhere on $(0,\tau)$ and (38) is satisfied.

System (37), (38) has had a wide treatment in the literature and we quote Sviridyuk and Efremov (1995) for arguments related to ours. Here we extend the method developed in the monograph by Favini and Yagi (1999) for solutions in $C[0,\tau;H]$ to solutions in $L^2(0,\tau;H)$.

To this end we shall assume that $\lambda = 0$ is a polar singularity of the resolvent $(\lambda + T)^{-1}$, where $T = ML^{-1}(\in \mathcal{L}(H))$, so that

$$\|(\lambda+T)^{-1}\|_{\mathcal{L}(H)} = \|L(\lambda L + M)^{-1}\|_{\mathcal{L}(H)} \le \frac{C}{|\lambda|^m},\ 0 < |\lambda| \le \varepsilon_0, \tag{39}$$

where m is an integer ≥ 1. Of course, (39) reads equivalently

$$\|L(\mu M + L)^{-1}\|_{\mathcal{L}(H)} \le C|\mu|^{m-1},\ |\mu| \ge \varepsilon_0^{-1}.$$

Then it is well known that the representation $H = N(T^m) \oplus R(T^m)$ holds, so that if P denotes the projection operator onto $N(T^m)$, then P commutes with T and system (37), (38) splits into the couple of problems

$$\frac{d}{dt}T_1 Pv(t) + Pv(t) = Pf(t),\ \ 0 < t < \tau, \tag{40}$$

and

$$\frac{d}{dt}T_2(1-P)v(t)+(1-P)v(t)=(1-P)f(t),\quad 0<t<\tau, \tag{42}$$

$$T_2(1-P)v(0)=T_2(1-P)v_0, \tag{43}$$

where T_1 denotes the restriction of T to $N(T^m)$ and T_2 is the restriction of T to $R(T^m)$; the new unknown $v(t)$ is clearly $v(t)=Lu(t)$ and $Lu_0=v_0$.

An important fact should be observed, namely that T_2 has a bounded inverse (in $\mathcal{L}(R(T^m))$ and thus necessarily each solution $(1-P)v(t)$ to (42), (43) has a derivative and in fact it satisfies

$$T_2\frac{d}{dt}(1-P)v(t)+(1-P)v(t)=(1-P)f(t),$$

$$(1-P)v(0)=(1-P)v_0,$$

i.e.,

$$\frac{d}{dt}(1-P)v(t)+T_2^{-1}(1-P)v(t)=T_2^{-1}(1-P)f(t),$$

$$(1-P)v(0)=(1-P)v_0.$$

Moreover, $(1-P)v(t)$ is given by

$$(1-P)v(t)=e^{-tT_2^{-1}}(1-P)v_0+\int_0^t e^{-(t-s)T_2^{-1}}T_2^{-1}(1-P)f(s)ds.$$

for all $f\in L^2(0,\tau;H)$.

Concerning the system (40), (41), we observe that $T_1^m=0$, and hence it is easily seen that the unique solution to (40) is guaranteed by the assumption that $f\in H^{m-1}(0,\tau;H)$ and is given by

$$Pv(t)=\sum_{j=0}^{m-1}(-1)^jT_1^jPf^{(j)}(t).$$

Furthermore, $Pv(t)$ satisfies (41) if and only if

$$\sum_{j=0}^{m-2}(-1)^jT_1^{j+1}f^{(j)}(0)=TPv_0=T_1Pv_0,\ m\ge 2.$$

Therefore, if $m=1$ (this is the case of $\lambda=0$ a simple pole for $(z+T)^{-1}$), then

$$v(t)=Pf(t)+e^{-tT_2^{-1}}(1-P)v_0+\int_0^t e^{-(t-s)T_2^{-1}}T_2^{-1}(1-P)f(s)ds,$$

and only $f\in L^2(0,\tau;H)$ is needed. Notice that

If $m \geq 2$, assumption $f \in H^{m-1}(0,\tau;H)$ assures that $Tv(t)$ has a limit as $t \longrightarrow 0$. However, if $f \in H_0^{m-1}(0,\tau;H)$, where

$$H_0^{m-1}(0,\tau;H) = \{f \in H^{m-1}(0,\tau;H);\ f^{(j)}(0) = 0, j = 0, 1, \ldots, m-2\}$$

then $Tv(t)$ converges to $T_2(1-P)v_0 \in R(T^{m+1}) = R(T^m)$. Observe that if $v_0 \in R(T^m)$ then $T_2(1-P)v_0 = T(1-P)v_0 = Tv_0$.

