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Controllability of semilinear wave equations with infinitely iterated logarithms

by

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Abstract: In a previous work we improved some earlier results of Imanuvilov, Li and Zhang, and of Zuazua, on the boundary exact controllability of semilinear wave equations by weakening the growth assumptions on the nonlinearity. Answering a question of Zuazua we give a still weaker, essentially optimal condition. Furthermore, we establish an approximate internal controllability result under the same growth assumptions.

Keywords: wave equation, semilinear equation, controllability.

1. Introduction and formulation of the main results

Fix a bounded open interval (a, b) and a positive number T. Given a function $f : \mathbf{R} \to \mathbf{R}$ of class C^1, consider the problem

$$\begin{cases} v_{tt} - v_{xx} - f(v) = 0 \quad \text{in} \quad (a,b) \times (0,T), \\ v(a,t) = h_a(t) \quad \text{and} \quad v(b,t) = h_b(t) \quad \text{for} \quad t \in (0,T). \end{cases} \tag{1}$$

We will obtain a boundary exact controllability result under suitable, rather weak growth assumptions on the nonlinearity f. In order to state our result, let us introduce the *iterated logarithm* functions $\log_j$ defined by the formulas

$$\log_0 s := s \quad \text{and} \quad \log_j s := \log(\log_{j-1} s), \quad j = 1, 2, \ldots,$$

and define the numbers e_j by the equations $\log_j e_j = 1$ for $j = 0, 1, \ldots$:

$$e_0 = 1, \quad e_1 = e, \quad e_2 = e^e, \quad e_3 = e^{e^e}, \ldots$$

We prove in the next section that the formula

$$L(x) := \prod_{k=0}^{\infty} \log_k(e_k + |x|) = (1 + |x|)\log(e + |x|)\log_2(e^e + |x|)\ldots \tag{2}$$

defines an everywhere finite, even function with $L(0) = 1$. Furthermore, $L(x)$ is increasing for $x \geq 0$, and $L(x) \to +\infty$ relatively slowly as $x \to +\infty$, so that

$$\int_0^\infty \frac{dx}{L(x)} = +\infty.$$

Let us also introduce the primitive F of f defined by

$$F(x) = \int_0^x f(s)\, ds, \quad x \in \mathbf{R}.$$

We have the

THEOREM 1.1 *Assume that there exists a positive number β such that*

$$|F(x)| \leq \beta L(x)^2 \quad \textit{for all} \quad x. \tag{3}$$

If $T > b - a$, then for any given

$$(u_0, u_1),\ (v_0, v_1) \in H^1(a, b) \times L^2(a, b)$$

there exist control functions

$$h_a,\ h_b \in H^1(0, T)$$

such that (1) has a global solution

$$v \in C([0, T]; H^1(a, b)) \cap C^1([0, T]; L^2(a, b))$$

satisfying the final conditions

$$v(T) = v_0 \quad \textit{and} \quad v'(T) = v_1 \quad \textit{in} \quad (a, b). \tag{4}$$

REMARKS 1

- *This theorem improves an earlier one obtained in Cannarsa, Komornik, and Loreti (1999). Instead of (3) we made there the stronger assumption*
$$|F(x)| \leq \beta L_n(x)^2 \quad \text{for all} \quad x, \tag{5}$$
for some positive integer n, where $L_n(s)$ is defined by the formula
$$L_n(x) := \prod_{k=0}^{n} \log_k(e_k + |x|) = (1 + |x|) \log(e + |x|) \ldots \log_n(e_n + |x|).$$
(We used a slightly different but equivalent condition.) E. Zuazua asked whether in (5) the term $L_n(x)$ could be replaced by some convergent series $\sum_{n=0}^{\infty} c_n L_n(x)$. Our theorem answers this question in particular.
- *Our results in Cannarsa, Komornik and Loreti (1999) also show that the assumption (3) of the above theorem is essentially optimal.*

Next we study the *internal* controllability of the problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = h \quad \text{in} \quad (a,b) \times (0,T), \\ u(a,t) = u(b,t) = 0 \quad \text{for} \quad t \in (0,T), \\ u(0) = u_0 \ \text{ and } \ u'(0) = u_1 \quad \text{in} \quad (a,b). \end{cases} \tag{6}$$

