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On boundary observability estimates for semi-discretizations of a dynamic network of elastic strings[1,2]

by

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Abstract: We consider a tripod as an exemplaric network of strings. We know that such a network is exactly controllable in the natural finite energy space, if, e.g., the simple nodes are controlled by Dirichlet controls in $H^1(0,T)$. Assume that we want to calculate the corresponding norm-minimal controls using semi-discretization in space. We then obtain a system of coupled second-order-in-time ordinary differential equations with three control inputs. Controllability of the latter system can easily been checked by Kalman's rank condition on each space discretization level h. One expects, as h tends to zero, that the exact controllability of the continuous system is revealed. This expectation is frustrated, as has been shown by Infante and Zuazua (1998) for a single string and by Zuazua (1999) for a membrane. Indeed, it was shown there that uniformity of observability estimates is lost in the limit. On the other hand, spectral filtering allows to cure this pathology. We show in this paper that similar results hold for our string network. The generalization to arbitrary networks of strings in the out-of-the-plane as well as in the in-plane or 3-d-setup is then a technical matter. Therefore, this paper essentially extends the existing results to semidiscretizations of wave equations on arbitrary irregular computational grids.

Keywords: network of strings, semidiscretization, lack of uniform observability, filtering.

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1. Introduction

The theory of exact controllability of 1-dimensional mechanical *single-link* systems like strings and beams is by now fairly complete. On the other hand, related problems for dynamic *networks* of strings and beams still offer a basket of open problems. This is particularily true for Euler-Bernoulli beam systems, see Leugering and Schmidt (1989) and very recently Dekoninck and Nicaise (1999). As a general reference see Lagnese, Leugering and Schmidt (1994). As control theory is intrinsically linked to applications, numerical simulations of controlled problems and numerical realization of control laws are extremely important. In addition, in cases where theoretical results are lacking, numerical evidence is very important in supporting research on the continuous level. Therefore, reliable numerical computations are also at the heart of control theory of infinite dimensional systems. The problem one is faced with, particularly in the case of hyperbolic equations, is that physical phenomena, such as finite speed of propagation, do not exist in finite dimensions. Therefore, control strategies for infinite dimensional problems which make extensive use of such phenomena might have little to do with those for finite dimensional ones. Our philosophy, therefore, is to stay as long as possible with the PDE-models, use control designs on that level and then discretize.

In this context it has recently been observed by Infante and Zuazua (1998), and Zuazua (1999), that approaching exactly controllable systems by naive semi-discretizations and use of standard finite-dimensional control strategies can be dangerous. In particular, they show that as the space discretization parameter h (in 1-d, or the 'size' of a finite element in 2-d) tends to zero, the uniformity of the corresponding observability estimates gets lost. It is shown that the approximation properties of the eigenfrequencies of the semi-discrete system are responsible for this lack of uniformity. More precisely, asymptotic gaps in the spectrum, which are essential for the application of moment techniques, do no longer hold for the semi-discrete approximations. As the spectral gap appears in the lower bound of the controllability Grammian – the discrete counterpart of the HUM operator – such that the lower bound tends to zero with the gap, bounded invertibility is lost in the limit. The cure for this pathology is to filter the frequencies of the finite dimensional problems according to the discretization level. It appears that such a filtering technique is strongly related to Tychonov regularizations of the controllability problem. This relation is very important from the numerical point of view. The semi-discretization of related problems and the corresponding Tychonov regularizations have been investigated by Vasilyev, Kurzhanskii and Potapov (1993) for a single string and by Leugering (1999) for networks. We further note that this pathology extends to other important optimal control problems, namely *time optimal control* problems with pointwise norm bounds on the control.

Since the insight into the spectrum of general networks of strings (and mandatory to us. Also, as networks of strings (and beams) after semi-discretization can be considered as computational domains of continuous structures such as membranes (and plates) on very irregular grids, we considered the extension of the abovementioned results to such networks as a useful contribution to the subject, even if finally the results are very similar.

Further research in this area is devoted to domain decomposition techniques on the semi-discrete level, see Leugering (1999) (see also Lagnese, 1999, Lagnese and Leugering, 1999, and Leugering 1997, 1999a, b).

Let us now consider a simple star as shown below:

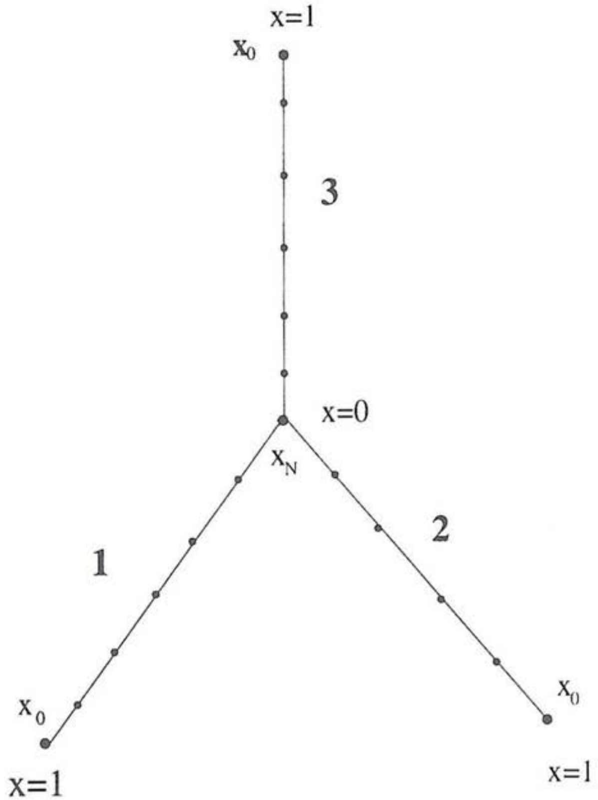

Figure 1. A star shaped simple network of strings

We will consider classical semi-discretization in the space of equations governing the motion of this star viewed as a system of three prestretched strings or three 1-d elastic elements. We will investigate boundary observation estimates, and finally we are going to define a filtering device for the frequencies of the

This paper is organized as follows. In the second section we discuss the continuous model and its main properties. This is followed by an analysis of the spectrum of the Laplace operator on the graph. In section three we turn to the semi-discrete model and after clarifying the notation we present an analysis of the spectrum of the corresponding Laplacian on the computational graph. The next section contains the main *blow-up* result, the definition of an appropriate subclass of solutions, the *filtered solutions*, for which we then derive the boundary observability estimate. We discuss then open points and possible generalizations. Finally the appendix consists of the proofs of the main lemmas.

2. The continuous model and its spectral analysis

We denote by $u_k(x,t),\ k=1,2,3$ the out-of-the-plane displacement of string number k. All strings are coupled at $x=0$ and are supposed to satisfy non-homogeneous Dirchlet conditions at the other end $x=1$. We take all physical constants equal to 1. Then, the system of equations governing the motion of the tripod (star, divinig rod) is given by:

$$
\begin{aligned}
\ddot{u}_k - u_k'' &= 0, \quad 0<x<1,\ 0<t<T,\ k=1,2,3, &(1)\\
u_k(1,t) &= v_k(t), \quad 0<t<T,\ k=1,2,3, &(2)\\
u_k(0,t) &= u_m(0,t), \quad 0<t<T,\ k,m=1,2,3, &(3)\\
\sum_{k=1}^{3} u_k'(0,t) &= 0, \quad 0<t<T &(4)\\
u_k(x,0) &= u_{k0}(x),\ \dot{u}_k(x,0)=u_{k1}(x),\quad 0<x<1,\ k=1,2,3. &(5)
\end{aligned}
$$

We have denoted u_x, u_t by u', $\dot{u}$. Some remarks should be made:

REMARK 2.1
- *Well posedness and regularity of solutions to system (1)–(5) have been proven by Lagnese, Leugering and Schmidt (1994) for general networks, see also Leugering (1999c) for this particular setup. some details will be discussed below.*
- *Condition (4) can be replaced by $\sum_{k=1}^{3} u_k'(0,t) = m\ddot{u}_k(0,t)$, where m denotes a point mass at the multiple node. For such systems see Hansen and Zuazua (1995), Ming Wei (1993) and Leugering (1998).*

It is clear that we take this system as an exemplaric model in order to keep the notation as simple as possible, while still dealing with a nontrivial graph i.e. a non-serial situation. The generalization to arbitrary graphs in the spirit of Lagnese, Leugering and Schmidt (1994) is possible but appears to be some features of the continuous one. The energy of solutions is given by

$$E(t) = \frac{1}{2} \sum_{k=1}^{3} \int_0^1 \left[| \dot{u}_k(x,t) |^2 + | u'_k(x,t) |^2 \right] dx \tag{6}$$

and it is conserved along time, i.e.

