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Characterization of the space of solutions of the membrane shell equation for arbitrary $C^{1,1}$ midsurfaces[1]

by

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Abstract: A comprehensive convergence theory of dynamical thin shell models for the purposes of control theory relies heavily on a thorough analysis of the static model and the complete specification of the spaces of solutions of the asymptotic solution for general midsurfaces ranging from the plate to arbitrary $C^{1,1}$ midsurfaces.

In this paper the existence of solution to the *membrane shell equation* is studied in a bounded open connected domain ω (Lipschitzian when ω has a boundary γ) in a $C^{1,1}$ midsurface for homogeneous Neumann boundary conditions or homogeneous Dirichlet boundary conditions on a part γ_0 of γ. It is proved that its tangential part is solution of the *reduced membrane shell equation* in $H^1(\omega)^N$ (resp. $H^1_{\gamma_0}(\omega)^N$) unique up to an element of a finite dimensional subspace, while its normal component belongs to a weighted $L^2(\omega)$ space by the pointwise norm of the second fundamental form. It is also shown that the reduced equation is equivalent to the equation for the projection onto the linear subspace of vector functions whose *linear change of metric tensor* is orthogonal to the second fundamental form of the midsurface.

Keywords: shells, membrane shell equation, asymptotic shell models.

1. Introduction

Thin shells are two-dimensional approximations of three-dimensional elastic bodies as their thickness becomes small. From the mathematical and practical

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points of view, such models make sense provided that the loading conditions are chosen in such a way that the three-dimensional model and the two-dimensional approximations both converge to some *asymptotic shell* model. A fairly large and rich body of literature addresses this issue for the static model and it is possible to identify which of the thin shell models have good asymptotic properties. They can be used in the optimal design of static shells as long as the asymptotic properties of the optimal thin shell are preserved through the optimization process.

In their standard version the associated control problems make use of dynamical models. They are obtained by including a kinetic energy term in the energy balance (potential energy and work of the applied loads of the static model). For the models to be completely coherent and accurate, it is necessary to show that as the thickness goes to zero, the dynamical three-dimensional model also converges to an asymptotic dynamical model. Good dynamical thin shell models should converge to an appropriate asymptotic dynamical model. A comprehensive convergence theory of dynamical thin shell models relies heavily on a thorough analysis of the static model and the complete specification of the spaces of solutions of the asymptotic solution for general midsurfaces ranging from the plate to arbitrary $C^{1,1}$ midsurfaces.

In this paper we focus our attention on the membrane shell equation which is one of the pieces of the *big puzzle*. In recent papers (Delfour, 1998, 1999a, Delfour and Zolésio, 1997b, 1999) it was established that the polynomial $P(2,1)$ model is both pertinent and basic in the theory of *thin shells*. It was shown in Delfour (1998) that its solution converges to the solution of a coupled system of variational equations. For the plate and the bending dominated shell it yields (as the thickness $2h$ goes to zero) the *membrane shell equation* and the *asymptotic bending equation*.

The first variational equation of this asymptotic coupled system coincides with the variational equation characterizing the *asymptotic* $P(0,1)$ *model*. It was shown in Delfour (1998) that this equation decomposes into two equations: a first equation containing the *Love-Kirchhoff* group of terms and a second equation which coincides with the classical *membrane shell equation*. The detailed correspondence with the covariant form of the membrane shell equation is given in Delfour and Zolésio (1997). The decomposition is achieved by variable elimination which results in the explicit introduction of an *effective compliance* C_{eP} associated with the initial three-dimensional compliance C. This two-dimensional effective compliance inherits the properties of continuity, symmetry and coercivity of the initial three-dimensional compliance.

In this paper the *membrane shell equation* is studied in a bounded open connected domain ω (Lipschitzian when ω has a boundary γ) in a $C^{1,1}$ submanifold of codimension one for homogeneous Neumann boundary conditions or homogeneous Dirichlet boundary conditions on a part γ_0 of γ when γ_0 has non-zero H_{N-2} Hausdorff measure. It is a companion paper to Delfour (1998) where

$P(0,1)$ model and the membrane shell equation are defined as completions of the appropriate quotient spaces. It gives a complete characterization of the space E^P without extra condition on the second fundamental form. Such a characterization is currently available for the *plate* and for *uniform elliptic shells* in Destuynder (1980), Ciarlet and Lods (1996c), Ciarlet and Sanchez-Palencia (1993, 1996). It also shows that we can always associate with the vector functions of the space E^P an equivalence class of tangential components which turns out to be solutions of the *reduced membrane shell equation*. This reduced equation is also connected with a projection onto a linear subspace of elements of E^P whose *linear change of metric tensor* is orthogonal to the second fundamental form. Another consequence of the characterization of E^P is the fact that in the asymptotic convergence of the solution of the $P(2,1)$ model we now know that the tangential component of the displacement of the midsurface strongly converges in $H^1(\omega)^N$ and the normal component in a weighed $L^2(\omega)$ space by the pointwise norm of the second fundamental form. The characterization of the spaces E^0 and E^P given in this paper, and the one of E^{01} given in Delfour (1999a) for the $P(2,1)$ model, sharpen and complete the abstract results of Delfour (1998). Some of the results have been announced without proofs in Delfour (1999b).

Finally for $N=3$, this paper extends to arbitrary second fundamental forms D^2b the available existence of solutions obtained in Ciarlet and Lods (1994a, 1996a) for $g^0=0$, homogeneous Dirichlet boundary conditions on the whole boundary, the special constitutive law $C^{-1}\varepsilon = 2\,\mu\,\varepsilon + \lambda\,\mathrm{tr}\,\varepsilon\, I$ and the *uniform ellipticity* of the two-dimensional C^2-midsurface ω. However in the case of uniform elliptic shells uniqueness does not so far seem to follow directly in an obvious way from the techniques used in the present paper. The first existence and uniqueness result seems to be due to Destuynder (1980) under relatively strong conditions. For a domain ω with a C^3 boundary γ in an analytic midsurface, the existence and uniqueness of solutions $(\hat{v}^0_\Gamma, \hat{v}^0_n)$ in $H^1_0(\omega)^3 \times L^2(\omega)$ was established in Ciarlet and Sanchez-Palencia (1993, 1996). The conditions were relaxed in Ciarlet and Lods (1994a, 1996a): the midsurface is of class C^2 and the boundary γ is Lipschitz for the existence (midsurface C^5 and the boundary γ of class C^4 for existence and uniqueness).

Notation, assumptions and background material

The inner product in $\mathbf{R}^N$ and the double inner product in $\mathcal{L}(\mathbf{R}^N;\mathbf{R}^N)$ (space of $N\times N$ matrices or tensors) are denoted as

$$x\cdot y = \sum_{i=1}^N x_i\, y_i, \quad A\cdot\cdot B = \sum_{i=1}^N\sum_{j=1}^N A_{ij}\, B_{ij}.$$

${}^*M$ denotes the transpose of of an arbitrary $k\times m$ matrix M.