Set

$$\ell(x) := L(x)/(1+|x|) = \prod_{k=1}^{\infty} \log_k(e_k+|x|) = \log(e+|x|) \log_2(e_2+|x|) \ldots \tag{7}$$

for brevity. Applying Theorem 1.1 we shall prove the following *approximate* controllability result:

THEOREM 1.2 *Assume (3) again and let $T > b - a$. Furthermore, assume that there exists another positive number β' such that*

$$|f'(x)| \leq \beta' \ell(x)^2 \quad \text{for all} \quad x. \tag{8}$$

Let $0 < \delta < (b-a)/2$ and let

$$(u_0, u_1),\ (v_0, v_1) \in H_0^1(a,b) \times L^2(a,b)$$

be fixed. Then, for any $\varepsilon > 0$ there exists a control function

$$h \in L^\infty(0,T; L^2(a,b))$$

with

$$h(x,t) = 0 \quad \text{for any} \quad a + \delta < x < b - \delta,$$

such that (6) has a global solution

$$u \in C([0,T]; H_0^1(a,b)) \cap C^1([0,T]; L^2(a,b))$$

satisfying

Under an additional assumption concerning the support of the initial and final data, we also have an *exact* controllability result:

THEOREM 1.3 *Assume (3) again and let $T > b - a$. Let $0 < \delta < (b-a)/2$ and let*

$$(u_0, u_1),\ (v_0, v_1) \in H^1(a,b) \times L^2(a,b)$$

be fixed so that all four functions vanish outside the interval $(a+\delta, b-\delta)$. Then there exists a control function

$$h \in L^\infty(0,T; L^2(a,b)) \tag{10}$$

with

$$h(x,t) = 0 \quad \textit{whenever} \quad a+\delta < x < b-\delta, \tag{11}$$

such that (6) has a global solution

$$u \in C([0,T]; H_0^1(a,b)) \cap C^1([0,T]; L^2(a,b))$$

satisfying the final conditions

$$u(T) = v_0 \quad \textit{and} \quad u'(T) = v_1 \quad \textit{in} \quad (a,b). \tag{12}$$

The authors are grateful to E. Zuazua for his question leading to Theorem 1.1 above.

2. Infinitely iterated logarithms

Let us observe that (e_j) is a strictly increasing sequence of positive numbers, rapidly tending to infinity. Note that

$$e_0 = 1 \quad \text{and} \quad \log_j e_l = e_{l-j} \text{ for all } l \geq j \geq 0. \tag{13}$$

The purpose of this section is to establish some properties of the function $L(x)$, defined in the introduction, which we will be using in the sequel:

PROPOSITION 2.1 *The formula (2) defines an even, everywhere finite function $L(x)$ which is increasing for $x \geq 0$. We have $L(x) \geq L(0) = 1$ for all x and*

$$\int_0^\infty \frac{dx}{L(x)} = +\infty. \tag{14}$$

Finally, for every $\alpha > 0$ and $\delta > 0$ there exists a constant $c(\alpha, \delta) > 0$ such that

$$L(x)^2 \leq \delta |x|^{2+2\alpha} + c(\alpha, \delta) \quad \textit{for all} \quad x. \tag{15}$$

LEMMA 2.2 *Let* $0 \le x \le e^2 - e \approx 4.67$. *Then*

$$\log(e_l + x) \le 2e_{l-1} \tag{16}$$

for any integer $l \ge 1$. *Moreover, for any integer* $k \ge 2$.

$$\log_k(e_l + x) \le \Big(1 + \prod_{j=2}^{k} e_{l-j}^{-1}\Big) e_{l-k} \tag{17}$$

for all integers $l \ge k$.

Proof: Note that (16) formally coincides with (17) for $k = 1$. Hence we must prove (17) for all integers $1 \le k \le l$. Fix a positive integer l arbitrarily. We prove (17) by induction over k for $k = 1, \dots, l$.