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{7}$$

We have chosen Dirichlet boundary controls at all simple nodes, partly for convenience, partly to simplify the comparison with the results of Infante and Zuazua (1998) and Zuazua (1999). However, for reasons of completeness we would like to discuss the possible choices of boundary conditions and controls for a general graph in some detail.

REMARK 2.2 (REMARKS ON CONTROLLABILITY OF NETWORKS OF STRINGS) *We consider a general simply connected graph G as a reference configuration of a network of strings. We apply controls at simple nodes (the leaves), that is, at nodes with edge degree equal to 1. We can have Dirchlet controls and Neumann controls (or Robin-type controls). In order to stay in a natural finite energy space we take Dirichlet controls from $H^1(0,T)$, whereas Neumann controls are assumed to be in $L^2(0,T)$. In the statements below we assume that the control time is sufficiently large, such that the signals can travel from controlled nodes accross the entire graph and back. With this convention we may state the following known facts in a somewhat informal way:*

- *Let G be a tree with all simple nodes controlled (either Dirichlet, Neuman or Robin conditions). This is the simplest of all cases, and this one is discussed in this paper as a model problem. Then one has exact controllability of finite energy solutions.*
- *Let G be a rooted tree with the root being clamped. If all the leaves are controlled one obtains exact controllability of finite energy solutions, for Dirichlet or Neumann boundary controls or a mixture of both. See Lagnese, Leugering and Schmidt (1994).*
- *Let G be a rooted tree with the root being clamped. If all but one of the leaves are controlled, and the remaining leaf satisfies the Neumann boundary conditions, then again exact controllability obtains, see Leugering (1998).*
- *Let G be a rooted tree with the root being clamped. If again all but one of the leaves are controlled, and the remaining leaf satisfies a Dirichlet boundary condition, and if, in addition, the path connecting these two simple nodes consists of strings having rationally dependent optical lengths, then even approximate controllability fails to hold. The same applies to Neumann conditions, see Lagnese, Leugering and Schmidt (1994). However, if the lengths of the strings are in a certain class (in fact Roth's class) of*

constants are set to 1, then it can be shown that exact controllability holds in smoother spaces, see Leugering and Zuazua (1999). This appears to be true even if only the root or any other simple node is controlled.

- *Let G be a graph containing circuits. Assume that the strings constituting such a circuit have rationally dependent optical lengths, then, even if all nodes are controlled, approximate controllability fails to hold. However, if those strings have their mutual length in a class described above, exact controllability may hold in smoother spaces. This problem is investigated in Leugering and Zuazua (1999b).*
- *If we consider the problems above in the context of Timoshenko beams we obtain similar results. If, however, we consider those problems for networks of Euler-Bernoulli beams, only the very first result, namely exact controllability from all simple nodes is known to hold. See Leugering and Schmidt (1989). That is to say, even exact controllability of a carpernter's square of Euler-Bernoulli beams in the plane appears to be an open problem. See also Dekoninck and Nicaise (1999) for recent results on some scalar beam networks.*

As mentioned above, we are much more modest here and consider only three coupled strings as in (1)-(5).

In the situation of the first problem, where all simple nodes are under control, we can easily derive observability estimates using energy multipliers. For the sake of easier reference, and also because the semi-discrete case is developed in complete analogy with the continuous case, we outline the arguments.

In order to obtain energy estimates we use simple multipliers $m_i(x)$

$$\begin{aligned} 0 &= \int_0^T \int_0^1 \sum_{i=1}^3 (\ddot{u}_i - u_i'') m_i u_i' \, dx \, dt \\ &= \int_0^1 \sum_{i=1}^3 \dot{u}_i m_i u_i' \, dx \Big|_0^T - \int_0^T \int_0^1 \sum_{i=1}^3 m_i \frac{1}{2} \frac{d}{dx} \dot{u}^2 \, dx \, dt \\ &\quad -\frac{1}{2} \int_0^T \sum_{i=1}^3 u_i'^2 m_i \Big|_0^1 dt + \frac{1}{2} \int_0^T \int_0^1 \sum_{i=1}^3 m_i' u_i'^2 \, dx \, dt \\ &= \int_0^1 \sum_{i=1}^3 \dot{u}_i m_i u_i' \, dx \Big|_0^T - \frac{1}{2} \int_0^T \sum_{i=1}^3 m_i \dot{u}^2 \Big|_0^1 dt \\ &\quad -\frac{1}{2} \int_0^T \sum_{i=1}^3 m_i u_i'^2 \Big|_0^1 dt \\ &\quad +\frac{1}{2} \int_0^T \int_0^1 \sum_{i=1}^3 m_i' (\dot{u}_i^2 + u_i'^2) \, dx \, dt. \end{aligned}$$

Hence we obtain the general identity

$$\frac{1}{2}\int_0^T \sum_{i=1}^3 m_i(\dot u_i^2 + u_i'^2)\big|_0^1\, dt = \int_0^1 \sum_{i=1}^3 \dot u_i m_i u_i'\, dx\Big|_0^T + \frac{1}{2}\int_0^T \int_0^1 \sum_{i=1}^3 m_i'(\dot u^2 + u'^2)\, dx\, dt \tag{8}$$

which relates the 'energy trace' to total energy. Alluding to our particular boundary and transmission conditions and assuming that $m_i(x) = -1 + 2x$ we obtain a direct energy inequality:

$$\frac{1}{2}\int_0^T \sum_{i=1}^3 \{\dot u_i(0,t)^2 + u_i'(0,t)^2 + \dot u_i(l,t)^2 + u_i'(l,t)\}\, dt \\ \leq CE(0). \tag{9}$$

Inequality (9) establishes a so-called 'hidden regularity', as finite energy solutions have L^2-traces of the velocity and Neuman data, a fact which is not directly seen from trace-theorems. On the other hand, by transposition, we obtain $H \times V^\star$-regularity of u with initial data $(u_0, u_1) \in H \times V^\star$, $v \in L^2(0,T)^3$, where the spaces V and H are defined by:

$$\begin{aligned} H &:= \prod_{i=1}^3 L^2(0,1) \\ V &:= \left\{ u \in \prod_{i=1}^3 H^1(0,1) \big| u_1(0) = u_2(0) = u_3(0), u_i(1) = 0,\ i = 1,2,3 \right\}. \end{aligned}$$

In equation (8) we may also take $m_i(x) = x$, and derive an indirect energy inequality:

$$E(0) \leq C(T) \sum_{k=1}^3 \int_0^T \mid u_k'(1,t) \mid^2 dt \tag{10}$$

for $T > 2$ and for some $C(T) > 0$, where the u_k, $k = 1,2,3$ satisfy (1)-(5) for $v_k(t) = 0$, $t \in (0,T)$.

In order to illustrate the meaning of the crucial inequality (10), we define the control-to-state operator L_T carrying the controls v_k into the final data at time T. We formulate this in terms of $L^2(0,T)$-controls by transposition.

$$\left\{ L_T : \prod_{i=1}^3 L^2(0,T) \longrightarrow V^\star \times H \right. \tag{11}$$

We observe immediately that $L_T^\star$ is given by

$$\begin{cases} L_T^\star : V \times H \longrightarrow \prod_{i=1}^{3} L^2(0,T) \\ L_T^\star(\varphi_0, \varphi_1) = (\varphi_i'(l,\cdot))_{i=1,2,3}. \end{cases} \tag{12}$$

Exact controllability is equivalent to the surjectivity of L_T which is, in turn, equivalent to the condition

$$\|L_T^\star(\varphi_0, \varphi_1)\| \geq \gamma \|\varphi_0, \varphi_1\|_{V\times H}. \tag{13}$$

For $T > 2$ this gives exact controllability.

The norm minimal controls realizing the transfer from zero initial conditions to given final conditions $(u_T, \dot{u}_T)$ are given via the right-inverse of L_T

$$v = L_T^\star (L_T L_T^\star)^{-1}(-\dot{u}_T, u_T). \tag{14}$$

Thus, all comes down to solving the symmetric problem

$$L_T L_T^\star(\varphi_0, \varphi_1) = (-\dot{u}_T, u_T) \tag{15}$$

for the data $(\varphi_0, \varphi_1) \in V \times H$.