In any case, the solution u to (37) is unique and it is given by

$$u(t) = \sum_{j=0}^{m-1}(-1)^j L^{-1}T_1^j P f^{(j)}(t) + L^{-1}e^{-tT_2^{-1}}(1-P)v_0 +$$
$$+\int_0^t L^{-1}e^{-(t-s)T_2^{-1}}T_2^{-1}(1-P)f(s)ds.$$

Clearly, if $f \in H^m(0,\tau;H)$, then u is more regular, in the sense that $u \in H^1(0,\tau;\mathcal{D}(M))$, and equation (37) holds in the stronger sense

$$M\frac{du}{dt} + Lu(t) = f(t), \quad 0 < t < \tau \tag{44}$$

almost everywhere on $(0,\tau)$. Since $m \geq 1$, u is strongly continuous at $t=0$, so that

$$u(0) = \sum_{j=0}^{m-1}(-1)^j L^{-1}T_1^j P f^{(j)}(0) + L^{-1}(1-P)v_0.$$

Therefore, if $f \in H_0^m(0,\tau;H)$, then the Cauchy problem (44),(45), where

$$u(0) = u_0, \tag{45}$$

has a unique solution $u \in H^1(0,\tau;H)$, $\dfrac{du}{dt} \in L^2(0,\tau;\mathcal{D}(M))$ provided that $Lu_0 \in R(T^m)$.

It is readily seen that also the problem

$$M\frac{du}{dt}(t) - Lu(t) = f(t), \quad 0 < t < \tau, \tag{46}$$

a.e. in $(0,\tau)$, with

$$u(\tau) = \bar{u} \in H \tag{47}$$

admits a solution $u \in H^1(0,\tau;H)$, $\dfrac{du}{dt} \in L^2(0,\tau;\mathcal{D}(M))$ if

$$f \in H_\tau^m(0,\tau;H) = \{f \in H^m(0,\tau;H);\ f^{(j)}(\tau) = 0,\ j = 0,\ldots,m-1\}$$

5. The linear quadratic optimal control problem

The analysis of Section 4 concerning equation (37) clarifies the difference between the case when $z = 0$ is a simple pole or a higher order pole for the resolvent $(z + T)^{-1}$.

Therefore we shall describe two different, although related, optimal control problems.

We begin with the case where $m \geq 2$. Here we have, as above, three real Hilbert spaces H, U, Z with norms $\|\cdot\|_H, \|\cdot\|_U, \|\cdot\|_Z$, respectively, and corresponding inner products $\langle,\rangle_H$, $\langle,\rangle_U$, $\langle,\rangle_Z$. We assume that the closed linear operators M, L in H satisfy the same assumptions as in Section 4 with $m \geq 2$, $T \in ML^{-1}$, $B \in \mathcal{L}(U, H)$, $C \in \mathcal{L}(H, Z)$, $N_q \in \mathcal{L}(U)$ is a self-adjoint positive definite operator for $q = 0, \ldots, m-1$. Finally, let $\mathcal{U}$ be a closed convex subset of $H_0^{m-1}(0, \tau; U)$ and let $f \in H_0^{m-1}(0, \tau; H), y_0 \in \mathcal{D}(L), y_0(\cdot) \in L^2(0, \tau; H)$. We shall consider the initial-value problem in $L^2(0, \tau; H)$

$$\frac{d}{dt}(My) = -Ly + f + Bu, \quad 0 < t < \tau, \tag{48}$$

$$My(0) = My_0 \; (= Tx_0), \tag{49}$$

where $u \in \mathcal{U}$. We know from Section 4 that (48), (49) has a unique solution $y = y(u)$. Define the cost functional

$$J(u) = \int_0^\tau |C(y(u)(t) - y_0(t))|_Z^2 dt + \sum_{q=0}^{m-1} \int_0^\tau \left\langle N_q u^{(q)}(t), u^{(q)}(t) \right\rangle_U dt.$$

Then the optimal control problem consists in finding $u^* \in \mathcal{U}$ such that

$$J(u^*) = \inf_{u \in \mathcal{U}} J(u). \tag{50}$$

We have

THEOREM **5.1.** *Under the above hypotheses there exists a unique optimal control $u^* \in \mathcal{U}$ for* (48), (49), (50).

Proof. First of all, we observe that the bracket

$$[u, v] = \sum_{q=0}^{m-1} \int_0^\tau \left\langle N_q u^{(q)}(t), v^{(q)}(t) \right\rangle_U dt$$

is a continuous bilinear coercive form on $H_0^{m-1}(0, \tau; U)$.

Moreover, since B induces a continuous operator from $H_0^{m-1}(0, \tau; U)$ into $H_0^{m-1}(0, \tau; H)$ and $f \in H_0^{m-1}(0, \tau; H)$, the mapping $u \longrightarrow y(u)$, where according to Section 4, with the same notation,

$$y(u)(t) = \sum_{j=0}^{m-1} (-1)^j L^{-1} T_1^j P(f^{(j)}(t) + Bu^{(j)}(t)) + L^{-1} e^{-tT_2^{-1}} x_0 +$$

$$+ \int_0^t L^{-1} e^{-(t-s)T_2^{-1}} T_2^{-1} (1 - P)(f(s) + Bu(s)) ds$$

is continuous from $H_0^{m-1}(0,\tau;U)$ into $L^2(0,\tau;H)$.