The proof for $k = 1$ is straightforward:

$$\log(e_l + x) \le \log e_l^2 = 2e_{l-1}.$$

Now assume that (17) holds true for some $1 \le k < l$. Then, using also the inequality $\log(1 + y) \le y$, we have

$$\begin{aligned}
&\log_{k+1}(e_l + x) \le \log\Big[\Big(1 + \prod_{j=2}^{k} e_{l-j}^{-1}\Big) e_{l-k}\Big] \\
&= \log\Big(1 + \prod_{j=2}^{k} e_{l-j}^{-1}\Big) + e_{l-k-1} \le e_{l-k-1} + \prod_{j=2}^{k} e_{l-j}^{-1} \\
&= \Big(1 + \prod_{j=2}^{k+1} e_{l-j}^{-1}\Big) e_{l-k-1}.
\end{aligned}$$

■

LEMMA 2.3 *If* $x \ge 0$, *then*

$$\log(e_l + x^2) \le 2\log(e_l + x) \tag{18}$$

for $l = 1, 2, \dots$. *Moreover, for any integer* $k \ge 2$,

$$\log_k(e_l + x^2) \le \Big(1 + \prod_{j=2}^{k} e_{l-j}^{-1}\Big) \log_k(e_l + x) \tag{19}$$

for all integers $l \ge k$.

Proof: Similarly as above, (18) formally coincides with (19) for $k = 1$. Hence we must prove (19) for all integers $1 \le k \le l$. Fix a positive integer l arbitrarily.

The proof for $k=1$ is easy:

$$\log(e_l+x^2)\le\log(e_l+x)^2=2\log(e_l+x).$$

Now assume (19) for some $1\le k<l$. Then we have

$$\begin{aligned}\log_{k+1}(e_l+x^2)&\le\log\Big[\Big(1+\prod_{j=2}^{k}e_{l-j}^{-1}\Big)\log_k(e_l+x)\Big]\\&=\log\Big(1+\prod_{j=2}^{k}e_{l-j}^{-1}\Big)+\log_{k+1}(e_l+x)\le\log_{k+1}(e_l+x)+\prod_{j=2}^{k}e_{l-j}^{-1}\\&\le\Big(1+\prod_{j=2}^{k+1}e_{l-j}^{-1}\Big)\log_{k+1}(e_l+x).\end{aligned}$$

■

LEMMA 2.4 *The infinite product*

$$\ell(x)=\prod_{k=1}^{\infty}\log_k(e_k+|x|)=\log(e+|x|)\log_2(e^e+|x|)\dots \tag{20}$$

converges for every real number x. Furthermore, the function ℓ is even, strictly increasing for $x\ge0$, and it has the following additional properties:

$$\ell(x)\ge\ell(0)=1\quad\textit{for all}\quad x, \tag{21}$$

$$\ell(x^2)\le C\ell(x)\quad\textit{for all}\quad x \tag{22}$$

where the constant C is defined by the convergent infinite product

$$C:=\prod_{k=1}^{\infty}\Big(1+\prod_{j=2}^{k}e_{k-j}^{-1}\Big)=2\prod_{k=2}^{\infty}\Big(1+\prod_{j=2}^{k}e_{k-j}^{-1}\Big), \tag{23}$$

and

$$\frac{\ell(x)}{x^\alpha}\to0\quad\textit{as}\quad x\to+\infty \tag{24}$$

for every $\alpha>0$.

Proof: Since $e_{k-2}\to+\infty$, the series

$$\sum_{k=2}^{\infty}\prod_{j=2}^{k}e_{k-j}^{-1}$$

converges by the ratio test, and therefore the infinite products in (23) converge, Applying the inequalities (16) and (17) of Lemma 2.2 with $l = 1$ and $l = k$, respectively, we conclude that $1 \leq \ell(x) \leq C$ if $|x| \leq e^2 - e$.