The illposedness of the problem of exact-controllability for semi-discrete approximations becomes more apparent when looking at (15). After semi-discretization the HUM-operator appearing in (15) becomes a symmetric matrix with condition number depending on the spatial step size h. If that condition number (which is the ratio of the largest and the smallest eigenvalue) becomes very large, the problem is numerically ill-conditioned. We will see later that cutting Fourier series at a given level corresponds precisely to cutting off of the smallest eigenvalue of the semi-discrete HUM-operator, hence this technique, which has been proposed by E.Zuazua is equivalent to an adaptive *truncated singular value decomposition*.

One might also consider the strongly related *Tychonov regularization* method which corresponds to

$$\begin{cases} L_T L_T^\star(p_T, \dot{p}_T) + \dfrac{1}{k}(p_T, \dot{p}_T) = (-\dot{u}_T, u_T) \\ v = L_T^\star(\varphi_T, \dot{\varphi}_T) = (\varphi_i'(l,\cdot))_{i=1,2,3}. \end{cases} \tag{16}$$

In fact, this system is the optimality condition for the penalized optimal control problem

$$\left\{ \min_f \left\{ \frac{1}{2}\int_0^T \sum_{i=1}^{3} f_i^2 \, dt + \frac{k}{2}\{\|u(T) - z_0\|_H^2 + \|\dot{u}(T) - z_1\|_{V^\star}^2\} =: J_k(f) \right\} \right. \tag{17}$$

Indeed, given the negative results in the sprit of Infante and Zuazua (1998), for the corresponding semi-discrete optimal control problem and then use a Lagrangian approach. See Leugering (1999c) for more details.

The problem we want to consider is whether or not an estimate like (10) continues to be true for the semi-discrete model to be developed below, with the Neumann derivative in (10) replaced with a finite difference approximation uniformly in the discretization parameter h, and $C(T)$ independent of h. In order to investigate this problem we need some information about the spectrum of the continuous as well as the semi-discrete model. The spectrum of Sturm-Liouville operators on graphs has been investigated by von Below (1988) and Nicaise (1993). After discretization the Laplacian on the graph turns into the so called Laplacian-matrix associated with the computational graph. Spectra of such Laplacian matrices, in the general case, are still under investigation; see for instance Merris (1998) and Grone and Zimmermann (1990), and the bibliographies therein. Our example is quite simple, and, therefore, we give the arguments directly.

2.1. Spectral analysis for the continuous case

We consider

$$\begin{aligned} -\varphi_k'' &= \lambda\varphi_k, \quad \varphi_k(1) = 0, \quad k = 1,2,3, && (18)\\ \varphi_k(0) &= \varphi_l(0), \quad k,l = 1,2,3, && (19)\\ \sum_{k=1}^{3}\varphi_k'(0) &= 0, && (20)\end{aligned}$$

with

$$\varphi_k = A_k \sin\sqrt{\lambda}(1-x) + B_k\cos\sqrt{\lambda}(1-x). \tag{21}$$

The Dirichlet data at $x = 1$ imply $B_k = 0$, and the continuity at $x = 0$ implies

$$A_k \sin\sqrt{\lambda} = A_l \sin\sqrt{\lambda}. \tag{22}$$

Now two cases have to be distinguished:

1. $A_k = A_l = A$, $k,l = 1,2,3$ and $\sin\lambda \neq 0$
2. $\sin\lambda = 0$ which implies $\sqrt{\lambda} = k\pi$.

As we shall see, the first case corresponds to the global eigen-modes and the second case to the local ones. We use the balance of forces at $x = 0$ which implies

$$\sum_{k=1}^{3} A_k\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{23}$$

Using case 1 and 2 we obtain:

2. $\sum_{k=1}^{3} A_k = 0$, from which we obtain a two-parameter family

$$
\begin{aligned}
A_1 &= A_2 = C \\
A_1 &= -2C \qquad A_2 = A_3 = C = \sqrt{\frac{1}{3}}(-1)^{k+1}.
\end{aligned}
$$

To sum up, the two cases are

1. $\lambda_m = \left(\frac{m\pi}{2}\right)^2$ m odd, $\varphi_k^m(x) = \sqrt{\frac{2}{3}} \cos \frac{m}{2}\pi x$, $k = 1, 2, 3$
2. $\lambda_m = (m\pi)^2$, $m \in \mathbf{N}$, $\varphi_1^m(x) = \varphi_2^m(x) = \frac{1}{\sqrt{3}} \sin m\pi x$, $\varphi_3^m(x) = -\frac{2}{\sqrt{3}} \sin m\pi x$.

3. The semi-discrete model

Now we turn to the semi-discrete model. We devide each edge into $N+2$ points, and define the stepsize $h = 1/(N+1)$ as well as the discrete coordinates $x_j = jh,\ j = 0, 1, 2, .., N+1$. We then approximate the evaluation of $u_k(x_j, t)$ by $y_{k,j}$. We employ the standard finite difference scheme to the local wave equations on the strings as follows:

$$
h^2 \ddot{y}_{k,j} = y_{k,j+1} - 2y_{kj} + y_{k,j-1} \quad j = 1..N, \quad k = 1, 2, 3 \tag{24}
$$

with the the following conditions:

- initial conditions
$$y_{k,j}(0) = u_{k0}(x_j),\ \dot{y}_{k,j} = u_{k1}(x_j), \quad j = 0, ..., N+1,\ k = 1, 2, 3 \tag{25}$$
- Dirichlet conditions at $x = 1$ correspond to
$$y_{kN+1} = 0 \quad k = 1, 2, 3 \tag{26}$$
- continuity at $x = 0$ that leads to
$$y_{k,0} = z, \quad k = 1, 2, 3, \tag{27}$$
with an unknown z
- balance of forces at $x = 0$: in order to obtain a second order consistent model, we introduce a fictitious boundary point at $j = -1$ and then use the local equations to eliminate this point. This is done as follows:
$$\sum_{k=1}^{3} u_k'(0) \approx \sum_{k=1}^{3} \frac{y_{k,1} - y_{k,-1}}{2h} = 0. \tag{28}$$
In order to eliminate $y_{k,-1}$, we are using the wave equation at $x = 0$ with a new mass $\frac{2}{3} =: m_{k0}$
$$h^2 m_{k0} \ddot{z} = y_{k,1} - 2z - y_{k,-1}, \tag{29}$$
hence
$$y_{k,-1} = 2y_{k,0} - y_{k,1} + h^2 m_{k0} \ddot{z}, \tag{30}$$
which leads to
$$0 = \sum^{3} \frac{y_{k,1} - y_{k-1}}{} = \sum^{3} \frac{2y_{k,1} - 2z - h^2 m_{k0} \ddot{z}}{} \tag{31}$$

$$= \frac{1}{h}\sum_{k=1}^{3} y_{k,1} - \frac{3}{h}z - \frac{3}{2}hm_{k0}\ddot{z}.$$

The choice of $m_{k0} = \frac{2}{3}$ is plausible as we have an artificial interval of length $2h$ and we average over the number of incident edges. This choice leads to the condition:

$$h^2\ddot{z} = \sum_{k=1}^{3} y_{k,1} - 3z \tag{32}$$

REMARK 3.1 *Note that in the case of a rectangular grid, the interior points have edge degree 4. Then the balance condition above would read*

$$h^2\ddot{z} = \sum_{k=1}^{4} y_{k,1} - 4z \tag{33}$$

Which is exactly the semi-discrete 2-d wave equation with the classical five-point-star finite difference scheme. At this point the intrinsic similarity with Zuazua (1999) becomes apparent.