A summary of some of the main definitions and results which are necessary

The submanifold Γ of codimension one is specified as the boundary of a subset Ω of $\mathbf{R}^N$. It is assumed that ω is a bounded (relatively) open subset of Γ and that Γ is of class $C^{1,1}$ in some neighborhood of ω. This is equivalent to saying that the algebraic distance function $b = b_\Omega$ of Ω is $C^{1,1}$ in that neighborhood. Its gradient ∇b coincides with the unit exterior normal n to Γ on ω and its Hessian matrix $D^2 b$ with the second fundamental form. $P \stackrel{\text{def}}{=} I - n\,{}^*n$ will denote the orthogonal projection onto the tangent plane to ω ($[n\,{}^*n]_{ij} = n_i\, n_j$). Further assume that ω is Lipschitzian and connected when ω has a non-empty boundary γ. For a detailed account of the intrinsic differential calculus on a $C^{1,1}$-submanifold, the reader is referred to the now available lecture notes Delfour and Zolésio (1997), Delfour (1998). Finally, it will be convenient to introduce the following notation for the decompositions of an $N \times N$ matrix τ into its tangential and normal parts along ω

$$\tau^P \stackrel{\text{def}}{=} P\tau P, \quad \tau_{nn} \stackrel{\text{def}}{=} \tau n \cdot n,$$
$$[t]\tau = \tau^P + (P\tau n)^* n + n^*(P\tau n) + \tau_{nn}\, n^* n$$

and the spaces of symmetrical matrices (or tensors)

$$\mathrm{Sym}_N \stackrel{\text{def}}{=} \{\tau \in \mathcal{L}(\mathbf{R}^N; \mathbf{R}^N) :{}^* \tau = \tau\}$$
$$\mathrm{Sym}_N^P \stackrel{\text{def}}{=} \{\tau \in \mathrm{Sym}_N : \tau\, n = 0\} \Rightarrow \forall \tau \in \mathrm{Sym}_N,\ \tau^P \in \mathrm{Sym}_N^P.$$

2. The membrane shell equation

It was shown in Delfour (1998) that the membrane shell equation can be obtained by decomposition of the variational equation of the asymptotic $P(1,0)$ model which also yields the typical group of terms occurring in the *Love-Kirchhoff condition.* It involves an effective compliance C_{eP} which retains the properties of the three-dimensional compliance C. For the purposes of this paper it is convenient to start with the following assumption on the effective compliance.

ASSUMPTION 2.1 *The* effective compliance *(Delfour, 1998) is a tensor valued function $C_{eP}: \omega \to \mathcal{L}(Sym_N^P, Sym_N^P)$ such that for all $X \in \omega$, $C_{eP}(X)$ is a linear bijective and symmetrical transformation of Sym_N^P,*

$$C_{eP}^{-1} \in L^\infty(\omega; \mathcal{L}(Sym_N^P, Sym_N^P)), \text{ and}$$
$$\exists \alpha > 0 \text{ such that } \forall X \in \omega, \forall \tau \in Sym_N^P, \quad C_{eP}^{-1}(X)\tau \cdot\cdot\, \tau \geq \alpha \|\tau\|^2.$$

The *membrane shell variational equation* is given by: for all $v^0 \in H^1(\omega)^N$

where the right-hand side is specified by a linear functional ℓ^P. Associate with ε_Γ^P the space

$$V \stackrel{\text{def}}{=} \{v \in L^2(\omega)^N : v_\Gamma \in H^1(\omega)^N\} \subset H \stackrel{\text{def}}{=} L^2(\omega)^N \tag{2}$$

and define E^P as the *completion* of the quotient space $V/\ker \varepsilon_\Gamma^P$ with respect to the norm associated with the inner product

$$\int_\omega \varepsilon_\Gamma^P(u) \cdot\cdot\, \varepsilon_\Gamma^P(v)\, d\Gamma. \tag{3}$$

Similarly for homogeneous Dirichlet boundary conditions on a part γ_0 of γ, denote by $E_{\gamma_0}^P$ the completion of the quotient space

$$V_{\gamma_0}/\ker \varepsilon^P, \quad V_{\gamma_0} \stackrel{\text{def}}{=} \{v \in L^2(\omega)^N : v_\Gamma \in H_{\gamma_0}^1(\omega)^N\}$$
$$H_{\gamma_0}^1(\omega) \stackrel{\text{def}}{=} \{f \in H^1(\omega) : f|_{\gamma_0} = 0\}$$

with respect to the norm generated by the scalar product (3). By Assumption 2.1 on C_{eP}, the bilinear term in (1) is continuous and coercive in E^P. We obtain the following general existence and uniqueness theorem where uniqueness in the space E^P means a unique equivalence class in the quotient space, that is – the solution is unique up to an element of $\ker \varepsilon^P$.

THEOREM 2.1 *Let Assumption 2.1 on C_{eP} be verified.*

(*i*) *Given $\ell^P \in (E^P)'$, that is – there exists $c > 0$ such that for all $v^0 \in H^1(\omega)^N$*

$$|\ell^P(v^0)| \le c\, \|\varepsilon_\Gamma^P(v^0)\|_{L^2(\omega)} \tag{4}$$

the variational equation: to find $\hat{v}^0 \in E^P$ such that for all $v^0 \in H^1(\omega)^N$

$$\int_\omega [C_{eP}^{-1} \varepsilon_\Gamma^P(\hat{v}^0)] \cdot\cdot\, \varepsilon_\Gamma^P(v^0)\, d\Gamma = \ell^P(v^0) \tag{5}$$

has a unique solution $\hat{v}^0$ in E^P.

(*ii*) *Assume that ω is connected and that γ_0 is a subset of γ with strictly positive H_{N-2} measure. Given $\ell^P \in (E_{\gamma_0}^P)'$, that is – there exists $c > 0$ such that for all $v^0 \in H_{\gamma_0}^1(\omega)^N$*

$$|\ell^P(v^0)| \le c\, \|\varepsilon_\Gamma^P(v^0)\|_{L^2(\omega)} \tag{6}$$

the variational equation: to find $\hat{v}^0 \in E_{\gamma_0}^P$ such that for all $v^0 \in H_{\gamma_0}^1(\omega)^N$

$$\int_\omega [C_{eP}^{-1} \varepsilon_\Gamma^P(\hat{v}^0)] \cdot\cdot\, \varepsilon_\Gamma^P(v^0)\, d\Gamma = \ell^P(v^0) \tag{7}$$

has a unique solution $\hat{v}^0$ in $E_{\gamma_0}^P$.

3. The reduced membrane shell equation

The membrane shell equation can be decomposed into a system of two equations. For test functions $v \in V$, that is, $(v_\Gamma^0, v_n^0) \in H^1(\omega)^N \times L^2(\omega)$,

$$\begin{aligned} &\exists c>0,\, \forall v_n^0 \in H^1(\omega), \quad |\ell^P(v_n^0\, n)| \le c\, \|v_n^0\, D^2 b\|_{L^2(\omega)} \le c'\, \|v_n^0\|_{L^2(\omega)} \\ &\Rightarrow\ \exists f^P \in L^2(\omega) \text{ such that } \ell^P(v_n^0\, n) = \int_\omega f^P\, v_n^0\, d\Gamma \end{aligned}$$

It will be convenient to introduce the function

$$f_n^P(X) \overset{\text{def}}{=} \begin{cases} f^P(X)/\|D^2 b(X)\|, & \text{if } \|D^2 b(X)\| \neq 0 \\ 0, & \text{if } \|D^2 b(X)\| = 0 \end{cases}$$

By construction, $f_n^P\, \|D^2 b\| \in L^2(\omega)$.

Denote by $H_t^1(\omega)^N$ (resp. $H_{\gamma_0 t}^1(\omega)^N$) the subspace $\{v \in H^1(\omega)^N$ (resp. $H_{\gamma_0}^1(\omega)^N)$: $v \cdot n = 0\}$ of tangential vectors. The decomposition yields the two equations

$$\begin{aligned} &\left[C_{eP}^{-1}\varepsilon_\Gamma^P(\hat{v}^0)\right]\cdot\cdot\, D^2 b = f^P = f_n^P\, \|D^2 b\| \\ &\forall v_\Gamma^0 \in H_t^1(\omega)^N, \quad \int_\omega \left[C_{eP}^{-1}\varepsilon_\Gamma^P(\hat{v}^0)\right]\cdot\cdot\, \varepsilon_\Gamma^P(v_\Gamma^0)\, d\Gamma = \ell^P(v_\Gamma^0) \end{aligned} \tag{8}$$

where by condition (4) on ℓ^P

$$\exists c>0,\, \forall v_\Gamma^0 \in H_t^1(\omega)^N, \quad |\ell^P(v_\Gamma^0)| \le c\, \|\varepsilon_\Gamma^P(v_\Gamma^0)\|_{L^2(\omega)}.$$