The functions

$$\begin{aligned}\pi(u,v) &= \langle C[y(u)-y(0)], C[y(v)-y(0)]\rangle_{\mathcal{Z}} + [u,v],\\ \ell(u) &= \langle C[y_0(\cdot)-y(0)], C[y(u)-y(0)]\rangle_{\mathcal{Z}},\end{aligned}$$

where $\mathcal{Z} = L^2(0,\tau;Z)$, are well defined and it is readily seen that

$$J(u) = \pi(u,u) - 2\ell(u) + \|C[y_0(\cdot)-y(0)]\|_{\mathcal{Z}}^2.$$

is continuous and coercive which concludes the proof. ■

Remark 5.1 A similar technique was used in the paper by Sviridyuk and Efremov (1995).

Let us discuss the case of $m=1$. Then we know that for all $f \in L^2(0,\tau;H)$ and any $y_0 \in \mathcal{D}(L)$, the solution $y = y(u)$ to (48), (49) exists and it is given by

$$\begin{aligned}&y(t) = y(u)(t) = L^{-1}P[f(t)+Bu(t)] + L^{-1}e^{-tT_2^{-1}}(1-P)Ly_0 +\\ &+\int_0^t L^{-1}e^{-(t-s)T_2^{-1}}T_2^{-1}(1-P)[f(s)+Bu(s)]ds.\end{aligned}$$

Notice that $T(1-P)Ly_0 = TLy_0 = My_0$.

Hence, $u \to y(u)$ is a continuous mapping from $L^2(0,\tau;U)$ into $L^2(0,\tau;H)$. Take $C \in \mathcal{L}(H;Z)$, $N_0 = N = N^* > 0$, $N \in \mathcal{L}(U)$, $y_0(\cdot) \in L^2(0,\tau;H)$. At last, let $\mathcal{U}$ be a closed convex subset of $L^2(0,\tau;U)$. The cost functional J has then the form

$$J(u) = \int_0^\tau |C(y(u)(t) - y_0(t))|_Z^2 dt + \int_0^\tau \langle Nu(t), u(t)\rangle_U \, dt.$$

Since $J(u) = \pi(u,u) - 2\ell(u) + \|C(y_0(\cdot) - y(0))\|_{\mathcal{Z}}^2$, where $\pi(u,v)$ and $\ell(u)$ were previously defined, with $m=1$, we get the following result:

THEOREM **5.2.** *Let $m=1$ and let $y_0 \in \mathcal{D}(L)$. Then, under the above assumptions the optimal control problem* (5.3) *for* (48), (49) *has a unique solution.*

Our next step consists in extending the analysis of Lions (1968) to arrive at the results close to Theorem 2.1 and Remark 2.3 in Lions (1968), pp.114–115. Here the situation is rather more delicate because there is a possible lack of regularity in the solution $y(u)$ and much caution must be used. By Lions (1968, Theorem 1.2, p.9), we know that the optimal control u, whose existence and uniqueness is guaranteed by Theorem 5.2, is characterized by $\pi(u, v-u) \geq \ell(v-u)$ for all $v \in \mathcal{U}$, and in particular, if $\mathcal{U} = L^2(0,\tau;U)$, by $\pi(u,\varphi) = \ell(\varphi)$ for all $\varphi \in L^2(0,\tau;U)$. Now

$$0 \;\leq\; \pi(u, v-u) - \ell(v-u) = \langle C(y(u)-y(0)), C(y(v-u)-y(0))\rangle_{\mathcal{Z}} +$$

$$
\begin{aligned}
&= \langle C(y(u)-y_0(\cdot)), C(y(v-u)-y(0))\rangle_{\mathcal{Z}} + \langle Nu, v-u\rangle_{L^2(0,\tau;U)} = \\
&= \langle C(y(u)-y_0(\cdot)), C(y(v)-y(u))\rangle_{\mathcal{Z}} + \langle Nu, v-u\rangle_{L^2(0,\tau;U)} = \\
&= \int_0^\tau \langle C^*C(y(u)(t)-y_0(t)), y(v)(t)-y(u)(t)\rangle_H\, dt + \langle Nu, v-u\rangle_{L^2(0,\tau;U)},
\end{aligned}
$$

for all $v \in \mathcal{U}$.
Assume the existence of the adjoint state $p(u) \in L^2(0,\tau;\mathcal{D}(L^*)) \cap H^1_\tau(0,\tau;H)$ satisfying

$$
-M^*\frac{dp}{dt} + L^*p = C^*C(y(u)-y_0(\cdot)), \quad 0<t<\tau, \tag{51}
$$

$$
p(u)(\tau) = 0. \tag{52}
$$

Notice that (51) yields $M^*\frac{dp}{dt} \in L^2(0,\tau;H)$, but from Section 4 we know that more regularity is needed for $y(u)$ to conclude that such a solution $p(u)$ exists.