The complete system (24),(25),(26),(27), (32), therefore, reads like

$$\left\{\begin{array}{l} h^2\ddot{y}_{k,j} = y_{k,j+1} - 2y_{kj} + y_{k,j-1},\ j = 1,..,N,\ k = 1,2,3,\ 0 < t < T, \\ h^2\ddot{z} = \sum_{k=1}^{3} y_{k,1} - 3z, \quad 0 < t < T, \\ y_{k,N+1} = 0, \quad k = 1,2,3,\ 0 < t < T, \\ y_{k,0} = z, \quad k = 1,2,3,\ 0 < t < T, \\ y_{k,j} = u_{k0}(x_j),\ \dot{y}_{k,j} = u_{k1}(x_j), \quad j = 0,...,N+1,\ k = 1,2,3 \end{array}\right. \tag{34}$$

We have taken homogeneous boundary data here, as we will be dealing with the adjoint problem throughout the paper. We introduce a weighting factor at the origin.

$$\left\{\begin{array}{ll} \rho_{kj} = 1, & j = 1,..,N \\ \rho_{k0} = \frac{1}{3}, & j = 0. \end{array}\right. \tag{35}$$

We then define the semidiscrete energy

$$E_h(t) = \frac{h}{2}\sum_{k=1}^{3}\sum_{j=0}^{N}\left[\rho_{kj} \mid \dot{y}_{k,j}(t) \mid^2 + \left|\frac{y_{k,j+1}(t) - y_{k,j}(t)}{h}\right|^2\right]. \tag{36}$$

We introduce the local stiffnes matrix (mod $\frac{1}{h^2}$) for each individual string together with coupling vectors.

$$T_h \ := \ \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{pmatrix} \tag{37}$$

The overall stiffness-matrix (again mod $\frac{1}{h^2}$) then becomes

$$L_h = -\begin{pmatrix} c & v_h^T & v_h^T & v_h^T \\ v_h & T_h & & \\ v_h & & T_h & \\ v_h & & & T_h \end{pmatrix}. \tag{38}$$

The nodal vectors are arranged as follows

$$y = (z, y_{11}, \ldots, \ldots, \ldots, y_{1N}, y_{21}, \ldots, \ldots, \ldots, y_{2n}, y_{31}, \ldots, \ldots, \ldots, y_{3N})^T.$$

Denote by $D = diag(3, 2 \ldots 2, 2 \ldots 2, 2 \ldots 2)$ the diagonal matrix containing the edge degrees for the grid points. Denote further by A the vertex–to–vertex adjacency matric. Then the semi-discrete equation (34) reads as:

$$h^2 \, \ddot{y} + \underbrace{(D - A)}_{L_h} y = 0, \tag{39}$$

where $L_h = D - A$ is the so-called Laplacian–matrix of the graph. (This is completely general and can be written down for arbitrary graphs.) We now turn to the spectral analysis of the semidiscrete model.

3.1. Spectral analysis

We look for the eigenvalues of the Laplacian-matrix L_h in the form

$$L_h \varphi = -\lambda \varphi h^2, \tag{40}$$

and proceed with the following Ansatz $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$, $\varphi_{k,0} = z$, $\quad k = 1, 2, 3$, $\varphi_{kj} = A_k \sin \mu(N+1-j)h + B_k \cos \mu(N+1-j)h$, $\quad j = 0 \ldots N+1$, $\quad k = 1, 2, 3$.

At this point we now take into account the restriction imposed by the first configuration (i. e. the three controlled Dirichlet nodes). Otherwise we would have to change the matrix introduced above in the appropriate locations by applying some rank-one-updates.

Our Dirichlet conditions imply

$$\varphi_{k,N+1} = 0 \quad \Longrightarrow B_k = 0 \quad k = 1, 2, 3. \tag{41}$$

Using continuity at $k = 0$:

$$A_k \sin \mu = A_j \sin \mu \quad k, j = 1, 2, 3, \tag{42}$$

gives us

$$i.) \;\; A_k \;\; = \;\; A \;\; k = 1, 2, 3, \qquad \sin \mu \neq 0,$$

Taking the balance of forces at $j = 0$ into account gives

$$\begin{aligned} \sum_{k=1}^{3} \varphi_{k1} - 3\varphi_{i0} &= -\lambda h^2 \varphi_{i0} & (43) \\ i.)\ 3\sin\mu Nh - 3\sin\mu &= -\lambda h^2 \sin\mu & (44) \\ 3(1-\cos\mu h - \cot\mu\sin\mu h) &= \lambda h^2, & (45) \end{aligned}$$

so we are left with a discrete 'elliptic' problem of the form

$$\varphi_{k,j+1} - 2\varphi_{k,j} + \varphi_{k,j-1} = \lambda h^2 \varphi_{k,j},\ j = 1,..,N,\ k = 1,2,3, \qquad (46)$$

with

$$\begin{aligned} \varphi_{k,j\pm1} &= A_k \left\{\sin\mu\,(N+1-j)\,h\cos\mu h \pm \sin\mu h\cos\mu(N+1-j)h\right\}, \\ \sin\mu\,(N+1-j)\,h(2\cos\mu h - 2) &= -\lambda h^2 \sin\mu\,(N+1-j) \end{aligned}$$

or

$$\lambda h^2 = 4\sin^2\frac{\mu h}{2}. \qquad (47)$$

Hence λ is an eigenvalue in case i.). We now consider case ii.) which gives

$$2(1-\cos\mu h) = 3(1-\cos\mu h - \cot\mu\sinh\mu) \qquad (48)$$

and hence

$$\tan\frac{\mu h}{2} = 3\cot\mu. \qquad (49)$$

Observe, that we know from the continuous model that $\left(k+\frac{1}{2}\right)\pi = (2k+1)\frac{\pi}{2}$ is the new string of eigenvalues. But, because $\cot\left((2k+1)\frac{\pi}{2}\right) = 0$, we expect $\mu = \frac{k\pi}{2} - \epsilon_h$, ϵ_h being a small number depending on h. We use a perturbation argument first order in ϵ and obtain

i) $\lambda_k \doteq \frac{4}{h^2}\sin^2\left(\frac{k\pi}{4(N+1)} - \frac{k\pi}{24(N+1)^2}\right)$,

$\varphi^m_{kj} = A_m \sin\mu_m h\,(N+1-j) = A_m\sin\mu_m\left(1-\frac{j}{N+1}\right)$

ii) $\mu = m\pi$, then

$$\begin{aligned} \lambda_m &= \frac{2}{h^2}(1-\cos m\pi h) = \frac{4}{h^2}\sin\frac{m\pi}{2(N+1)}, & (50) \\ \varphi^m_{kj} &= A_m \sin\frac{m\pi}{N+1}(N+1-j) = -A_m\left((-1)^m\sin\frac{m\pi j}{N+1}\right) \end{aligned}$$

$$\sum_{k=1}^{3} A_k = 0,\ A_1 = A_2 = C,\ A_3 = -2C,\ \text{or}\, A_1 = -2C, A_2 = A_3 = C.$$

Of course, the constants are then used to properly normalize the eigen-

4. Useful lemmas

We now present the essential tools to prove our main results, that is the *blow-up* result and *the modified boundary control estimate* which concerns a class of filtered solutions being defined in analogy with Infante and Zuazua (1998), Zuazua (1999), as

$$\mathcal{C}_h(\gamma) := \left\{ u = \sum_{\lambda_k(h) \leq \gamma h^{-2}} \left[a_k \sin\left(\sqrt{\lambda_k(h)}t\right) + b_k \cos\left(\sqrt{\lambda_k(h)}t\right)\right] \varphi^k \right. \\ \left. \text{with } a_k, b_k \in \mathbf{R} \right\}. \tag{51}$$

This definition is justified by the spectral analysis of Section 3.1. and the results in Infante and Zuazua (1998). Next, a series of useful lemmas is presented which is important for both the blow-up and the control estimate. Note that these lemmas are very similar to the ones obtained in Infante and Zuazua (1998). The main difference is the treatment of the inner multiple node at $x = 0$. Nevertheless, the treatment of the multiple node depends on the discretization at that point, and even though similar results can easily be anticipated, actually proving them is a different matter. In addition, we believe that these lemmas will be useful for further numerical analysis all by themselves. Therefore, we give the proofs in the appendix.