Also, by application of (18) and (19) with $l = 1$ and $l = k$, respectively, (22) follows for all x. Next we use this inequality to show that $\ell(x)$ is finite for every x. We already know this for $|x| \leq e^2 - e =: a$. Given an arbitrary x, choose a positive integer n such that $|x| \leq a^{2^n}$. This is possible because $a > 1$. Applying (22) n times we obtain that

$$\ell(x) \leq \ell(a^{2^n}) \leq C^n \ell(a) < +\infty.$$

Finally, we prove (24). Since

$$\frac{\ell(x^2)}{x^{2\alpha}} \leq \frac{C\ell(x)}{x^{2\alpha}} = \frac{C}{x^\alpha}\frac{\ell(x)}{x^\alpha},$$

we have, writing $a_n := a^{2^n}$ for brevity,

$$\sup_{a_{n+1} \leq x \leq a_{n+2}} \frac{\ell(x)}{x^\alpha} \leq \frac{C}{a_{n+1}^\alpha} \sup_{a_n \leq x \leq a_{n+1}} \frac{\ell(x)}{x^\alpha}$$

for every n. Choosing a sufficiently large positive integer m such that $a_{m+1}^\alpha \geq 2C$, it follows that

$$\sup_{a_n \leq x \leq a_{n+1}} \frac{\ell(x)}{x^\alpha} \leq 2^{m-n} \sup_{a_m \leq x \leq a_{m+1}} \frac{\ell(x)}{x^\alpha}$$

for every $n \geq m$. Hence (24) follows. ∎

Now we are ready to prove Proposition 2.1.

Proof: [Proposition 2.1] Since $L(x) = (1 + |x|)\ell(x)$, all properties but (14) and (15) follow easily from the preceding lemma.

For the proof of (15) observe that $L(x)/x^{2+\alpha}$ tends to zero as $x \to +\infty$ because

$$0 \leq \frac{L(x)}{x^{1+\alpha}} \leq 2\frac{\ell(x)}{x^\alpha}$$

for all $x \geq 1$, and the last expression tends to zero by (24). Now (15) easily follows by applying the Young inequality.

Turning to the proof of (14), assume, on the contrary, that the integral converges. Then

$$\int_{e_n}^{\infty} \frac{dx}{L(x)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{25}$$

By performing the change of variable $x = e^t$ we obtain the equalities

$$\int^{\infty} \frac{dx}{} = \int^{\infty} \frac{dx}{} = \int^{\infty} \frac{e^t\ dt}{} = \int^{\infty} \frac{e^t}{} \frac{dt}{}$$

Observe that we have

$$\frac{e^t}{1+e^t} \geq \frac{e^{e_n}}{1+e^{e_n}} = \frac{e_{n+1}}{1+e_{n+1}} \geq 1 - e_{n+1}^{-1}$$

and

$$\log_k(e_k + e^t) \leq \log_k(e_k e^t) = \log_{k-1}(e_{k-1} + t)$$

for all $k \geq 1$ and $t \geq e_n$. Therefore we deduce from the above equalities the following inequalities:

$$\int_{e_{n+1}}^{\infty} \frac{dx}{L(x)} \geq (1 - e_{n+1}^{-1}) \int_{e_n}^{\infty} \frac{dt}{L(t)}.$$

It follows by induction that

$$\int_{e_n}^{\infty} \frac{dx}{L(x)} \geq \Big(\prod_{j=2}^{n} (1 - e_j^{-1}) \Big) \int_{e_1}^{\infty} \frac{dx}{L(x)}$$

for $n = 2, 3, \ldots$.

Since the series $\sum e_j^{-1}$ clearly converges (because $e_j \to +\infty$ very quickly) and since every e_j is greater than 1, we have

$$A := \prod_{j=2}^{\infty} (1 - e_j^{-1}) > 0$$

and therefore

$$\int_{e_n}^{\infty} \frac{dx}{L(x)} \geq A \int_{e}^{\infty} \frac{dx}{L(x)} > 0$$

for all n. This contradicts (25). ■

3. Proof of Theorem 1.1

In our previous paper, the proof of the above mentioned weaker result was based on two important properties of the functions L_k. One of them was the divergence of the integral of $1/L_k$; we have already shown that the same property also holds for the function L. The other property was the estimate (26) below for the functions L_k instead of L. Thus, after having proved the following lemma, Theorem 1.1 can be proved by repeating the arguments given in Cannarsa, Komornik and Loreti (1999). So, we only need to prove the