In the case of the plate ($D^2 b = 0$), $\varepsilon_\Gamma^P(v^0) = \varepsilon_\Gamma^P(v_\Gamma^0) + v_n^0\, D^2 b = \varepsilon_\Gamma^P(v_\Gamma^0)$ and there is only the variational equation

$$\forall v_\Gamma^0 \in H^1(\omega)^N, \quad \int_\omega \left[C_{eP}^{-1}\varepsilon_\Gamma^P(\hat{v}_\Gamma^0)\right]\cdot\cdot\, \varepsilon_\Gamma^P(v_\Gamma^0)\, d\Gamma = \ell^P(v_\Gamma^0)$$

which completely specifies $\hat{v}_\Gamma^0 \in H_t^1(\omega)^N/\ker \varepsilon_\Gamma^P$ (resp. $H_{\gamma_0 t}^1(\omega)^N$) and $\hat{v}_n^0$ is arbitrary. This result generalizes to $C^{1,1}$ midsurfaces without adding extra conditions on $D^2 b$. It turns out that the second equation (8) specifies the *tangential part* $\hat{v}_\Gamma^0$ of $\hat{v}^0$ up to an element of some appropriate equivalence class providing a natural decomposition of the membrane shell equation into an equation for the equivalence class of $\hat{v}_\Gamma^0$ and an equation for $\hat{v}_n^0$ again modulo another equivalence class. In the case of the plate the corresponding equivalence class for $\hat{v}_n^0$ is so big that there is no information on $\hat{v}_n^0$ and we have uniqueness for $\hat{v}_\Gamma^0$ in the case of homogeneous Dirichlet boundary conditions on a part of the boundary.

Denote by $[v]_E$ the equivalence class of v in E^P (resp. $E_{\gamma_0}^P$). Let

$$\omega_0 \overset{\text{def}}{=} \left\{x \in \omega : D^2 b(x) = 0\right\} \text{ and } \omega_+ \overset{\text{def}}{=} \omega \backslash \omega_0.$$

For $v \in V$ (resp. V_{γ_0}) define the function

$$(\ ) \overset{\text{def}}{=} \begin{cases} v - \dfrac{C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot\, D^2 b}{[illegible]}\, n, & \text{in } \omega_+ \end{cases} \tag{9}$$

Using the identity $\varepsilon_\Gamma^P(v) = \varepsilon_\Gamma^P(v_\Gamma) + v_n\, D^2 b$, it is easy to verify that for all $v \in V$ (resp. V_{γ_0})

$$\varepsilon_\Gamma^P(\pi_S(v)) = \begin{cases} \varepsilon_\Gamma^P(v) - \dfrac{C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot\, D^2 b}{C_{eP}^{-1} D^2 b\cdot\cdot\, D^2 b}\, D^2 b, & \text{in } \omega_+ \\ \varepsilon_\Gamma^P(v), & \text{in } \omega_0 \end{cases} \tag{10}$$

For each $v_\Gamma \in H_t^1(\omega)^N$ (resp. $H_{\gamma_0 t}^1(\omega)^N$) define the tensor

$$\tilde\varepsilon_\Gamma^P(v_\Gamma) \stackrel{\text{def}}{=} \varepsilon_\Gamma^P(\pi_S(v_\Gamma)) = \begin{cases} \varepsilon_\Gamma^P(v_\Gamma) - \dfrac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2 b}{C_{eP}^{-1} D^2 b\cdot\cdot\, D^2 b}\, D^2 b, & \text{in } \omega_+ \\ \varepsilon_\Gamma^P(v_\Gamma), & \text{in } \omega_0 \end{cases} \tag{11}$$

the quotient space

$$V^P \stackrel{\text{def}}{=} H_t^1(\omega)^N / \ker \tilde\varepsilon_\Gamma^P \quad (\text{resp. } V_{\gamma_0}^P \stackrel{\text{def}}{=} H_{\gamma_0 t}^1(\omega)^N / \ker \tilde\varepsilon_\Gamma^P) \tag{12}$$

and the space

$$U \stackrel{\text{def}}{=} \pi_S(H_t^1(\omega)^N) \quad (\text{resp. } U_{\gamma_0} \stackrel{\text{def}}{=} \pi_S(H_{\gamma_0 t}^1(\omega)^N)). \tag{13}$$

Consider the *reduced membrane shell equation*: to find $v_\Gamma \in V^P$ (resp. $V_{\gamma_0}^P$) such that for all $w_\Gamma \in H_t^1(\omega)^N$ (resp. $H_{\gamma_0 t}^1(\omega)^N$)

$$\int_\omega C_{eP}^{-1}\tilde\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\,\tilde\varepsilon_\Gamma^P(w_\Gamma)\, d\Gamma = \ell^P(\pi_S(w_\Gamma)) \tag{14}$$

By condition (4) on ℓ^P there exists $c > 0$ such that for all $w_\Gamma \in H_t^1(\omega)^N$

$$|\ell^P(\pi_S(w_\Gamma)| \le c\, \|\varepsilon_\Gamma^P(\pi_S(w_\Gamma))\|_{L^2(\omega)} = c\, \|\tilde\varepsilon_\Gamma^P(w_\Gamma)\|_{L^2(\omega)}$$

and equation (14) has a unique solution in the completion of the quotient space $V^P \stackrel{\text{def}}{=} V/\ker\tilde\varepsilon_\Gamma^P$ with respect to the topology generated by the norm $\|\tilde\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega)}$. We now give a sharper existence theorem for the reduced membrane shell equation and the membrane shell equation. This theorem is based on a characterization of the elements of the spaces E^P and $E_{\gamma_0}^P$.

THEOREM 3.1 *Let Assumption 2.1 on C_{eP} and (4) on ℓ^P be verified.*

(i) There exists a solution $\hat v_\Gamma$ in $H_t^1(\omega)^N$ (resp. $H_{\gamma_0 t}^1(\omega)^N$) *to the* reduced membrane shell equation *(14) unique up to an element of* $\ker\tilde\varepsilon_\Gamma^P$ *and*
$$[\pi_S(\hat v_\Gamma)]_E = [\pi_S(\hat v)]_E \tag{15}$$
where $[\hat v]_E$ is the solution of the membrane shell equation (5) (resp. *(7)*) *in E^P* (resp. $E_{\gamma_0}^P$).

(ii) There exists a solution $\hat u$ such that $\hat u_\Gamma \in H_t^1(\omega)^N$ (resp. $H_{\gamma_0 t}^1(\omega)^N$) *and $\hat u_n\|D^2 b\| \in L^2(\omega)$ to the membrane shell equation (5)* (resp. *(7)*) *which is*

(iii) $\ker \tilde{\varepsilon}^P_\Gamma$ *is finite dimensional. When* $D^2 b \neq 0$ *almost everywhere in* ω, $\ker \varepsilon^P_\Gamma$ *is also finite dimensional and*

$$\ker \varepsilon^P_\Gamma = \left\{ v : v_\Gamma \in \ker \tilde{\varepsilon}^P_\Gamma \text{ and } v_n = -\frac{C^{-1}_{eP}\varepsilon^P_\Gamma(v_\Gamma)\cdot\cdot D^2 b}{C^{-1}_{eP} D^2 b \cdot\cdot D^2 b} \right\} \tag{16}$$

This theorem necessitates the following theorem on the structure of the spaces E^P and $E^P_{\gamma_0}$ which follows from a sequence of lemmas.

THEOREM 3.2 *Let Assumption 2.1 on* C_{eP} *be verified.*

(i) $\ker \tilde{\varepsilon}^P_\Gamma$ *is finite dimensional and the space* $V^P = V/\ker \tilde{\varepsilon}^P_\Gamma$ (resp. $V^P_{\gamma_0} = V_{\gamma_0}/\ker \tilde{\varepsilon}^P_\Gamma$) *is complete for the norm* $\|\tilde{\varepsilon}^P_\Gamma(v_\Gamma)\|_{L^2(\omega)}$.