Multiplying both sides of (51) by $y(v)-y(u)$, and taking into account that

$$
\begin{aligned}
&\int_0^\tau \left\langle -M^*\frac{dp(u)}{dt}, y(v)-y(u)\right\rangle_H dt = \\
&-\int_0^\tau \left\langle \frac{dp(u)}{dt}, M(y(v)-y(u))\right\rangle_H dt = \\
&= \int_0^\tau \left\langle p(u), \frac{d}{dt}(My(v)-My(u))\right\rangle_H dt
\end{aligned}
$$

and

$$
\int_0^\tau \langle L^*p(u), y(v)-y(u)\rangle_H\, dt = \int_0^\tau \langle p(u), L[y(v)-y(u)]\rangle_H\, dt
$$

yields

$$
\begin{aligned}
&\int_0^\tau \langle C^*C(y(u)(t)-y_0(t)), y(v)(t)-y(u)(t)\rangle_H\, dt = \\
&= \int_0^\tau \left\langle -M^*\frac{dp(u)}{dt} + L^*p(u), y(v)(t)-y(u)(t)\right\rangle_H dt = \\
&= \int_0^\tau \left\langle p(u)(t), \left(\frac{d}{dt}M + L\right)(y(v)-y(u))\right\rangle_H dt = \\
&= \int_0^\tau \langle p(u)(t), Bv(t)-Bu(t)\rangle_H\, dt = \int_0^\tau \langle B^*p(u)(t), v(t)-u(t)\rangle_U\, dt = \\
&= \langle B^*p(u), v-u\rangle_{L^2(0,\tau;U)}.
\end{aligned}
$$

Therefore, an admissible control u satisfying the following system

d

$$
\begin{aligned}
&-M^* \frac{dp(u)}{dt} + L^* p(u) = C^* C(y(u) - y_0(\cdot)), \quad 0 < t < \tau, \\
&My(u)(0) = My_0, \; p(u)(\tau) = 0, \; My(u) \in H^1(0,\tau;H)), \; p(u) \in H^1(0,\tau;H), \\
&\langle B^* p(u) + Nu, v - u\rangle_{L^2(0,\tau;U)} \geq 0, \text{ for all } v \in \mathcal{U},
\end{aligned}
$$

is necessarily the unique optimal control for (48), (49), (50). In particular, if $\mathcal{U} = L^2(0,\tau;U)$, the last inequality reduces to $u = -N^{-1}B^*p$, so that we have (compare with Theorem 2.2)

THEOREM **5.3.** *Let $m = 1$, $\mathcal{U} = L^2(0,\tau;U)$. Under the assumptions above, if the degenerate two-point problem*

$$\frac{d}{dt}(My) + Ly + BN^{-1}B^*p = f, \; 0 < t < \tau, \tag{53}$$

$$-M^* \frac{dp}{dt} + L^* p = C^* C(y - y_0(\cdot)), \; 0 < t < \tau \tag{54}$$

$$My(0) = My_0, \; p(T) = 0, \tag{55}$$

has a solution (y,p), $y \in L^2(0,\tau;\mathcal{D}(L))$, $My \in H^1(0,\tau;H)$, $p \in L^2(0,\tau;\mathcal{D}(L^*)) \cap H^1(0,\tau;H)$, then $u = -N^{-1}B^*p$ is the unique optimal control for the problem (48), (49), (50).

Now, the system (53)$\sim$(55) does not seem to be too satisfactory because (54) requires more regularity for p (and hence for y) than one expects. Clearly, one should like to substitute (54) with the less restrictive differential equation

$$-\frac{d}{dt}(M^*p) + L^*p = C^*C(y - y_0(\cdot)), \tag{56}$$

and (55) with the boundary conditions

$$My(0) = My_0, \; M^*p(\tau) = 0. \tag{57}$$

In fact we shall exploit the special features due to the presence of a simple pole 0 for $(\lambda + T)^{-1}$, $(\lambda + S)^{-1}$, where $T = ML^{-1}$, $S = M^*L^{*-1}$, to show that if the pair (y,p) satisfies the system (53), (56), (57), then $u = -N^{-1}B^*p$ is the optimal control.

Let P denote the projection onto $N(ML^{-1})$ and let Q be the projection onto $N(M^*L^{*-1})$. Then (56), (57) reads equivalently

$$
\begin{aligned}
-\frac{d}{dt}\widetilde{S}(1-Q)q(t) + (1-Q)q(t) &= (1-Q)C^*C(y(t) - y_0(t)), &(58)\\
Qq(t) &= QC^*C(y(t) - y_0(t)), &(59)
\end{aligned}
$$

on $(0,\tau)$ where $\widetilde{S}$ is the restriction of S to $R(S)$ and $q(t) = L^*p(t)$. Since $\widetilde{S}$ has a bounded inverse $\widetilde{S}^{-1} \in \mathcal{L}(R(S))$, equation (58) takes the form

$$\begin{aligned} &-\frac{d}{dt}(1-Q)q(t) + \widetilde{S}^{-1}(1-Q)q(t) = \\ &= \widetilde{S}^{-1}(1-Q)C^*C(y(t) - y_0(t)), 0 < t < \tau \end{aligned} \tag{60}$$