LEMMA 4.1 (CONTROL ESTIMATE FOR THE EIGENVECTORS)
For any eigenvector $\varphi = (\varphi_1, \cdots, \varphi_N)$ of system (40) the following identity holds:

$$h \sum_{k=1}^{3} \sum_{j=0}^{N} \left| \frac{\varphi_{k,j+1} - \varphi_{k,j}}{h} \right|^2 \leq \frac{2}{4 - \lambda h^2} \sum_{k=1}^{3} \left| \frac{\varphi_{k,N}}{h} \right|^2. \tag{52}$$

Proof: See the Appendix

LEMMA 4.2 (IDENTITIES FOR THE EIGENVALUES)
For any eigenvector φ with eigenvalue λ of (40) the following identity holds:

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \left| \frac{\varphi_{k,j} - \varphi_{k,j+1}}{h} \right|^2 = \lambda \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{kj} \mid \varphi_{k,j} \mid^2. \tag{53}$$

If φ^m and φ^ℓ are eigenvectors associated to eigenvalues $\lambda_m \neq \lambda_\ell$ it follows that

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \left(\varphi_{k,j}^m - \varphi_{k,j+1}^m\right)\left(\varphi_{k,j}^\ell - \varphi_{k,j+1}^\ell\right) = 0. \tag{54}$$

LEMMA 4.3 (CONSERVATION OF ENERGY)
For any $h > 0$ and y being the solution of (34) we have

$$E_h(t) = E_h(0), \quad \forall t \in [0, T]. \tag{55}$$

Proof: See the Appendix

LEMMA 4.4 (DISCRETE MULTIPLIER IDENTITY)
For any $h > 0$ and y being the solution of (34) we have

$$\begin{aligned} \frac{h}{2}\int_0^T \left[\sum_{j=0}^{N} \dot{y}_{k,j}\dot{y}_{k,j+1} + \sum_{j=0}^{N}\left|\frac{y_{k,j+1}-y_{k,j}}{h}\right|^2\right] dt \\ + X_{kh}(t)\Big|_0^T = \frac{L}{2}\int_0^T \left|\frac{y_{k,N}(t)}{h}\right|^2 dt, \end{aligned} \tag{56}$$

with

$$X_{kh}(t) = h\sum_{j=0}^{N} j\left(\frac{y_{k,j+1}-y_{k,j-1}}{2}\right)\dot{y}_{k,j}. \tag{57}$$

Proof: See the Appendix

LEMMA 4.5 (EQUIPARTITION OF ENERGY)
For any $h > 0$ and y being the solution of (34) the following identity holds:

$$\begin{aligned} -h\sum_{k=1}^{3}\sum_{j=0}^{N}\int_0^T \rho_{kj} \mid \dot{y}_{k,j} \mid^2 dt + \\ h\sum_{k=1}^{3}\sum_{j=0}^{N}\int_0^T \left|\frac{y_{k,j}-y_{k,j+1}}{h}\right|^2 dt + Y_h(t)\Big|_0^T = 0, \end{aligned} \tag{58}$$

with

$$Y_h(t) = h\sum_{k=1}^{3}\sum_{j=1}^{N} \rho_{kj}\dot{y}_{kj}y_{kj}. \tag{59}$$

Proof: See the Appendix

We observe that Lemma 4.4 is not quite the straightforward semi-discretization of its continuous counterpart (8), in that mixed velocity terms rather than just squares appear. This fact makes it necessary to absorb more terms using conservation of energy, Lemma 4.5, in order to obtain the following important inequality:

$$T\left(1 - \frac{\Lambda h^2}{4}\right) E_h(0) + Z_h(t)\Big|^T \leq \frac{1}{2}\sum^{3} \int^T \left|\frac{y_{k,N}(t)}{h}\right|^2 dt \tag{60}$$

with

$$Z_h(t) = X_h(t) - \frac{\Lambda h^2}{8} Y_h(t) \tag{61}$$
$$= h \sum_{k=1}^{3} \sum_{j=1}^{N} \dot{y}_{k,j} \left[j \frac{(y_{k,j+1} - y_{k,j-1})}{2} - \frac{\Lambda h^2}{8} y_{k,j} \right],$$

for every solution of (34) in which Λ is the largest eigenvalue entering its Fourier expansion. This is the crucial point: we observe (once again) that controllability and discretization do not commute!

The idea of deriving this inequality is to start from Lemma 4.4 and summing for $k = 1, 2, 3$, using the conservation of the energy Lemma 4.3, that is $E_h(t) = E_h(0)$ gives us

$$\frac{h}{2} \sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \dot{y}_{kj} \dot{y}_{k,j+1} - \rho_{kj} |\dot{y}_{kj}|^2 dt + T E_h(0) + \sum_{k=1}^{3} X_{hk}(t) \Big|_0^T = \sum_{k=1}^{3} P_{hk} \tag{62}$$

where $P_{hk} = \frac{1}{2} \int_0^T \left(\frac{y_{k,N}}{h} \right)^2 dt$.

A short algebraic calculation shows that

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \dot{y}_{k,j} \dot{y}_{k,j+1} - \rho_{kj} |\dot{y}_{kj}|^2 dt \geq - \sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \frac{1}{2} |\dot{y}_{k,j} - \dot{y}_{k,j+1}|^2 dt \tag{63}$$

which leads to

$$T E_h(0) - \frac{h}{4} \sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T |\dot{y}_{k,j} - \dot{y}_{k,j+1}|^2 dt + \sum_{k=1}^{3} X_{hk}(t) \Big|_0^T \leq \sum_{k=1}^{3} P_{hk}. \tag{64}$$

Now use will be made of the fact that our solutions are filtered, that is – we have cut the high frequencies of the Fourier spectrum.

Let Λ be the largest eigenvalue in the Fourier development of y. Then

$$y = \sum_{|\mu_m| \leq \sqrt{\Lambda}} a_m e^{i \mu_m t} \varphi^m \tag{65}$$

with $\mu_m = \sqrt{\lambda_m}$ for $m > 0$ and $\mu_{-m} = -\mu_m$. Therefore

$$\dot{y} = i \sum_{|\mu_m| \leq \sqrt{\Lambda}} a_m \mu_m e^{i \mu_m t} \varphi^m. \tag{66}$$

Thus

$$\sum^{3} \sum^{N} |\dot{y}_{kj} - \dot{y}_{k,j+1}|^2 = \sum^{3} \sum^{N} \left| \sum a_m \mu_m e^{i \mu_m t} \left(\varphi^m_{k,j} - \varphi^m_{k,j+1} \right) \right|^2$$

$$= \sum_{k=1}^{3} \sum_{j=0}^{N} \sum_{|\mu_m| \le \sqrt{\Lambda}} \mu_m^2 \mid a_m \mid^2 \left| \varphi_{k,j}^m - \varphi_{k,j+1}^m \right|^2 \tag{67}$$
$$+ \sum_{k=1}^{3} \sum_{j=0}^{N} \sum_{\substack{|\mu_k| \le \sqrt{\Lambda} \\ |\mu_\ell| \le \sqrt{\Lambda} \\ \mu_m \neq \mu_\ell}} \mu_m \mu_\ell a_m \bar{a}_\ell e^{i(\mu_m - \mu_\ell)t} \left(\varphi_{k,j}^m - \varphi_{k,j+1}^m \right) \left(\varphi_{k,j}^\ell - \varphi_{k,j+1}^\ell \right).$$

Using the identities (53) and (54) of Lemma 4.2 the term in (67) can be rewritten as

$$\begin{aligned} \sum_{k=1}^{3} \sum_{j=0}^{N} \mid \dot{y}_{k,j} - \dot{y}_{k,j+1} \mid^2 &\le \Lambda \sum_{|\mu_m| \le \sqrt{\Lambda}} \mid a_m \mid^2 \lambda_m h^2 \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{kj} \left| \varphi_{k,j}^m \right|^2 \\ &= \Lambda h^2 \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{kj} \left| \dot{y}_{k,j} \right|^2. \end{aligned}$$

Therefore

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \left[\dot{y}_{k,j} \dot{y}_{k,j+1} - \rho_{kj} \left| \dot{y}_{k,j} \right|^2 \right] dt \ge -\frac{\Lambda h^2}{2} \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{kj} \int_0^T \left| \dot{y}_{k,j} \right|^2 dt. \tag{68}$$

So, we are left with

$$T E_h(0) - \frac{\Lambda h^2}{4} \sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \rho_{kj} |\dot{y}_{k,j}|^2 dt + \sum_{k=1}^{3} X_{hk}(t) \Big|_0^T \le \sum_{k=1}^{3} P_{hk}. \tag{69}$$

Applying the equipartition of energy as stated in Lemma 4.5 we can easily derive that

$$h \sum_{k=1}^{3} \sum_{j=0}^{N} \int_0^T \rho_{kj} \left| \dot{y}_{k,j} \right|^2 dt = \int_0^T E_h(t) dt + \frac{1}{2} Y_h(t) \Big|_0^T = T E_h(0) + \frac{1}{2} Y_h(t) \Big|_0^T. \tag{70}$$

Combining (68) and (69) we deduce that

$$T \left(1 - \frac{\Lambda h^2}{4} \right) E_h(0) + Z_h(t) \Big|_0^T \le \frac{1}{2} \sum_{k=1}^{3} \int_0^T \left| \frac{y_{k,N}(t)}{h} \right|^2 dt \tag{71}$$

with

$$\begin{aligned} Z_h(t) &= X_h(t) - \frac{\Lambda h^2}{8} Y_h(t) \qquad (72) \\ &= h \sum_{k=1}^{3} \sum_{j=0}^{N} \dot{y}_{k,j} \left[j \frac{(y_{k,j+1} - y_{k,j-1})}{} - \frac{\Lambda h^2}{} y_{\dots} \right] \end{aligned}$$

for every solution of (24) in which Λ is the largest eigenvalue entering its Fourier expansion.