LEMMA 3.1 *Let Ω be a bounded open domain in $\mathbf{R}^{\mathbf{N}}$. Given $\varepsilon > 0$ arbitrarily, there is a constant $c(\varepsilon)$ such that*

$$\|L(u)\| \leq \varepsilon \|\nabla u\| + c(\varepsilon) L(\|u\|) \tag{26}$$

Proof: Assume for simplicity that $N \geq 3$: the cases of $N = 1, 2$ are analogous and simpler. We recall that by the Sobolev imbedding theorem there exists a constant S such that

$$\|u\|_{2N/(N-2)}^2 \leq S\|\nabla u\|^2$$

for all $u \in H_0^1(\Omega)$.

Given $\delta > 0$ arbitrarily, by Proposition 2.1 there exists a constant $c(\delta) > 0$ such that

$$L(x)^2 \leq \delta |x|^{(2N+4)/N} + c(\delta)$$

for all real x. Since

$$\frac{N}{2N+4} = \alpha\frac{1}{2} + (1-\alpha)\frac{N-2}{N}$$

if $\alpha = 2/(N+2)$, applying the interpolational inequality we have, denoting by $|\Omega|$ the volume of Ω,

$$\|L(u)\|^2 \leq \delta\|u\|_{(2N+4)/N}^{(2N+4)/N} + c(\delta)|\Omega|$$
$$\leq \delta\big(\|u\|^{2/(N+2)}\|u\|_{N/(N-2)}^{N/(N+2)}\big)^{(2N+4)/N} + c(\delta)|\Omega|$$
$$= \delta\|u\|^{4/N}\|u\|_{N/(N-2)}^2 + c(\delta)|\Omega|$$
$$\leq \delta S\|u\|^{4/N}\|\nabla u\|^2 + c(\delta)|\Omega|$$

Since $L \geq 1$ everywhere in case of $\|u\| \leq 1$, hence we deduce the estimate

$$\|L(u)\|^2 \leq \delta S\|\nabla u\|^2 + c(\delta)|\Omega|L_k(\|u\|)^2, \tag{27}$$

and (26) follows by choosing $\delta = S^{-1}\varepsilon^2$.

Henceforth assume that $\|u\| > 1$. Let us note that

$$\ell(ab) \leq C\ell(a) + C\ell(b) \tag{28}$$

for all real numbers a and b. Indeed, assuming for example that $|a| \geq |b|$, using (22) we have

$$\ell(ab) \leq \ell(a^2) \leq C\ell(a) < C\ell(a) + C\ell(b).$$

Now given $u \in H_0^1(\Omega)$ such that $\|u\| > 1$, setting $v := u/\|u\|$ and applying (28) we have

$$\int_\Omega L(u)^2\,dx = \int_{|u|\leq\|u\|} L(u)^2\,dx + \int_{|u|>\|u\|} L(u)^2\,dx \leq |\Omega|L(\|u\|)^2 +$$
$$\int_{|u|>\|u\|} (1+|u|)^2\ell(u)^2\,dx \leq |\Omega|L(\|u\|)^2 +$$
$$2C^2\int_{|u|>\|u\|} (1+|u|)^2\ell(v)^2\,dx + 2C^2\int_{|u|>\|u\|} (1+|u|)^2\ell(\|u\|)^2\,dx$$

Since $\|u\| > 1$ implies that

$$1 + |u| \le \|u\|(1 + |v|),$$

we have

$$I_1 \le \|u\|^2 \int_{|u|>\|u\|} L(v)^2 \, dx \le \|u\|^2 \|L(v)\|^2.$$

Furthermore, since $|u| > \|u\| > 1$ implies that

$$1 + |u| \le 2|u|,$$

we have

$$I_2 \le 4L(\|u\|)^2.$$

Substituting them into (29) we find that

$$\|L(u)\|^2 \le (|\Omega| + 8C^2)\, L(\|u\|)^2 + 2C^2 \|u\|^2 \|L(v)\|^2.$$

Applying (27) for v and using the inequality $L(x) \ge |x|$ we obtain that

$$\begin{aligned} \|L(u)\|^2 &\le (|\Omega| + 8C^2) L(\|u\|)^2 + 2C^2 \|u\|^2 (\delta S \|\nabla v\|^2 + c(\delta)|\Omega| L(1)^2) \\ &\le 2C^2 \delta S \|\nabla u\|^2 + \{|\Omega| + 8C^2 + 2C^2 c(\delta)|\Omega| L(1)^2\} L(\|u\|)^2. \end{aligned}$$