so that (58), (59), (60) guarantee that in fact

$$S\frac{d}{dt}(1-Q)q(t) + q(t) = C^*C(y(t) - y_0(t))$$

holds. Therefore,

$$\begin{aligned} &0 \le \pi(u, v-u) - \ell(v-u) = \\ &= \int_0^\tau \langle C^*C[y(u)(t) - y_0(t)], y(v)(t) - y(u)(t)\rangle_H \, dt + \\ &+ \langle Nu, v-u\rangle_{L^2(0,\tau;U)} = \\ &= \int_0^\tau \left\langle -S\frac{d}{dt}(1-Q)q(u)(t), y(v)(t) - y(u)(t)\right\rangle_H dt + \\ &+ \int_0^\tau \langle p(u)(t), L^*[y(v)(t) - y(u)(t)]\rangle_H \, dt + \\ &+ \langle Nu, v-u\rangle_{L^2(0,\tau;U)} . \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_0^\tau \left\langle -S\frac{d}{dt}(1-Q)q(u)(t), y(v)(t) - y(u)(t)\right\rangle_H dt = \\ &= -\int_0^\tau \left\langle \frac{d}{dt}(1-Q)q(u)(t), L^{-1}M[y(v)(t) - y(u)(t)]\right\rangle_H dt, \end{aligned}$$

$(1-Q)q(u)(\tau) = 0$, and there exists the $\lim_{t\to 0}(1-Q)q(u)(t)$. This yields

$$\begin{aligned} &\int_0^\tau \left\langle -S\frac{d}{dt}(1-Q)q(u)(t), y(v)(t) - y(u)(t)\right\rangle_H dt = \\ &= \int_0^\tau \left\langle L^{*-1}(1-Q)q(u)(t), \frac{d}{dt}M[y(v)(t) - y(u)(t)]\right\rangle_H dt = \\ &= \int_0^\tau \left\langle L^{*-1}(1-Q)q(u)(t), \frac{d}{dt}ML^{-1}(1-P)[Ly(v)(t) - Ly(u)(t)]\right\rangle_H dt = \\ &= \int_0^\tau \left\langle L^{*-1}(1-Q)q(u)(t), ML^{-1}\frac{d}{dt}(1-P)[Ly(v)(t) - Ly(u)(t)]\right\rangle_H dt = \\ &= \int_0^\tau \left\langle M^*L^{*-1}(1-Q)q(u)(t), L^{-1}\frac{d}{dt}(1-P)L[y(v)(t) - y(u)(t)]\right\rangle dt = \end{aligned}$$

$$= \int_0^\tau \left\langle M^* L^{*-1} q(u)(t), L^{-1}\frac{d}{dt}(1-P)L[y(v)(t)-y(u)(t)]\right\rangle_H dt =$$
$$= \int_0^\tau \left\langle p(u)(t), \frac{d}{dt} M L^{-1} L[y(v)(t)-y(u)(t)]\right\rangle_H dt =$$
$$= \int_0^\tau \left\langle p(u)(t), \frac{d}{dt} M[y(v)(t)-y(u)(t)]\right\rangle_H dt.$$

It follows that

$$0 \le \pi(u, v-u) - \ell(v-u) =$$
$$= \int_0^\tau \left\langle p(u)(t), \left(\frac{d}{dt}M + L(y(v)(t)-y(u)(t))\right\rangle_H dt + \langle Nu, v-u\rangle_{L^2(0,\tau;U)} =$$
$$= \int_0^\tau \langle p(u)(t), B(v(t)-u(t))\rangle_H dt + \langle Nu, v-u\rangle_{L^2(0,\tau;U)} =$$
$$= \langle B^* p(u) + Nu, v-u\rangle_{L^2(0,\tau;U)}$$

for all admissible $v \in \mathcal{U}$.

We can now state the following improvement of Theorem 5.3.

THEOREM **5.4.** *Let $\lambda = 0$ be a simple pole for $(\lambda + T)^{-1}$, $(\lambda + S)^{-1}$, where $T = ML^{-1}$, $S = M^* L^{*-1}$. If the pair (y, p) satisfies (53), (56), (57), where $y_0 \in \mathcal{D}(L)$, $My \in H^1(0,\tau;H)$, $M^*p \in H^1(0,\tau;H)$, then $u = -N^{-1}B^*p$ is the unique optimal control for (48)∼(50) with $\mathcal{U} = L^2(0,\tau;U)$.*

The solvability of (53), (56), (57) will be considered elsewhere.