(ii) The space E^P (resp. $E^P_{\gamma_0}$) *is equal to*

$$\left\{ u_\Gamma + u_n\, n : \begin{array}{l} u_\Gamma \in H^1_t(\omega)^N (\text{resp. } H^1_{\gamma_0 t}(\omega)^N) \\ \text{and } u_n \, \|D^2 b\| \in L^2(\omega) \end{array} \right\} / \ker \varepsilon^P_\Gamma \tag{17}$$

Specifically for each $[v]_E \in E^P$, *there exists a unique* $[u_\Gamma]_V \in V^P = H^1_t(\omega)^N/\ker \tilde{\varepsilon}^P_\Gamma$ (resp. $V^P_{\gamma_0} = H^1_{\gamma_0 t}(\omega)^N/\ker \tilde{\varepsilon}^P_\Gamma$) *such that*

$$[\pi_S(u_\Gamma)]_E = [\pi_S(v)]_E \tag{18}$$

and for each u_Γ *in the equivalence class* $[v_\Gamma]_V$ *the* normal component

$$u_n \overset{def}{=} \begin{cases} \dfrac{C^{-1}_{eP}\varepsilon^P_\Gamma(v-u_\Gamma)\cdot\cdot D^2 b}{C^{-1}_{eP} D^2 b\cdot\cdot D^2 b}, & \text{in } \omega_0 \\ 0, & \text{in } \omega_+ \end{cases}$$

is such that $u_n \, \|D^2 b\| \in L^2(\omega)$ *and*

$$[u_\Gamma + u_n\, n]_E = [v]_E.$$

Conversely for all $u_\Gamma \in H^1_t(\omega)^N$ *and* $u_n \, \|D^2 b\| \in L^2(\omega)$

$$[u_\Gamma + u_n\, n]_E \in E^P.$$

(iii) When $D^2 b \neq 0$ *almost everywhere in* ω, *then* $\ker \varepsilon^P_\Gamma$ *is finite dimensional.*

Define the closed linear subspace

$$\begin{aligned} &S^P \overset{\text{def}}{=} \left\{ v \in E^P : C^{-1}_{eP}\varepsilon^P_\Gamma(v)\cdot\cdot D^2 b = 0 \right\} \\ &(\text{resp } S^P_{\gamma_0} \overset{\text{def}}{=} \left\{ v \in E^P_{\gamma_0} : C^{-1}_{eP}\varepsilon^P_\Gamma(v)\cdot\cdot D^2 b = 0 \right\}) \end{aligned} \tag{19}$$

of E^P (resp. $E^P_{\gamma_0}$). We first make sense of the map π_S on E^P.

LEMMA 3.1 *The map*

$$[v]_E \mapsto \pi_S([v]_E) \overset{def}{=} [\pi_S(v)]_E : E^P \to S^P \tag{20}$$

is well-defined, linear and continuous. Moreover

$$\overline{\pi_S(V)/\ker \varepsilon^P_\Gamma}^{E^P} = \overline{\pi_S(V/\ker \varepsilon^P_\Gamma)}^{E^P} = S^P \tag{21}$$

$$\forall v \in S^P, \quad \varepsilon^P_\Gamma(\pi_S(v)) = \varepsilon^P_\Gamma(v). \tag{22}$$

Proof. For each $v \in V$, $\varepsilon_\Gamma^P(v) = 0$ implies $\varepsilon_\Gamma^P(\pi_S(v)) = 0$ and hence

$$[v]_E = 0 \quad \Rightarrow [\pi_S(v)]_E = 0$$

So the map (20) is well-defined and linear from $V/\ker \varepsilon_\Gamma^P$ in E^P and

$$\pi_S(V)/\ker \varepsilon_\Gamma^P = \pi_S(V/\ker \varepsilon_\Gamma^P). \tag{23}$$

By direct computation it is easy to verify that

$$\forall v \in V, \quad C_{eP}^{-1}\varepsilon_\Gamma^P(\pi_S(v))\cdot\cdot\, D^2 b = 0$$

and hence $[\pi_S(v)]_E \in S^P$ so that

$$\pi_S \,:\, V/\ker \varepsilon_\Gamma^P \to \pi_S(V)/\ker \varepsilon_\Gamma^P \subset S^P.$$

The map π_S is also continuous. On ω_0 $\|\varepsilon_\Gamma^P(\pi_S(v))\|_{L^2(\omega_0)} = \|\varepsilon_\Gamma^P(v)\|_{L^2(\omega_0)}$ and on ω_+

$$\|\varepsilon_\Gamma^P(\pi_S(v))\|_{L^2(\omega\setminus\omega_0)} \leq \|\varepsilon_\Gamma^P(v)\|_{L^2(\omega\setminus\omega_0)} + \left\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2 b}{C_{eP}^{-1}D^2 b\cdot\cdot\, D^2 b} D^2 b \right\|_{L^2(\omega\setminus\omega_0)}$$

But from Assumption 2.1 on C_{eP}

$$\exists \alpha > 0,\, \forall X \in \omega,\, \forall \tau \in \mathrm{Sym}_N^P, \quad C_{eP}^{-1}(X)\tau\cdot\cdot\,\tau \geq \alpha\|\tau\|^2$$

which implies that

$$C_{eP}^{-1}\frac{D^2 b}{\|D^2 b\|}\cdot\cdot\,\frac{D^2 b}{\|D^2 b\|} \geq \alpha > 0 \text{ in } \omega_+$$

and

$$\begin{aligned} \left\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\,\frac{D^2 b}{\|D^2 b\|}}{C_{eP}^{-1}\frac{D^2 b}{\|D^2 b\|}\cdot\cdot\,\frac{D^2 b}{\|D^2 b\|}} \frac{D^2 b}{\|D^2 b\|} \right\|_{L^2(\omega_+)} &\leq \frac{1}{\alpha}\, \|C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega_+)} \\ &\leq c'\, \|\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega)}. \end{aligned}$$

So that for all $v \in V$

$$\|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|_{L^2(\omega)} \leq c'\, \|\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega)}.$$

Since E^P is the closure of $V/\ker \varepsilon_\Gamma^P$ with respect to the norm $\|\varepsilon_\Gamma^P(v)\|_{L^2(\omega)}$, π_S extends to a continuous linear map from E^P to S^P. Identity (21) is proved as follows. With any v in S^P we can associate a sequence $\{v^k\}$ in V such that

$$\varepsilon_\Gamma^P(v^k) \to \varepsilon_\Gamma^P(v) \text{ in } L^2(\omega)^{N\times N}.$$

By continuity of π_S

But

$$\varepsilon_\Gamma^P(\pi_S(v^k)) = \begin{cases} \varepsilon_\Gamma^P(v^k) - \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v^k)\cdot\cdot D^2b}{C_{eP}^{-1}D^2b\cdot\cdot D^2b}\, D^2b, & \text{in } \omega_+ \\ \varepsilon_\Gamma^P(v^k), & \text{in } \omega_0 \end{cases}$$

On ω_0

$$\varepsilon_\Gamma^P(\pi_S(v^k)) = \varepsilon_\Gamma^P(v^k) \to \varepsilon_\Gamma^P(v) \text{ in } L^2(\omega_0)^{N\times N}.$$

On $\omega\backslash\omega_0$

$$\left\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v^k)\cdot\cdot \frac{D^2b}{\|D^2b\|}}{C_{eP}^{-1}\frac{D^2b}{\|D^2b\|}\cdot\cdot\frac{D^2b}{\|D^2b\|}} \frac{D^2b}{\|D^2b\|}\right\|_{L^2(\omega\backslash\omega_0)} \le \frac{1}{c}\left\| C_{eP}^{-1}\varepsilon_\Gamma^P(v^k)\cdot\cdot\frac{D^2b}{\|D^2b\|}\right\|_{L^2(\omega\backslash\omega_0)}$$

and

$$\left\| C_{eP}^{-1}\varepsilon_\Gamma^P(v^k)\cdot\cdot\frac{D^2b}{\|D^2b\|}\right\|_{L^2(\omega\backslash\omega_0)} \to \left\| C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot\frac{D^2b}{\|D^2b\|}\right\|_{L^2(\omega\backslash\omega_0)} = 0$$

since $C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot D^2b = 0$ for v in S^P. Therefore

$$\varepsilon_\Gamma^P(\pi_S(v^k)) \to \varepsilon_\Gamma^P(v) \text{ in } L^2(\omega)^{N\times N}$$

and

$$\varepsilon_\Gamma^P(\pi_S(v^k)) \to \varepsilon_\Gamma^P(\pi_S(v)) = \varepsilon_\Gamma^P(v) \text{ in } L^2(\omega)^{N\times N}.$$