For $\delta = \varepsilon^2/(2CS)$ the lemma follows. ■

4. Proof of Theorem 1.2

Without loss of generality we assume that v_0 and v_1 vanish outside the interval $(a + \delta', b - \delta')$ for some $0 < \delta' < \delta$.

Applying Theorem 1.1 we obtain control functions h_a and h_b and a solution

$$v \in C([0, T]; H_0^1(a, b)) \cap C^1([0, T]; L^2(a, b))$$

of (1) satisfying (4). Notice that, in particular, v is bounded.

Let us consider, for every $0 < \sigma < \delta'$, a cutoff function $\chi_\sigma \in C^2(a, b)$ satisfying

$$\begin{aligned} &\chi_\sigma(x) = 1 \quad \text{if} \quad a + \sigma < x < b - \sigma, \\ &\chi_\sigma(x) = 0 \quad \text{if} \quad a < x < a + 2^{-1}\sigma \text{ or } b - 2^{-1}\sigma < x < b. \end{aligned}$$

We may choose it so that

$$|\chi'_\sigma(x)| \le c\sigma^{-1} \quad \text{and} \quad |\chi''_\sigma(x)| \le c\sigma^{-2} \tag{30}$$

A simple computation shows that for every $0 < \sigma < \delta'$ the function $v_\sigma := \chi_\sigma v$ solves

$$\begin{cases} v_{tt} - v_{xx} - f(v) = h_\sigma & \text{in} \quad (a,b)\times(0,T), \\ v(a,t) = v(b,t) = 0 & \text{for} \quad t \in (0,T), \\ v(0) = \chi_\sigma u_0 \text{ and } v'(0) = \chi_\sigma u_1 & \text{in} \quad (a,b) \end{cases} \tag{31}$$

with

$$h_\sigma = \chi_\sigma f(v) - f(\chi_\sigma v) - 2\chi_\sigma' v_x - \chi_\sigma'' v.$$

Hence,

$$h_\sigma \in L^\infty(0,T; L^2(a,b))$$

and

$$h(x,t) = 0 \quad \text{for any} \quad a+\delta < x < b-\delta.$$

Moreover,

$$v_\sigma(T) = v_0 \quad \text{and} \quad v_\sigma'(T) = v_1 \quad \text{in} \quad (a,b) \tag{32}$$

and

$$\|v_\sigma\|_\infty \le \|v\|_\infty. \tag{33}$$

Now, by the same method used in the proof of Cannarsa, Komornik and Loreti (1999) [Theorem 1.1], we conclude that the problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = h_\sigma & \text{in} \quad (a,b)\times(0,T), \\ u(a,t) = u(b,t) = 0 & \text{for} \quad t \in (0,T), \\ u(0) = u_0 \text{ and } u'(0) = u_1 & \text{in} \quad (a,b) \end{cases}$$

has a unique solution

$$u_\sigma \in C([0,T];\ H_0^1(a,b)) \cap C^1([0,T];\ L^2(a,b)).$$

The proof will be complete if we show that

$$\|u_\sigma(T) - v_\sigma(T)\|_{H^1(a,b)} < \varepsilon \quad \text{and} \quad \|u_\sigma'(T) - v_\sigma'(T)\|_{L^2(a,b)} < \varepsilon$$

if σ is chosen to be sufficiently small. For this purpose, let us set

$$w^\sigma = u_\sigma - v_\sigma.$$

Then, w^σ satisfies

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = \varphi(t,x,w(x,t))w(x,t) & \text{in} \quad (a,b)\times(0,T), \\ w(a,t) = w(b,t) = 0 & \text{for} \quad t \in (0,T), \end{cases} \tag{34}$$

where φ is given by

$$\varphi(t,x,w) = \int_0^1 f'\big(v_\sigma(x,t) + \lambda w\big) d\lambda\,,$$

so that

$$\varphi(t,x,w^\sigma(x,t))w^\sigma(x,t) = f(u_\sigma(x,t)) - f(v_\sigma(x,t)).$$