We finish our list of lemmas by the following crucial estimate on $Z_h(t)$:

LEMMA 4.6 (ESTIMATE ON $Z_h(t)$) *For any $h > 0, t \in [0,T]$ and y solution of (34) in which Λ is the upper bound on the eigenvalues entering its Fourier development, it follows that*

$$
\mid Z_h(t) \mid \leq \sqrt{1 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}} E_h(0). \tag{73}
$$

Proof: The proof is an adaptation of the corresponding one in Infante and Zuazua (1998).

5. Main results

In the first sub-section we are going to prove that without appropriate low-pass filters applied to the semi-discrete system (34) there is no observability estimate that is uniform in the stepsize h. In the second sub-section we proceed to show that an h-adaptive low-pass filtering, which is equivalent to a truncated singular value decomposition of the controllability Grammian, restores uniformity on the class of filtered data.

5.1. The blow up result

THEOREM 5.1 (BLOW UP RESULT FOR UNFILTERED SOLUTIONS) *For any $T > 0$, we have*

$$
\sup_{y \text{ solution of } (24)} \left[\frac{E_h(0)}{\int_0^T \sum_{k=1}^{3} \mid \frac{y_{k,N}(t)}{h} \mid^2 dt} \right] \to \infty \text{ as } h \to 0. \tag{74}
$$

Proof: Using Fourier expansion $y = e^{i\sqrt{\lambda_N(h)}t}\varphi^N$ of the solution and Lemma 4.1 an essential inequality of the form

$$
\frac{E_h(0)}{\sum_{k=1}^{3} \int_0^T \left| \frac{y_{k,N}(t)}{h} \right|^2 dt} \leq \frac{2}{T\,(4 - \lambda_N(h)h^2)} \tag{75}
$$

is derived. Thus, as

$$
\lim_{h \to 0} \lambda_N(h)h^2 \to 4 \tag{76}
$$

We start with the derivation of (75): We insert $y = e^{i\sqrt{\lambda_N(h)}t}\varphi^N$ in the expression $\frac{E_h(0)}{\sum_{k=1}^{3}\int_0^T\left|\frac{y_{k,N}(t)}{h}\right|^2 dt}$ which gives

$$\frac{E_h(0)}{\sum_{k=1}^{3}\int_0^T\left|\frac{y_{kN}(t)}{h}\right|^2 dt} = \frac{\frac{h}{2}\sum_{k=1}^{3}\sum_{k=1}^{3}\lambda_N|\varphi_{k,j}^N|^2 + \left|\varphi_{k,j+1}^N - \varphi_{k,j}^N\right|^2}{T\sum_{k=1}^{3}\left|\frac{\varphi_{k,N}^N}{h}\right|^2}. \tag{77}$$

Here we used conservation of energy and the fact that $\int\limits_0^T\left|\frac{\varphi_{k,N}^N}{h}\right|^2 dt = T\left|\frac{\varphi_{k,N}^N}{h}\right|^2$. Now we use Lemma 4.1 for the denominator and Lemma 4.2 for the numerator which leads to

$$\frac{E_h(0)}{\sum_{k=1}^{3}\int_0^T|y_{k,N}(t)/h|^2\,dt} \leq \frac{\frac{h}{2}2\sum_{k=1}^{3}\sum_{k=1}^{3}\left|\varphi_{k,j+1}^N - \varphi_{k,j}^N\right|^2}{T\frac{h(4-\lambda h^2)}{2}\sum_{k=1}^{3}\sum_{k=1}^{3}\left|\varphi_{k,j+1}^N - \varphi_{k,j}^N\right|^2} \tag{78}$$

which is the desired inequality (75) and therefore the proof is complete. ∎

5.2. The control estimate

THEOREM 5.2 (CONTROL ESTIMATE FOR FILTERED SOLUTIONS) *Assume that $0 < \gamma < 4$. Then, there exists $T(\gamma) \geq 2$ such that for all $T > T(\gamma)$ there exists a constant $C = C(T,\gamma) > 0$ with*

$$E_h(0) \leq C(T,\gamma)\sum_{k=1}^{3}\int_0^T\left|\frac{y_{k,N}(t)}{h}\right|^2 dt, \tag{79}$$

for every solution of (34) in the class $\mathcal{C}_h(\gamma)$.

Proof: The estimate (71) and Lemma 4.6 allow us to derive

$$\left[T(1-\Lambda h^2/4) - 2\sqrt{1-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}\right]E_h(0) \leq \frac{1}{2}\sum_{k=1}^{3}\int_0^T\left|\frac{y_{k,N}(t)}{h}\right|^2 dt. \tag{80}$$

Now, taking into account that these solutions are filtered, which implies $\Lambda = \gamma/h^2$ in the class of solutions $\mathcal{C}_h(\gamma)$ of system (24) we deduce that

$$E_h(0) \leq \frac{1}{\quad\cdots\quad}\sum^{3}\int^{T}\left|\frac{y_{k,N}(t)}{\cdots}\right|^2 dt$$

Given that

$$T > 2\frac{\sqrt{1 - \frac{\gamma h^2}{16} + \frac{3\gamma}{16\lambda_1}}}{(1 - \frac{\gamma}{4})}$$

we have the desired result. ■

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6. Appendix: Proofs of the Lemmas

Proof of Lemma 4.1: We start with the equation $-\frac{\varphi_{k,j+1}-2\varphi_{k,j}+\varphi_{k,j-1}}{h^2} = \lambda\varphi_{k,j}$, multiply it by $\frac{\varphi_{k,j+1}-\varphi_{k,j-1}}{2}j$ and sum over $j = 1,..,N$. Observe that this factor is precisely a second order approximation to the continuous energy multiplier times the spatial derivative xu'. After some calculus, the left hand is given by

$$-\sum_{j=1}^{N} \frac{\varphi_{k,j+1} - 2\varphi_{k,j} - \varphi_{k,j-1}}{h^2} \frac{\varphi_{k,j+1} - \varphi_{k,j-1}}{2} j \tag{81}$$

$$-\sum_{j=1}^{N} \left[\varphi_{k0}^2 + 2\sum_{j=1}^{N} \varphi_{k,j}^2 - \frac{1}{2h}\varphi_{k,N}^2 - \sum_{j=0}^{N} \varphi_{k,j}\varphi_{k,j+1} \right].$$

The right hand side turns out to be equal to

$$\lambda \frac{1}{h^2} \sum_{j=1}^{N} \frac{\varphi_{k,j+1} - \varphi_{k,j}}{2} j\varphi_{k,j} = -\lambda \frac{1}{2} \sum_{j=0}^{N} \varphi_{k,j+1}\varphi_{k,j}. \tag{82}$$

equation

$$\left[\sum_{k=1}^{3}\sum_{j=0}^{N}\left|\frac{\varphi_{k,j}-\varphi_{k,j+1}}{h}\right|^{2}-\frac{1}{h^{3}}\sum_{k=1}^{3}\varphi_{k,N}^{2}\right] = -\lambda\sum_{k=1}^{3}\sum_{j=0}^{N}\varphi_{k,j}\varphi_{k,j+1} \tag{83}$$