Then, for each $v \in S^P$ there exists a sequence $\{v^k\}$ in V such that

$$[\pi_S(v^k)]_E \to [v]_E \text{ in } E^P.$$

This establishes (21) and extends (22) from $(V/\ker\varepsilon_\Gamma^P)\cap S^P$ to S^P. Moreover from (22) for all $v\in E^P$ $\varepsilon_\Gamma^P(\pi_S(\pi_S(v))) = \varepsilon_\Gamma^P(\pi_S(v))$, $[\pi_S(\pi_S(v))]_E = [\pi_S(v)]_E$, and π_S is a projection. ■

LEMMA 3.2 *The map $\pi_S : H_t^1(\omega)^N \to U$ is a continuous linear bijection and U is closed for the topology generated by the norm*

$$\|u\|_U = \left\{ \|\varepsilon_\Gamma^P(u)\|_{L^2(\omega)}^2 + \|u_\Gamma\|_{L^2(\omega)}^2 + \|u_n\,\|D^2b\|\|_{L^2(\omega)}^2 \right\}^{1/2}.$$

The space V^P is complete for the norm $\|\tilde\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega)}$ and $\ker\tilde\varepsilon_\Gamma^P$ is finite dimensional.

Proof. By definition π_S is surjective. It is injective since $\pi_S(v_\Gamma)=0$ implies that $v_\Gamma = 0$ on ω_0 while on $\omega\backslash\omega_0$

$$v_\Gamma - \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\frac{D^2b}{\|D^2b\|}}{\ldots}\frac{n}{\ldots} = 0 \quad \Rightarrow v_\Gamma = 0$$

since v_Γ is orthogonal to the second term which is a fortiori also equal to zero. By Korn's inequality (38) in Theorem 4.1 of the Appendix the space $H_t^1(\omega)^N$ endowed with the norm

$$\|v_\Gamma\|_{H^1} = \left\{\|\varepsilon_\Gamma^P(v_\Gamma)\|^2 + \|v_\Gamma\|^2\right\}^{1/2}$$

is complete. By definition of $\pi_S(v_\Gamma)$ and expression (10) of $\varepsilon_\Gamma^P(\pi_S(v_\Gamma))$

$$\begin{aligned}\|\pi_S(v_\Gamma)\|_U = & \left\{\|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|^2_{L^2(\omega)} + \|v_\Gamma\|^2_{L^2(\omega)}\right. \\ & \left. + \left\| \|D^2b\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot D^2b}{C_{eP}^{-1}D^2b\cdot\cdot D^2b}\right\|^2_{L^2(\omega\backslash\omega_0)}\right\}^{1/2} \leq c\,\|v_\Gamma\|_{H^1(\omega)}.\end{aligned}$$

But we shall show that there exist $\lambda > 0$ and $\alpha > 0$ such that

$$\begin{aligned}&\|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|^2_{L^2(\omega)} + \lambda\left\{\|v_\Gamma\|^2_{L^2(\omega)} + \left\| \|D^2b\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot D^2b}{C_{eP}^{-1}D^2b\cdot\cdot D^2b}\right\|^2_{L^2(\omega\backslash\omega_0)}\right\} \\ &\geq \alpha\left(\|\varepsilon_\Gamma^P(v_\Gamma)\|^2_{L^2(\omega)} + \|v_\Gamma\|^2_{L^2(\omega)}\right).\end{aligned} \tag{24}$$

Therefore U is closed for the chosen norm and π_S is an isomorphism. Define the Hilbert space H as the closure of U with respect to the norm

$$\|u\|_H^2 \stackrel{\text{def}}{=} \|u_\Gamma\|^2_{L^2(\omega)} + \|\|D^2b\|\,u_n\|^2_{L^2(\omega)}.$$

The injection of U into H is compact. Pick any bounded sequence $\{\pi_S(v_\Gamma^k)\}$ in U. By the previous equivalence of norms $\{v_\Gamma^k\}$ is a Cauchy sequence in $H_t^1(\omega)^N$ and there exists $v_\Gamma \in H_t^1(\omega)^N$ and a subsequence, still denoted $\{v_\Gamma^k\}$, which weakly converges to v_Γ in $H_t^1(\omega)^N$. Therefore

$$\begin{aligned}& v_\Gamma^k \to v_\Gamma \text{ in } L^2(\omega)^N\text{- strong} \\ & \varepsilon_\Gamma^P(v_\Gamma^k) \rightharpoonup \varepsilon_\Gamma^P(v_\Gamma) \text{ in } L^2(\omega)^{N\times N}\text{-weak} \\ & \Rightarrow \varepsilon_\Gamma^P(v_\Gamma^k)\cdot\cdot\frac{D^2b}{\|D^2b\|} \to \varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\frac{D^2b}{\|D^2b\|} \text{ in } L^2(\omega)\text{-strong.}\end{aligned}$$

Hence

$$\pi_S(v_\Gamma^k) \to \pi_S(v_\Gamma) \text{ in } H\text{-strong}$$

and the injection of U into H is compact. In addition for the continuous linear map $A: U \to U'$ defined as

$$\langle Au, v\rangle \stackrel{\text{def}}{=} \int_\omega C_{eP}^{-1}\varepsilon_\Gamma^P(u)\cdot\cdot\varepsilon_\Gamma^P(v)\,d\Gamma$$

$[A+\lambda I]^{-1}$ is compact and by Lemma 4.2 of the Appendix ker A is finite dimensional. But

Since π_S is a bijection, then $\ker \tilde{\varepsilon}_\Gamma^P$ is also finite dimensional and thence the topology on the quotient space V^P defined by the norm $\|\tilde{\varepsilon}_\Gamma^P(v_\Gamma)\|$ is complete. To complete the proof it remains to establish inequality (24). On ω_0

$$\|\pi_S(v_\Gamma)\|_H = \|v_\Gamma\|_{L^2(\omega_0)} \text{ and } \|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|_{L^2(\omega_0)} = \|\varepsilon_\Gamma^P(v_\Gamma)\|_{L^2(\omega_0)}.$$

On $\omega\backslash\omega_0$ the norm $\|\pi_S(v_\Gamma)\|_H^2$ is equal to

$$\|v_\Gamma\|^2_{L^2(\omega\backslash\omega_0)} + \left\| \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2b}{C_{eP}^{-1}D^2b\cdot\cdot\, D^2b}\|D^2b\| \right\|^2_{L^2(\omega\backslash\omega_0)}$$

and the $L^2(\omega\backslash\omega_0)$-norm $\|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|$ is equal to

$$\begin{aligned}
&\left\|\varepsilon_\Gamma^P\left(v_\Gamma - \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2b}{C_{eP}^{-1}D^2b\cdot\cdot\, D^2b}\, n\right)\right\|^2_{L^2} = \left\|\varepsilon_\Gamma^P(v_\Gamma) - \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2b}{C_{eP}^{-1}D^2b\cdot\cdot\, D^2b}\, D^2b\right\|^2_{L^2}\\
&\geq \frac{1}{2}\|\varepsilon_\Gamma^P(v_\Gamma)\|^2_{L^2} - \left\|\frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\, D^2b}{C_{eP}^{-1}D^2b\cdot\cdot\, D^2b}\|D^2b\|\right\|^2_{L^2}
\end{aligned}$$

Finally

$$\|\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\|^2 + \|\pi_S(v_\Gamma)\|^2 \geq \begin{cases} \|\varepsilon_\Gamma^P S(v_\Gamma)\|^2 + \|v_\Gamma\|^2 & \text{on } \omega_0 \\ \frac{1}{2}\|\varepsilon_\Gamma^P S(v_\Gamma)\|^2 + \|v_\Gamma\|^2 & \text{on } \omega\backslash\omega_0 \end{cases}$$