Example 5.1 We can illustrate the last result with a simple but enlightening example. We are required to minimize

$$J(u,v) = \int_0^\tau \{x(t)^2 + y(t)^2 + u(t)^2 + v(t)^2\} dt$$

over $u, v \in L^2(0,\tau)$, under the constraints

$$0 = -x(t) - v(t) + f(t), 0 < t < \tau,$$
$$y'(t) = -y(t) + u(t), 0 < t < \tau,$$
$$y(0) = 0.$$

Here $f \in L^2(0,\tau)$ is given. We take $H = L^2(0,\tau) \times L^2(0,\tau) = L^2(0,\tau;R^2)$,

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ L = N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C,\ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, $J(u,v) = \int_0^\tau \{[(f(t)-v(t))^2 + v(t)^2] + [y(t)^2 + u(t)^2]\} dt \ge 2\int_0^\tau [v(t)^2 - f(t)v(t) + f(t)^2/2] dt \ge \frac{1}{2}\int_0^\tau f(t)^2 dt$ takes its minimum in $(0, f/2) = (\bar{u}, \bar{v})$ and

On the other hand, the system (53), (56), (57) can be written as

$$\frac{d}{dt}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} f(t) \\ 0 \end{bmatrix}, 0 < t < \tau,$$

$$-\frac{d}{dt}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = -\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}, 0 < t < \tau,$$

$$y(0) = q(\tau) = 0,$$

i.e.,

$$0 = -x - p + f(t), \quad 0 < t < \tau, \tag{61}$$

$$y' = -y - q, \quad 0 < t < \tau, \tag{62}$$

$$0 = -p + x, \quad 0 < t < \tau, \tag{63}$$

$$-q' = -q + y, \quad 0 < t < \tau. \tag{64}$$

The two point boundary value problem (62), (64) has the unique trivial solution $y(t) = q(t) \equiv 0$. This yields $x(t) = p(t) = f(t)/2$ and

$$(u, v) = -B^*(p, q) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = (0, f/2)$$

is the optimal control, as desired.

Example 5.2 Let $\Omega \subset R^n$, $u \geq 2$, be a bounded domain of R^n with a smooth boundary $\partial\Omega$. In the cylinder $\Omega\times(0,\tau)$ consider the initial boundary value problem

$$\frac{\partial}{\partial t}(\lambda_0 - \Delta)y = \alpha\Delta y - \beta\Delta^2 y - f + u,$$

$$(\lambda_0 - \Delta)y(x, 0) = (\lambda_0 - \Delta)y_0(x), \ x \in \Omega,$$

$$y(x,t) = \Delta y(x,t) = 0, \ (x,t) \in \partial\Omega\times(0,\tau),$$

where λ_0 is the first negative eigenvalue of the Laplacian Δ, with Dirichlet boundary conditions, $\alpha, \beta > 0$, $f \in L^2(\Omega\times(0,\tau))$ is given, $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ is the given initial condition, and $u \in L^2(\Omega\times(0,\tau))$ is the control. Similar equations model the evolution of a free surface of a filtered fluid. The spectral properties of the involved operators are described in Sviridyuk and Efremov (1995), where a precise representation of the pencil $(\lambda M + L)^{-1}$ is also given. Of course, here L, M are the operators in $L^2(\Omega) = H$ associated with $-\alpha\Delta + \beta\Delta^2$ and $\lambda_0 - \Delta$, respectively. Then, $D(L) = \{u \in H^2(\Omega) : u \in H_0^1(\Omega), \Delta u \in H_0^1(\Omega) \cap H^2(\Omega)\}$, $\mathcal{D}(M) = H_0^1(\Omega) \cap H^2(\Omega)$.

By Sviridyuk and Efremov (1995), p.1888, one sees that $\lambda = 0$ is a simple pole for $L(\lambda L + M)^{-1}$. Therefore, Theorems 5.2 and 5.4 work for the cost functional

$$J(u) = \int_0^\tau |y(t)|^2_{L^2(\Omega)} dt + \int_0^\tau |u(t)|^2_{L^2(\Omega)} dt.$$

6. Null controllability of the degenerate heat equation

Consider the controlled system (see (13))

$$
\begin{aligned}
(dy)_t(x,t) - \Delta y(x,t) &= m(x)u(x,t), \qquad && (x,t) \in Q = \Omega\times(0,T)\\
(dy)(x,0) &= d(x)y_0(x), && x \in \Omega\\
y &= 0 \quad \text{in} \quad \Sigma, &&
\end{aligned}
\tag{65}
$$

where $d \in C^2(\overline{\Omega})$, $d(x) \geq 0$, $\forall x \in \Omega$ and m is the characteristic function of an open subset $\omega \subset \Omega$. As usually, Ω is an open and bounded subset of R^n with a smooth boundary $\partial\Omega$.

The existence result below is well known (see, e.g., Barbu, Favini, Romanelli, 1996). However, we recall it for convenience.

LEMMA **6.1.** *Let $y_0 \in H_0^1(\Omega)$. Then equation* (6.1) *has a unique weak solution $y \in L^2(0,T; H_0^1(\Omega))$ which satisfies*

$$
(dy)_t \in L^2(0,T; H^{-1}(\Omega)),\ dy \in C([0,T]; L^2(\Omega)). \tag{66}
$$

If $u \in W^{1,2}([0,T]; L^2(\Omega))$ then

$$
(dy)_t \in L^2(Q),\ y \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega))
$$

and $\sqrt{d}\ y \in C([0,T]; L^2(\Omega))$.