Using assumption (8) and estimate (33), we have that

$$|\varphi(t,x,w^\sigma)| \le \beta' \ell(w^\sigma)^2\,. \tag{35}$$

Applying Cannarsa, Komornik and Loreti (1999) [Lemma 3.1] to problem (34) we deduce that the energy of w^σ is bounded. So, w^σ is also bounded, and we obtain

$$\varphi(t,x,w^\sigma) \le C$$

for some constant $C > 0$ independent of σ. The standard energy estimates may then be used to the prove continuous dependence on initial conditions of the solution to (34), that is

$$E'_\sigma(t) := \frac{1}{2}\frac{d}{dt}\int_a^b |w_t^\sigma|^2 + |w_x^\sigma|^2 \, dx \le \frac{C}{2}\int_a^b |w^\sigma|^2 + |w_t^\sigma|^2 dx \le C' E_\sigma(t)$$

where C' denotes another positive constant independent of σ. Therefore,

$$E_\sigma(t) \le E_\sigma(0) e^{C't}$$

and the proof will be complete if we show that $E_\sigma(0) \to 0$ as $\sigma \to 0$. To prove the last claim we note that

$$E_\sigma(0) = \frac{1}{2}\int_a^b |(1-\chi_\sigma)u_1|^2 + |(1-\chi_\sigma)u_0'|^2 \, dx + \frac{1}{2}\int_a^b |\chi_\sigma' u_0|^2 \, dx\,.$$

Since the limit, as $\sigma \to 0$, of the first two terms in the above right-hand side is 0, we only need to consider the right-most term

$$\int_a^b |\chi_\sigma' u_0|^2 \, dx = \int_a^{a+\sigma} |\chi_\sigma' u_0|^2 \, dx + \int_{b-\sigma}^b |\chi_\sigma' u_0|^2 \, dx\,.$$

It is sufficient to establish the inequalities

$$\int_a^{a+\sigma} |\chi_\sigma' u_0|^2 \, dx \le C \int_a^{a+\sigma} |u_0'|^2 \, dx \tag{36}$$

and

$$\int_{b-\sigma}^b |\chi_\sigma' u_0|^2 \, dx \le C \int_{b-\sigma}^b |u_0'|^2 \, dx \tag{37}$$

For this we use Poincaré's inequality as follows. Since

$$|u_0(x)|^2 = \left| \int_a^x u_0'(t)\, dt \right|^2 \le (x-a) \int_a^x |u_0'(t)|^2\, dt \le \sigma \int_a^{a+\sigma} |u_0'|^2\, dx$$

for all $a < x < a + \sigma$, (36) follows by using (30):

$$\int_a^{a+\sigma} |\chi_\sigma' u_0|^2\, dx \le c^2 \sigma^{-2} \int_a^{a+\sigma} |u_0|^2\, dx \le c^2 \int_a^{a+\sigma} |u_0'|^2\, dx.$$

The proof of (37) is similar. ■

5. Proof of Theorem 1.3

We argue as in the previous proof and consider a *cutoff function* $\chi \in C^2(a, b)$ satisfying

$$\chi(x) = 1 \quad \text{if} \quad a + \delta < x < b - \delta,$$
$$\chi(x) = 0 \quad \text{if} \quad a < x < a + 2^{-1}\delta \text{ or } b - 2^{-1}\delta < x < b.$$

Then, applying Theorem 1.1 we obtain control functions h_a and h_b and a solution v of (1) for which (4) holds true. A simple computation shows that $u := \chi v$ solves (6) with

$$h = \chi f(v) - f(\chi v) - 2\chi_x v_x - \chi_{xx} v$$

and that conditions (10), (11) and (12) are also satisfied.

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