Choose $\lambda = 1$ and $\alpha = 1/2$. ∎

LEMMA 3.3 *The map*

$$\begin{aligned} [v_\Gamma]_V \mapsto\ & \pi_S([v_\Gamma]_V) \overset{def}{=} [\pi_S(v)]_E \\ & : V^P = H_t^1(\omega)^N / \ker \tilde{\varepsilon}_\Gamma^P \to U/\ker \varepsilon_\Gamma^P \end{aligned} \tag{25}$$

is a well-defined isomorphism, where $[v]_V$ denotes the equivalence class of v in V^P, V^P is endowed with the topology generated by the norm $\|\tilde{\varepsilon}_\Gamma^P(v_\Gamma)\|$ and $U/\ker\varepsilon_\Gamma^P$ by the norm $\|\varepsilon_\Gamma^P(v)\|$ on E^P. Moreover

$$S^P = U/\ker\varepsilon_\Gamma^P = \pi_S(H_t^1(\omega)^N)/\ker\varepsilon_\Gamma^P = \pi_S(V^P).$$

Proof. The map (25) is well-defined and injective since

$$\tilde{\varepsilon}_\Gamma^P(v_\Gamma) = \varepsilon_\Gamma^P(\pi_S(v_\Gamma))$$

implies

$$[v_\Gamma]_V = 0 \quad \Leftrightarrow \quad [\pi_S(v_\Gamma)]_E = 0.$$

It is surjective since for any $[u]$ in $U/\ker\varepsilon_\Gamma^P$, there exists $v_\Gamma \in H_t^1(\omega)^N$ such that

$$[u]_E = [\pi_S(v_\Gamma)]_E.$$

It is bi-continuous since by definition of $\tilde{\varepsilon}_\Gamma^P(v_\Gamma)$ the norms are equal

Since V^P is complete for that norm by Lemma 3.2, $U/\ker\varepsilon_\Gamma^P$ is necessarily closed in E^P. Finally, from (10) it is easy to see that for all $v_\Gamma \in H_t^1(\omega)^N$

$$C_{eP}^{-1}\varepsilon_\Gamma^P(\pi_S(v_\Gamma))\cdot\cdot\, D^2b = 0$$

and $U/\ker\varepsilon_\Gamma^P \subset S^P$. Moreover, from (21) in Lemma 3.1

$$\overline{\pi_S(V)/\ker\varepsilon_\Gamma^P}^{E^P} = \overline{\pi_S(V/\ker\varepsilon_\Gamma^P)}^{E^P} = S^P.$$

But $H_t^1(\omega)^N \subset V$ and

$$U/\ker\varepsilon_\Gamma^P = \pi_S(H_t^1(\omega)^N)/\ker\varepsilon_\Gamma^P \subset \pi_S(V)/\ker\varepsilon_\Gamma^P.$$

In the other direction first observe that we always have from (10) by using the identity $\varepsilon_\Gamma^P(v) = \varepsilon_\Gamma^P(v_\Gamma) + v_n\, D^2b$ for each $v \in V$

$$\varepsilon_\Gamma^P(\pi_S(v)) = \varepsilon_\Gamma^P(\pi_S(v_\Gamma)).$$

Therefore for each $v \in V$ we have $[\pi_S(v)]_E = [\pi_S(v_\Gamma)]_E$ and

$$\begin{aligned}&\pi_S(V)/\ker\varepsilon_\Gamma^P \subset \pi_S(H_t^1(\omega)^N)/\ker\varepsilon_\Gamma^P = U/\ker\varepsilon_\Gamma^P\\ &\Rightarrow \pi_S(H_t^1(\omega)^N)/\ker\varepsilon_\Gamma^P = U/\ker\varepsilon_\Gamma^P = \pi_S(V)/\ker\varepsilon_\Gamma^P \subset S^P.\end{aligned}$$

Finally, since $U/\ker\varepsilon_\Gamma^P$ is closed in E^P

$$\begin{aligned}&\pi_S(V)/\ker\varepsilon_\Gamma^P = U/\ker\varepsilon_\Gamma^P = \overline{\pi_S(V)/\ker\varepsilon_\Gamma^P}^{E^P} = S^P\\ &U/\ker\varepsilon_\Gamma^P = \pi_S(V)/\ker\varepsilon_\Gamma^P = \pi_S(V/\ker\varepsilon_\Gamma^P) = \pi_S(V^P) = S^P.\end{aligned}$$

This completes the proof. ■

LEMMA 3.4 *For each $v \in E^P$, the projection $[\pi_S(v)]_E$ is the unique solution in S^P of the variational equation: for all $w \in H^1(\omega)^N$*

$$\int_\omega C_{eP}^{-1}\varepsilon_\Gamma^P(\pi_S(v) - v)\cdot\cdot\,\varepsilon_\Gamma^P(w)\, d\Gamma = 0 \tag{26}$$

and there is a solution $v_\Gamma \in H_t^1(\omega)^N$ unique up to an element of $\ker\tilde\varepsilon_\Gamma^P$ to the variational equation: for all w_Γ in $H_t^1(\omega)^N$

$$\int_\omega C_{eP}^{-1}\tilde\varepsilon_\Gamma^P(v_\Gamma)\cdot\cdot\,\tilde\varepsilon_\Gamma^P(w_\Gamma)\, d\Gamma = \int_\omega C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot\,\varepsilon_\Gamma^P(\pi_S(w_\Gamma))\, d\Gamma. \tag{27}$$

Moreover

$$[\pi_S(v)]_E = [\pi_S(v_\Gamma)]_E. \tag{28}$$

All this remains true for Dirichlet boundary conditions on a part γ_0 of the bound-

Proof. Since S^P is a closed linear subspace of E^P we already know that the variational equation (26) has a unique solution in S^P. So, it is sufficient to show that $\pi_S(v)$ is a solution. It is easy to check that

$$\varepsilon_\Gamma^P(v-\pi_S(v)) = \begin{cases} \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(v)\cdot\cdot D^2b}{C_{eP}^{-1}D^2b\cdot\cdot D^2b}\, D^2b, & \text{if } D^2b(x)\neq 0 \\ 0, & \text{if } D^2b(x)=0 \end{cases}$$

But $C_{eP}^{-1}\varepsilon_\Gamma^P(w)\cdot\cdot D^2b = 0$ for all $w\in S^P$ and hence $C_{eP}^{-1}\varepsilon_\Gamma^P(v-\pi_S(v))\cdot\cdot D^2b=0$ and $\pi_S(v)$ is a solution of (26). By Lemma 3.3 for each $v\in E^P$ there exists a unique $[v_\Gamma]_V\in V^P$ such that $[\pi_S(v)]_E=[\pi_S(v_\Gamma)]_E$ and $\tilde\varepsilon_\Gamma^P(v_\Gamma)=\varepsilon_\Gamma^P(\pi_S(v_\Gamma))=\varepsilon_\Gamma^P(\pi_S(v))$. From property (22) in Lemma 3.1 for each $w\in S^P$ $\tilde\varepsilon_\Gamma^P(w)=\varepsilon_\Gamma^P(\pi_S(w))$ and there exist a unique $[w_\Gamma]_V\in V^P$ such that $\tilde\varepsilon_\Gamma^P(w_\Gamma)=\varepsilon_\Gamma^P(\pi_S(w_\Gamma))=\varepsilon_\Gamma^P(\pi_S(w))=\tilde\varepsilon_\Gamma^P(w)$. It is then sufficient to substitute these identities in (26) to obtain (27). ■

LEMMA 3.5

(i) *The space* E^P *is equal to*

$$\left\{u_\Gamma+u_n\,n : u_\Gamma\in H_t^1(\omega)^N \text{ and } u_n\,\|D^2b\|\in L^2(\omega)\right\}/\ker\varepsilon_\Gamma^P \tag{29}$$

Specifically for each $[v]_E\in E^P$, *there exists a unique* $[v_\Gamma]_V\in V^P=H_t^1(\omega)^N/\ker\tilde\varepsilon_\Gamma^P$ (resp. $V_{\gamma_0}^P=H_{\gamma_0 t}^1(\omega)^N/\ker\tilde\varepsilon_\Gamma^P$) *such that*

$$[\pi_S(v_\Gamma)]_E=[\pi_S(v)]_E \tag{30}$$

and for each u_Γ *in the equivalence class* $[u_\Gamma]_V$ *the* normal component

$$u_n \overset{def}{=} \begin{cases} \frac{C_{eP}^{-1}\varepsilon_\Gamma^P(\tilde v^0-u_\Gamma)\cdot\cdot D^2b}{C_{eP}^{-1}D^2b\cdot\cdot D^2b}, & \text{in } \omega_0 \\ 0, & \text{in } \omega_+ \end{cases} \tag{31}$$

is such that $u_n\,\|D^2b\|\in L^2(\omega)$ *and*

$$[u_\Gamma+u_n\,n]_E=[v]_E.$$

Conversely for all $u_\Gamma\in H_t^1(\omega)^N$ *and* $u_n\,\|D^2b\|\in L^2(\omega)$

$$[u_\Gamma+u_n\,n]_E\in E^P.$$

(ii) *When* $D^2b\neq 0$ *almost everywhere in* ω, *then* $\ker\varepsilon_\Gamma^P$ *is finite dimensional.*

The lemma remains true for Dirichlet boundary conditions on a part γ_0 *of the boundary with* $H_{t,\gamma_0}^1(\omega)^N$ *and* $E_{\gamma_0}^P$ *in place of* $H_t^1(\omega)^N$ *and* $H_{\gamma_0 t}^1(\omega)^N$.