Proof. One approximates (65) by

$$
\begin{aligned}
((d+\varepsilon)y)_t - \Delta y &= mu, \quad \text{in} \quad Q\\
(d+\varepsilon)y(x,0) &= (d+\varepsilon)y_0(x), \qquad x \in \Omega;\ y = 0 \text{ in } \Sigma
\end{aligned}
\tag{67}
$$

which has a unique solution $y_\varepsilon \in H^{2,1}(Q)$.

We have the apriori estimates

$$
\begin{aligned}
&\int_Q |\nabla y_\varepsilon(x,t)|^2 dx\,dt + \int_\Omega (d(x)+\varepsilon)y_\varepsilon^2(x,t)dx \leq\\
&\leq C\left(\int_\Omega (d(x)+\varepsilon))y_0^2(x)dx + \int_Q mu^2 dx\,dt\right).
\end{aligned}
\tag{68}
$$

Then, one obtains the desired result letting ε tend to zero in (67).

If $u \in W^{1,2}([0,T]; L^2(\Omega))$ then, multiplying (67) by y_t and Δy, we obtain the estimate

$$
\begin{aligned}
&\int_Q |\Delta y_\varepsilon|^2 dx\,dt + \int_\Omega d(x)|\nabla y_\varepsilon(x,t)|^2 dx \leq\\
&\leq C\left(\int_\Omega |\nabla y_0|^2 dx + \|u\|^2_{W^{1,2}([0,T];L^2(Q))}\right)
\end{aligned}
$$

THEOREM **6.1.** *For each* $y_0 \in H_0^1(\Omega)$ *there is* $u \in L^2(Q)$ *such that*

$$(dy^u)(x,T) = 0, \quad a.\ e.\ x \in \Omega \tag{69}$$

where y^u *is the solution to (65).*

This theorem ressembles some recent results on the null internal controllability of the heat equation (see Fursikov, Imanuvilov, 1996, Lebeau, Robbiano, 1995). In particular, we derive by Theorem 6.1 the boundary controllability of equation (65).

THEOREM **6.2.** *Let* $y_0 \in H_0^1(\Omega)$. *Then there is* $v \in L^2(\Sigma)$ *such that the solution* y_v *to equation*

$$\begin{aligned} (dy)_t(x,t) - \Delta y(x,t) = 0 \quad & in \quad Q \\ (dy)(x,0) = d(x)y_0(x) \quad & in \quad \Omega \\ y = v \quad & in \quad \Sigma \end{aligned} \tag{70}$$

satisfies $(dy^v)(x,T) = 0$ *a.e.* $x \in \Omega$.

Proof of Theorem 6.2. Let $\widetilde{\Omega}$ be an open set such that $\Omega \subset \widetilde{\Omega}$ and set $\omega = \widetilde{\Omega} \backslash \overline{\Omega}$, $\widetilde{Q} = \widetilde{\Omega} \times (0,T)$. We shall apply Theorem 6.1 on $\widetilde{Q}$ where y_0 and d are suitably extended to $\widetilde{\Omega}$. Accordingly, there are $\widetilde{y}$ and $\widetilde{u} \in L^2(\widetilde{Q})$ such that $(d\widetilde{y})(x,T) = 0$ a.e. $x \in \Omega$. Then the restriction y of $\widetilde{y}$ to Q satisfies (6.5) with $v = \gamma_0(\widetilde{y})$. (Here γ_0 is the trace of $\widetilde{y} \in L^2(0,T; H_0^1(\Omega))$ to $\partial\Omega \subset \widetilde{\Omega}$.) This completes the proof of Theorem 6.2. ■

In order to prove Theorem 6.1 we need a Carleman's type estimate for the solutions to homogeneous equation

$$(d(x)y)_t - \Delta y = 0 \quad \text{in} \quad Q \tag{71}$$

$$y = 0 \quad \text{on} \quad \Sigma. \tag{72}$$

Let $\omega_0 \subset\subset \omega$ and let $\psi \in C^2(\overline{\Omega})$ be such that

$$\psi(x) > 0,\ \forall x \in \Omega,\ \psi|_{\partial\Omega} = 0,\ |\nabla\psi(x)| > 0,\ \forall x \in \Omega_0 = \Omega \setminus \omega_0.$$

The existence of such a function ψ has been proved in Fursikov, Imanuvilov (1996). We set

$$\varphi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}$$

where $\lambda > 0$. The proof of Lemma 6.2 below is essentially the same as that of Lemma 1.2 in Fursikov, Imanuvilov (1996) and so it will be omitted.

LEMMA **6.2.** *There exists* $s_0, \lambda_0 > 0$ *such that for* $s \geq s_0$ *and* $\lambda \geq \lambda_0$ *we have*

$$\int_Q (e^{2s\alpha}(\varphi^{-1}(dy)_t^2 + \varphi^3 y^2 + \varphi|\nabla y|^2)dx\,dt \leq C\int_{Q\omega} y^2 e^{2s\alpha}\varphi^3 dx\,dt \tag{73}$$

COROLLARY **6.1.** *There is C independent of y such that*

$$\int_\Omega d(x)y^2(x,T)dx \le C\int_{Q_\omega} y^2 dx\, dt \tag{74}$$

for each solution y to 70.