On the other hand, after multiplication by $\varphi_{k,j}$ and summation over $k = 1, 2, 3$ we have

$$-\frac{1}{h^{2}}\sum_{k=1}^{3}\sum_{j=1}^{N}\left(\varphi_{k,j+1}-2\varphi_{k,j}+\varphi_{k,j-1}\right)\varphi_{k,j} = \lambda\sum_{k=1}^{3}\sum_{j=1}^{N}\varphi_{k,j}^{2}, \tag{84}$$

which is equivalent to

$$\frac{1}{h^{2}}\left\{2\sum_{k=1}^{3}\sum_{j=0}^{N}(\rho_{kj}\varphi_{k,j}^{2}-\varphi_{k,j}\varphi_{k,j+1})-2\varphi_{0}^{2}+\varphi_{0}\sum_{k=1}^{3}\varphi_{k,1}\right\} = \lambda\sum_{k=1}^{3}\sum_{j=1}^{N}\varphi_{k,j}^{2}, \tag{85}$$

where φ_0 is identified with the common value of $\varphi_{k,0}$. Now the eigenvalue problem at node 0 gives

$$-\frac{1}{h^{2}}\sum_{k=1}^{3}\left(\varphi_{k,1}-\varphi_{0}\right)\varphi_{0} = \lambda\varphi_{0}^{2}. \tag{86}$$

If we add (86) and (85) we obtain, again after some calculus,

$$\sum_{k=1}^{3}\sum_{j=0}^{N}\left|\frac{\varphi_{k,j}-\varphi_{k,j+1}}{h}\right|^{2} = \lambda\sum_{k=1}^{3}\sum_{j=0}^{N}\rho_{k,j}\varphi_{k,j}^{2} =: \frac{\lambda}{h}. \tag{87}$$

Notice that we have normalized the eigen-elements by

$$h\sum_{k=1}^{3}\sum_{j=0}^{N}\rho_{k,j}\varphi_{k,j}^{2} = 1.$$

$$\left[\sum_{k=1}^{3}\sum_{j=0}^{N}\left|\frac{\varphi_{k,j}-\varphi_{k,j+1}}{h}\right|^2-\frac{1}{h^3}\sum_{k=1}^{3}\varphi_{k,N}^2\right] \tag{88}$$
$$=\lambda\sum_{k=1}^{3}\sum_{j=0}^{N}\left(\rho_{kj}\varphi_{k,j}^2-\varphi_{k,j}\varphi_{k,j+1}\right)-\lambda\sum_{k=1}^{3}\sum_{j=0}^{N}\rho_{kj}\varphi_{k,j}^2$$
$$\lambda\left[\frac{1}{2}\sum_{k=1}^{3}\sum_{j=0}^{N}|\varphi_{k,j}-\varphi_{k,j+1}|^2-\frac{\varphi_0^2}{2}\right]-\frac{\lambda}{h}$$

Now we make use of (87) and obtain

$$\begin{aligned}2\sum_{k=1}^{3}\left|\frac{\varphi_{kN}}{h}\right|^2 &= \lambda\left[(4-\lambda h^2)+h\varphi_0^2\right]\\ &\geq \lambda\left(4-\lambda h^2\right).\end{aligned} \tag{89}$$

Then,

$$\lambda\leq\frac{2}{(4-\lambda h^2)}\sum_{k=1}^{3}\left|\frac{\varphi_{kN}}{h}\right|^2, \tag{90}$$

thus

$$h\sum_{k=1}^{3}\sum_{j=0}^{N}\left|\frac{\varphi_{k,j}-\varphi_{k,j+1}}{h}\right|^2\leq\frac{2}{(4-\lambda h^2)}\sum_{k=1}^{3}\left|\frac{\varphi_{kN}}{h}\right|^2. \tag{91}$$

■

Proof of Lemma 4.2: The first part has already been demonstrated above.

Let λ_m, λ_ℓ be two different eigenvalues and φ_{jk}^m, φ_{jk}^ℓ be the corresponding eigenvectors. Then we have

$$\lambda_\ell\sum_{j=1}^{N}\varphi_{k,j}^m\varphi_{k,j}^\ell = -\frac{1}{h^2}\left(\varphi_{k,j+1}^m+\varphi_{k,j-1}^m-2\varphi_{k,j}^m\right)\varphi_{k,j}^\ell, \tag{92}$$

$$\lambda_k\sum_{j=1}^{N}\varphi_{k,j}^\ell\varphi_{k,j}^m = -\frac{1}{h^2}\left(\varphi_{k,j+1}^\ell+\varphi_{k,j-1}^\ell-2\varphi_{k,j}^\ell\right)\varphi_{k,j}^m. \tag{93}$$

Now,

$$\sum_{j=1}^{N}\left(\varphi_{k,j+1}^\ell+\varphi_{k,j-1}^\ell-2\varphi_{k,j}^\ell\right)\varphi_{k,j}^m=\varphi_{k,0}^\ell\varphi_{k,1}^m-\varphi_{k,1}^\ell\varphi_{k,0}^m \tag{94}$$
$$+\sum_{j=1}^{N}\left(\varphi_{k,j+1}^m+\varphi_{k,j-1}^m-2\varphi_{k,j}^m\right)\varphi_{k,j}^\ell,$$

and, therefore,

$$(\lambda_\ell - \lambda_m) \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{jk} \varphi_{k,j}^m \varphi_{k,j}^\ell = -\frac{1}{h^2} \left[\varphi_0^\ell \sum_{k=1}^{3} \varphi_{k,1}^m - \varphi_0^m \sum_{k=1}^{3} \varphi_{k,1}^\ell \right]$$
$$+ \frac{1}{h^2} (\lambda_\ell - \lambda_m) \varphi_0^m \varphi_0^\ell = 0, \tag{95}$$

from which we conclude

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{kj} \varphi_{k,j}^m \varphi_{k,j}^\ell = 0. \tag{96}$$

This lead us to

$$\sum_{k=1}^{3} \sum_{j=1}^{N} \left(\varphi_{k,j-1}^\ell + \varphi_{k,j+1}^\ell - 2\varphi_{k,j}^\ell \right) \varphi_{k,j}^m = \tag{97}$$
$$\sum_{k=1}^{3} \sum_{j=1}^{N} \left(\varphi_{k,j+1}^\ell + \varphi_{k,j-1}^\ell \right) \varphi_{k,j}^m + 2\varphi_0^m \varphi_0^\ell.$$

On the other hand

$$\lambda_m \sum_{k=1}^{3} \sum_{j=0}^{N} \rho_{jk} \varphi_{j,k}^m \varphi_{j,k}^\ell = \tag{98}$$
$$-\frac{1}{h^2} \left(-\varphi_0^m \varphi_0^\ell + \sum_{k=1}^{3} \varphi_{k,1}^m \varphi_{k,0}^\ell + \sum_{k=1}^{3} \sum_{j=1}^{N} \left(\varphi_{k,j+1}^m + \varphi_{k,j-1}^m \right) \varphi_{k,j}^\ell \right) = 0.$$

Therefore,

$$\sum_{k=1}^{3} \sum_{j=1}^{N} \varphi_{k,j+1}^m \varphi_{k,j}^\ell = - \sum_{k=1}^{3} \sum_{j=1}^{N} \varphi_{k,j+1}^\ell \varphi_{k,j}^m + \varphi_0^m \varphi_0^\ell, \tag{99}$$

and hence

$$\sum_{k=1}^{3} \sum_{j=0}^{N} \left(\varphi_{k,j}^m - \varphi_{k,j+1}^m \right) \left(\varphi_{k,j}^\ell - \varphi_{k,j+1}^\ell \right) \tag{100}$$
$$= \sum_{k=1}^{3} \sum_{j=0}^{N} \left(\varphi_{k,j}^m \varphi_{k,j}^\ell + \varphi_{k,j+1}^m \varphi_{k,j+1}^\ell - \varphi_{k,j+1}^m \varphi_{k,j}^\ell - \varphi_{k,j}^m \varphi_{k,j+1}^\ell \right)$$
$$= 2 \sum^{3} \sum^{N} \varphi_{k,j}^m \varphi_{k,j}^\ell + 3\varphi_0^\ell \varphi_0^m - \varphi_0^\ell \varphi_0^m = 0,$$

from which we deduce

$$\sum_{k=1}^{3}\sum_{j=0}^{N}\left(\varphi_{k,j}^{m}-\varphi_{k,j+1}^{m}\right)\left(\varphi_{k,j}^{\ell}-\varphi_{k,j+1}^{\ell}\right)=0. \tag{101}$$

■

Proof of Lemma 4.3:

PROOF SKETCH: The main idea is to take a time derivative of the energy, insert the equation and perform summation by parts.