Proof. (i) From Lemma 3.4 for each $[v]_E\in E^P$ there exists a unique $[v_\Gamma]_V\in V^P$ such that

$$[\pi_S(v_\Gamma)]_E=[\pi_S(v)]_E$$

Each element u_Γ of the class $[u_\Gamma]_V$ is a $H^1(\omega)^N$-function and we can associate

$L^2(\omega)$. Now for the vector function $u = u_\Gamma + u_n\, n$

$$
\begin{aligned}
&\varepsilon^P_\Gamma(v-u) = \varepsilon^P_\Gamma(v) - \varepsilon^P_\Gamma(u_\Gamma) - u_n\, D^2 b\\
&= \varepsilon^P_\Gamma(v) - \begin{cases} \dfrac{C^{-1}_{eP}\varepsilon^P_\Gamma(v)\cdot\cdot\, D^2b}{C^{-1}_{eP}D^2b\cdot\cdot\, D^2b} D^2 b, & \text{in } \omega\backslash\omega_0\\ 0, & \text{in } \omega_0\end{cases}\\
&-\varepsilon^P_\Gamma(u_\Gamma) + \begin{cases} \dfrac{C^{-1}_{eP}\varepsilon^P_\Gamma(u_\Gamma)\cdot\cdot\, D^2b}{C^{-1}_{eP}D^2b\cdot\cdot\, D^2b} D^2 b, & \text{in } \omega\backslash\omega_0\\ 0, & \text{in } \omega_0\end{cases}\\
&\Rightarrow\ \varepsilon^P_\Gamma(v-u) = \varepsilon^P_\Gamma(\pi_S(v) - \varepsilon^P_\Gamma(\pi_S(u) = 0 \quad \Rightarrow\ \varepsilon^P_\Gamma(u) = \varepsilon^P_\Gamma(v).
\end{aligned}
$$

and $[u]_E = [v]_E$. Conversely, for $u_\Gamma \in H^1_t(\omega)^N$ and $u_n\,\|D^2b\| \in L^2(\omega)$

$$\varepsilon^P_\Gamma(u_\Gamma + u_n\, n) = \varepsilon^P_\Gamma(u_\Gamma) + u_n\, D^2b \in L^2(\omega)^{N\times N}$$

and $[u]_E \in E^P$.

(ii) When $D^2b \neq 0$, then the identity

$$\varepsilon^P_\Gamma(u) = \varepsilon^P_\Gamma(u_\Gamma) + u_n\, D^2b = 0$$

uniquely determines the normal component u_n associated with $u_\Gamma \in \ker\tilde{\varepsilon}^P_\Gamma$ in (31). Then, $\ker\varepsilon^P_\Gamma$ is isomorphic to $\ker\tilde{\varepsilon}^P_\Gamma$ and hence finite dimensional. ∎

Proof. Proof of Theorem 3.1 (i). Going back to Lemma 3.4 if $[\hat{v}^0]_E \in E^P$ is the solution of the membrane shell equation (5): for all $v^0 \in H^1(\omega)^N$

$$\int_\omega C^{-1}_{eP}\varepsilon^P_\Gamma(\hat{v}^0)\cdot\cdot\,\varepsilon^P_\Gamma(v^0)\, d\Gamma = \ell^P(v^0)$$

then for all $w \in H^1(\omega)^N$

$$\int_\omega C^{-1}_{eP}\varepsilon^P_\Gamma(\pi_S(\hat{v}^0))\cdot\cdot\,\varepsilon^P_\Gamma(w)\, d\Gamma = \int_\omega C^{-1}_{eP}\varepsilon^P_\Gamma(\hat{v}^0)\cdot\cdot\,\varepsilon^P_\Gamma(w)\, d\Gamma = \ell^P(w)$$

Moreover, from Lemma 3.4 there exists a solution $\hat{u}_\Gamma$ in $H^1_t(\omega)^N$ unique up to an element of $\ker\tilde{\varepsilon}^P_\Gamma$ to the *reduced membrane shell equation*: for all $w_\Gamma \in H^1_t(\omega)^N$

$$\int_\omega C^{-1}_{eP}\tilde{\varepsilon}^P_\Gamma(\hat{u}_\Gamma)\cdot\cdot\,\tilde{\varepsilon}^P_\Gamma(w_\Gamma)\, d\Gamma = \ell^P(\pi_S(w_\Gamma)).$$

and $[\pi_S(\hat{v}^0)]_E = [\pi_S(\hat{u}_\Gamma)]_E$. Parts (ii) and (iii) follow from Lemma 3.5. The proof remains true for Dirichlet boundary conditions on a part γ_0 of the boundary with $H^1_{t,\gamma_0}(\omega)^N$ in place of $H^1_t(\omega)^N$. ∎

4. Earlier results on membrane shells and regularity

In the previous sections we have shown existence of solution with tangential component $\hat{v}^0_\Gamma$ in $H^1_t(\omega)^N$ (resp. $H^1_{\gamma_0 t}(\omega)^N$) and normal component $\hat{v}^0_n$ such that $\hat{v}^0_n\,\|D^2b\| \in L^2(\omega)$ unique up to an element of $\ker\varepsilon^P_\Gamma$ to the membrane has a non-empty boundary γ) in a $C^{1,1}$ midsurface for homogeneous Neumann or Dirichlet conditions or for shells without boundaries. The reduced membrane shell equation completely characterizes the tangential component up to an element of the finite dimensional subspace $\ker \tilde{\varepsilon}_\Gamma^P$.

For $N = 3$, this completely relaxes the condition on $D^2 b$ and generalizes the existence result in V and V_{γ_0} obtained by Ciarlet and Lods (1994a, 1996a) for $g^0 = 0$, homogeneous Dirichlet boundary conditions on the whole boundary, the special constitutive law $C^{-1}\varepsilon = 2\,\mu\,\varepsilon + \lambda\,\mathrm{tr}\varepsilon\, I$ and the *uniform ellipticity* of the 2-dimensional C^2-midsurface ω:

$$\exists \nu > 0 \text{ such that } \forall \xi \in \mathbf{R}^3, \quad D^2 b(X)\xi_\Gamma \cdot \xi_\Gamma \geq \nu\,|\xi_\Gamma|^2 \tag{32}$$

(recall from Delfour and Zolésio, 1994, that this means that ω is a domain which is locally contained in the boundary Γ of a uniformly strictly convex subset of $\mathbf{R}^N$ – observe that $D^2 b(X)\xi \cdot \xi = D^2 b(X)\xi_\Gamma \cdot \xi_\Gamma$).

However, so far, uniqueness in the case of uniform elliptic shells does not seem to follow directly in an obvious way from the techniques used in this paper. The first existence and uniqueness result seems to be due to Destuynder (1980) under relatively strong conditions. For a domain ω with a C^3 boundary γ in an analytic midsurface, the existence and uniqueness of solutions $(\hat{v}_\Gamma^0, \hat{v}_n^0)$ in $H_0^1(\omega)^3 \times L^2(\omega)$ was established by Ciarlet and Sanchez-Palencia (1993, 1996). The conditions were relaxed by Ciarlet and Lods (1994a, 1996a): the midsurface is of class C^2 and the boundary γ is Lipschitz for the existence (midsurface C^5 and the boundary γ of class C^4 for existence and uniqueness). See also Sanchez-Hubert and Sanchez-Palencia (1997).