Proof. By (70) we see that the function $t \to \int_\Omega d(x)y^2(x,t)dx$ is decreasing. Hence

$$\begin{aligned}&\int_\Omega d(x)y^2(x,T)dx \le \int_\Omega d(x)y^2(x,t)dx \le \\ &\le C_t\int_\Omega e^{2s\alpha(x,t)}y^2(x,t)dx,\ \forall t\in(0,T).\end{aligned}$$

Integrating on $(a, a+\varepsilon)$ where $0 < a < a+\varepsilon < T$ and using Carleman's estimate (73) we obtain (74) as claimed. ■

Proof of Theorem 6.1. Consider the optimal control problem

$$\textit{Minimize} \int_Q u^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d(x)y^2(x,T)dx \textit{ subject to } (65). \tag{75}$$

It is readily seen that (75) has a unique solution (y_λ, u_λ). Moreover, it satisfies the equations

$$u_\lambda(x,t) = m(x)p_\lambda(x,t)\text{a.e.}(x,t)\in Q \tag{76}$$
$$(dp_\lambda)_t + \Delta p_\lambda = 0 \text{ in } Q \tag{77}$$
$$(dp_\lambda)(x,T) = -\frac{1}{\lambda}dy_\lambda(x,T) \text{ in } \Omega$$

and therefore

$$\begin{aligned}&\int_Q mp_\lambda^2(x,t)dx\, dt = \\ &= \int_\Omega d(x)(p_\lambda(x,T)y_\lambda(x,T) - p_\lambda(x,0)y_\lambda(x,0))dx.\end{aligned} \tag{78}$$

This is obvious if $d > 0$ in $\overline{\Omega}$ because as mentioned earlier, in this case $p_\lambda, y_\lambda \in H^{2,1}(Q)$. In the general case we replace (75) by

$$\textit{Minimize} \int_Q u^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d_\varepsilon y^2(x,T)dx \quad \textit{subject to } (67). \tag{79}$$

Let $(y^\varepsilon, u^\varepsilon)$ be the corresponding solution to (79) and let p^ε be the solution to the dual system

$$\begin{aligned} d_\varepsilon(p^\varepsilon)_t + \Delta p^\varepsilon &= 0 \quad \text{in} \quad Q \\ d_\varepsilon(x)p^\varepsilon(x,T) &= -\frac{1}{\lambda} d_\varepsilon(x) y^\varepsilon(x,T) \quad \text{in} \quad Q \\ p^\varepsilon &= 0 \quad \text{in} \quad \Sigma. \end{aligned} \tag{80}$$

We have

$$mp^\varepsilon = u^\varepsilon, \quad \text{a.e. in } Q.$$

By (80) we see that

$$\int_Q (u^\varepsilon)^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d_\varepsilon(x)(y^\varepsilon)^2(x,T)dx \le$$
$$\le \int_Q u^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d_\varepsilon(x)(\bar{y}^\varepsilon)^2(x,T)dx$$

for any $u \in L^2(Q)$ where $\bar{y}^\varepsilon$ is the solution to (67). This implies that, on a subsequence,

$$\begin{aligned} u_\varepsilon &\longrightarrow u_\lambda \text{ weakly in } L^2(Q) \\ y_\varepsilon &\longrightarrow y_\lambda \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \\ \sqrt{d_\varepsilon}y^\varepsilon &\longrightarrow \sqrt{d}y_\lambda \text{ strongly in } C([0,T];L^2(\Omega)). \end{aligned}$$

(See Lemma 6.1.)

Similarly for the solutions p^ε to (80). Since (76) is obviously satisfied for y^ε and p^ε we get it for y_λ and p_λ by letting ε tend to zero.

Now by (75), (76) we have

$$\int_Q mp_\lambda^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d(x)y_\lambda^2(x,T)dx = \int_\Omega d(x)p_\lambda(x,0)y_0(x)dx.$$

By estimate (73) we see that

$$\int_Q u_\lambda^2 dx\, dt + \frac{1}{\lambda}\int_\Omega d(x)y_\lambda^2(x,T)dx \le C, \ \forall \lambda > 0.$$

Thus on a subsequence, again denoted λ,

$$\begin{aligned} u_\lambda &\longrightarrow u^* \text{ weakly in } L^2(Q) \\ \sqrt{d}y_\lambda(\cdot,T) &\longrightarrow 0 \text{ strongly in } L^2(\Omega)). \end{aligned} \tag{81}$$

Letting λ tend to zero in the equations

$$(dy_\lambda)_t - \Delta y_\lambda = mu_\lambda \quad \text{in } Q$$

and recalling the estimate (68) we infer that

$$y_\lambda \longrightarrow y^* \text{ weakly in } L^2(0,T;H_0^1(\Omega))$$

where y^* is the solution to (65) for $u = u^*$. By (81) it is also clear that $(dy)(x,T) = 0$, a.e. $x \in \Omega$. This completes the proof. ■

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