Recall

$$E_h(t)=\frac{h}{2}\left\{\sum_{k=1}^{3}\sum_{j=1}^{N}\rho_{jk}\left|\dot{y}_{ik}\right|^2+\left|\frac{y_{k,j+1}-y_{k,j}}{h}\right|^2\right\}, \tag{102}$$

and take a derivative with respect to t

$$\dot{E}_h(t)=h\sum_{k=1}^{3}\sum_{j=0}^{N}\left\{\rho_{kj}\dot{y}_{k,j}\ddot{y}_{k,j}+\frac{y_{k,j+1}-y_{k,j}}{h}\frac{\dot{y}_{k,j+1}-\dot{y}_{k,j}}{h}\right\}. \tag{103}$$

Upon summation by parts, regrouping terms, and using the differential equation we obtain

$$\begin{aligned}\dot{E}_h(t) \;=\; & h\sum_{k=1}^{3}\left\{\frac{1}{h^2}\left(y_{k,1}\dot{y}_{k,1}-\dot{y}_{k,1}y_{k,0}\right)\right. \\ & +\sum_{j=0}^{N}\frac{1}{h^2}\left(y_{k,j+1}-2y_{kj}+y_{k,j+1}\right)\dot{y}_{kj} \\ & +\frac{1}{h^2}\left[-y_{k,1}\dot{y}_{k,1}+2\sum_{j=1}^{N}y_{kj}\dot{y}_{kj}+\dot{y}_{k,1}y_{k,0}\right. \\ & \left.\left.-\sum_{j=1}^{N}y_{k,j-1}\dot{y}_{kj}-\sum_{j=1}^{N}y_{k,j+1}\dot{y}_{kj}\right]\right\} \\ & =0.\end{aligned} \tag{104}$$

■

Proof of Lemma 4.4: We multiply equation (34) by $\frac{y_{k,j+1}-y_{k,j-1}}{2}j$, sum over $j=1$ to N and integrate between 0 and T which leads to

$$\sum_{j=1}^{N}\int_0^T\ddot{y}_{k,j}\,\frac{y_{k,j+1}-y_{k,j-1}}{2}j\,dt= \tag{105}$$

$$\frac{1}{h^2}\sum\int^T\left(y_{k,j+1}-2y_{k,j}+y_{k,j-1}\right)\frac{y_{k,j+1}-y_{k,j-1}}{2}j\,dt$$

Now we expand both sides separately: The left hand side is integrated by parts with respect to time

$$\sum_{j=1}^{N} \dot{y} \; \frac{y_{k,j+1} - y_{k,j-1}}{2} \frac{j}{2}\bigg|_0^T - \sum_{j=1}^{N} \int_0^T \dot{y} \; \frac{\dot{y}_{k,j+1} - \dot{y}_{k,j-1}}{2} \frac{j}{2} dt \tag{106}$$

$$= \frac{1}{2} \sum_{j=1}^{N} \int_0^T \dot{y}_{k,j} \; \dot{y}_{k,j+1} dt + \sum_{j=1}^{N} \dot{y} \; \frac{y_{k,j+1} - y_{k,j-1}}{2} \frac{j}{2}\bigg|_0^T .$$

The right hand side gives

$$\frac{1}{2h^2} \sum_{j=1}^{N} \int_0^T \left(y_{k,j+1}^2 - y_{k,j-1}^2\right) j dt - \frac{1}{h^2} \sum_{j=1}^{N} \int_0^T y_{k,j} \left(y_{k,j+1} - y_{k,j-1}\right) j dt. \tag{107}$$

Summation by parts results in

$$\frac{1}{2h^2} \int_0^T -y_{k,0}^2 + y_{k,N}^2 \left(N-1\right) dt - \frac{1}{h^2} \sum_{j=1}^{N-1} \int_0^T y_{k,j}^2 dt \tag{108}$$

$$+\frac{1}{h^2} \int_0^T y_{k,1} y_{k,0} dt + \frac{1}{h^2} \sum_{j=1}^{N-1} \int_0^T y_{k,j} y_{k,j+1} dt,$$

and hence

$$\frac{1}{2h^2} \int_0^T -y_{k,0}^2 - y_{k,1}^2 + 2 y_{k,1} y_{k,0} + y_{k,N}^2 (N+1) dt \tag{109}$$

$$-\frac{1}{2h^2} \sum_{j=1}^{N} \int_0^T \left|y_{k,j} - y_{k,j+1}\right|^2 dt.$$

To sum up we have

$$\frac{1}{2} \sum_{j=0}^{N} \int_0^T \dot{y}_{k,j} \dot{y}_{k,j+1} dt + \frac{1}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{y_{k,j} - y_{k,j+1}}{h^2} \right| dt \tag{110}$$

$$+\sum_{j=0}^{N} \dot{y}_{k,j} \; \frac{y_{k,j} - y_{k,j-1}}{2} \frac{j}{2}\bigg|_0^T = \frac{1}{2h} \int\limits_0^T \left|\frac{y_{k,N}}{h}\right| dt \tag{111}$$

■

Proof of Lemma 4.5: We multiply the semi-discrete equation (34) by $y_{k,j}$, equation (32) by z, sum over $j = 1$ to N, and integrate between 0 and T.

$$\sum_{j=1}^{N} \int_0^T \ddot{y}_{k,j} \, y_{k,j} dt = \sum_{j=1}^{N} \int_0^T \frac{1}{h^2} \left(y_{k,j+1} - 2y_{k,j} + y_{k,j-1}\right) y_{k,j} dt \tag{112}$$

$$\sum_{k=1}^{3}\int_0^T \ddot{z}\, zdt - \frac{1}{h^2}\sum_{k=1}^{3}\int_0^T (y_{k,1}-z)\, zdt = 0 \tag{113}$$

Now the left hand side of the first equation gives, using integration by parts in time:

$$\sum_{k=1}^{N}\int_0^T \ddot{y}_{k,j}\, y_{k,j}dt = \sum_{j=1}^{N} \dot{y}_{k,j}\, y_{k,j}\Big|_0^T - \sum_{j=1}^{N}\int_0^T \dot{y}_{k,j}^{2}\, dt, \tag{114}$$

while summation by parts gives for the right hand side

$$\sum_{j=1}^{N}\int_0^T \frac{1}{h^2}(y_{k,j+1}-2y_{jk}+y_{k,j-1})\, y_{k,j}dt \tag{115}$$
$$= (-y_{k,0}(y_{k,1}-y_{k,0}) - \sum_{j=0}^{N}|y_{k,j+1}-y_{k,j}|^2)\frac{1}{h^2}.$$

Combining now equation (114) and (115) leads to

$$-h\sum_{j=1}^{N}\int_0^T \dot{y}_{k,j}^{2}\, dt + h\sum_{j=0}^{N}\int_0^T \left|\frac{y_{k,j+1}-y_{k,j}}{h}\right|^2 dt + \sum_{j=1}^{N}\dot{y}_{k,j}\, y_{k,j}\Big|_0^T$$
$$-h\int_0^T \left(\frac{z}{h}\right)^2 dt + h\int_0^T \frac{zy_{k,1}}{h^2}dt = 0. \tag{116}$$

The next step is to use equation (113) and to integrate it also in time which gives:

$$\int_0^T \ddot{z}\, zdt = \dot{z}\, z\Big|_0^T - \int\limits_0^T \dot{z}^2\, dt = \frac{1}{h^2}\sum_{k=1}^{3}\int_0^T (y_{k,1}-z)\, zdt. \tag{117}$$

Now multiplying the last equation by h adding it to equation (116) and summing up the resulting equation for $k = 1, 2, 3$ gives

$$-h\sum_{k=1}^{3}\sum_{j=1}^{N}\int_0^T \dot{y}_{k,j}^{2}\, dt - h\int\limits_0^T \dot{z}^2\, dt \tag{118}$$
$$+h\sum_{k=1}^{3}\sum_{j=1}^{N}\int_0^T \left|\frac{y_{k,j+1}-y_{k,j}}{h}\right|^2 dt + \sum_{k=1}^{3}\sum_{j=1}^{N}\dot{y}_{k,j}\, y_{k,j}\Big|_0^T + h\,\dot{z}\, z\Big|_0^T = 0$$

from which the result follows. ■