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Appendix

Given Ω in $\mathbf{R}^N$, $\Omega \neq \emptyset$ (resp. $\Gamma \stackrel{\text{def}}{=} \partial\Omega \neq \emptyset$) the *distance function* (resp. *oriented distance function*) is defined as

$$d_\Omega(x) \stackrel{\text{def}}{=} \inf_{y\in\Omega} |y-x| \text{ (resp. } b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\mathbf{R}^N-\Omega}(x)).$$

When Ω is a domain of class $C^{1,1}$ in $\mathbf{R}^N$, $b = b_\Omega$ is $C^{1,1}$ in a neighborhood of every point of Γ and the converse is true. Its gradient ∇b coincides with the exterior unit normal n to the boundary on Γ. The *projection* p onto Γ and the *orthogonal projection* P onto the *tangent plane* $T_x\Gamma$ are given by

$$p(x) \stackrel{\text{def}}{=} x - b(x)\,\nabla b(x), \quad P(x) \stackrel{\text{def}}{=} I - \nabla b(x)\,{}^*\nabla b(x),$$

where ${}^*V$ denotes the transpose of a column vector V in $\mathbf{R}^N$. Given $k > 0$ and an open domain ω in Γ, a *shell* of thickness $2k$ is the open domain

$$S_k(\omega) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N \colon \left|b_\Omega(x)\right| < k,\, p(x) \in \omega\right\}$$

in $\mathbf{R}^N$. When $\omega = \Gamma$ the shell has *no boundary*; otherwise we denote by γ the (relative) *boundary* of ω in Γ and by

$$\Sigma_k(\gamma) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N \colon \left|b_\Omega(x)\right| < k,\, p(x) \in \gamma\right\}$$

the *lateral boundary* of $S_k(\omega)$. Similarly for $\gamma_0 \subset \gamma$, we use the notation

$$\Sigma_k(\gamma_0) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N \colon \left|b_\Omega(x)\right| < k,\, p(x) \in \gamma_0\right\}$$

DEFINITION 4.1 (TANGENTIAL SOBOLEV SPACES) *Let ω be a bounded (relatively) open subset in Γ. Assume that there exists $h > 0$ such that b_Ω belongs to $C^{1,1}(S_h(\omega))$. Define*

$$W^{1,p}(\omega) \stackrel{def}{=} \left\{ f : \omega \to \mathbf{R} : \quad \begin{array}{l} \exists k,\, 0 < k \le h \\ \text{such that } f \circ p \in W^{1,p}(S_k(\omega)) \end{array} \right\} \tag{33}$$

The definition of $W^{1,p}(\omega)$ is independent of h.

LEMMA 4.1 *Let ω be a bounded open subset of Γ for which the assumptions of Definition 4.1 are verified and consider a function $f : \omega \to \mathbf{R}$. If there exists $0 < k \le h$ such that $f \circ p \in W^{1,p}(S_k(\omega))$, then*

$$f \circ p \in W^{1,p}(S_h(\omega)) \tag{34}$$

and for all $0 < k \le h$

Proof. Consider the map

$$y \mapsto R(y) \stackrel{\text{def}}{=} p(y) + \frac{k}{h} b(y)\nabla b(y) \,:\, S_h(\omega) \to S_k(\omega).$$

It is clearly well-defined since $p(R(y)) = p(y)$ and $|b(R(y))| = |k\, b(y)/h| < k$. Similarly the map

$$x \mapsto S(x) \stackrel{\text{def}}{=} p(x) + \frac{h}{k} b(x)\nabla b(x) \,:\, S_k(\omega) \to S_h(\omega)$$

is well-defined and $R(S(x)) = x$ and $S(R(y)) = y$. Moreover both R and $R^{-1} = S$ are Lipschitz continuous. From Nečas (1967, Lem. 3.1, pp. 65–66), the map

$$u \mapsto u \circ R \,:\, W^{1,p}(S_k(\omega)) \to W^{1,p}(S_h(\omega))$$

is an isomorphism. In particular, choosing $u = v \circ p \in W^{1,p}(S_k(\omega))$, we get $u \circ R = v \circ p \circ R \in W^{1,p}(S_h(\omega))$. But

$$v(p(R(y)) = v(p(p(y) + \frac{k}{h} b(y)\nabla b(y))) = v(p(y))$$

since ∇b, b and p are defined in $S_h(\omega)$. This implies that

$$v \circ p = (v \circ p) \circ R \in W^{1,p}(S_h(\omega)).$$

■

It can be shown that the Sobolev spaces defined in this way coincide with Sobolev spaces defined by local maps and that there is a direct relationship between tangential derivatives and covariant derivatives. The tangential gradient associated with an element f of the Sobolev space $W^{1,p}(\omega)$ is given by

$$\nabla_\Gamma f = \nabla(f \circ p)|_\omega \in L^p(\omega).$$

The same constructions apply to Sobolev spaces of vector functions $v \to \mathbf{R}^N$

$$D_\Gamma v = D(v \circ p)|_\omega \in L^p(\omega)^N \quad \varepsilon_\Gamma v = \varepsilon(v \circ p)|_\omega = \frac{1}{2}(D_\Gamma v + {}^* D_\Gamma v).$$

The *projected derivatives* are defined as

$$D_\Gamma^P v = P D_\Gamma(v) P \in L^p(\omega)^N \quad \varepsilon_\Gamma^P v = P \varepsilon_\Gamma(v) P.$$

The smoothness of the boundary γ of the domain ω in Γ is characterized by the smoothness of the *normal set* generated by the flow of the gradient of b through γ in a small neighborhood of ω. If γ is the (relative) boundary of ω in Γ, the *lateral boundary* of $S_h(\omega)$

$$\Sigma_h(\gamma) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N \,:\, |b_\Omega(x)| < h \text{ and } p(x) \in \gamma\} \tag{36}$$

is a submanifold of $\mathbf{R}^N$ of codimension one *normal* to Γ. The smoothness of the boundary γ of ω in Γ is characterized by the smoothness of the lateral boundary

DEFINITION 4.2 *Let ω be a bounded open subset of Γ which satisfies the assumptions of Definition 4.1.*

(i) Given an integer $k \geq 1$ and a real number $0 \leq \lambda \leq 1$, the boundary γ is $C^{k,\lambda}$ if there exist $h > 0$ and $0 < h' \leq h$ such that the piece $\Sigma_{h'}(\gamma)$ of the lateral boundary of $S_h(\omega)$ is $C^{k,\lambda}$.

(ii) The boundary γ is Lipschitzian if there exist $h > 0$ and $0 < h' \leq h$ such that the piece $\Sigma_{h'}(\gamma)$ of the lateral boundary of $S_h(\omega)$ is Lipschitzian.

(iii) The domain ω is connected if there exists h', $0 < h' < h$, such that $S_{h'}(\omega)$ is connected.

The above definitions correspond to the classical definitions in $\mathbf{R}^N$.

From previous considerations condition (i) is equivalent to statement that the oriented distance function $b_{S_h(\omega)}$ associated with the set $S_h(\omega)$ has the required smoothness in a neighborhood of $\Sigma_{h'}(\gamma)$.

DEFINITION 4.3 *Let ω be a bounded open subset of Γ which satisfies the assumptions of Definition 4.1. Further assume that ω is connected with a Lipschitzian boundary γ, and that γ_0 is an $(N-2)$-Hausdorff measurable subset of γ such that $H_{N-2}(\gamma_0) > 0$. Given $1 \leq p \leq \infty$, define*

$$W^{1,p}_{\gamma_0}(\omega) \overset{def}{=} \{f : \omega \to \mathbf{R} : f \circ p \in W^{1,p}_{\Sigma_h(\gamma_0)}(S_h(\omega))\}$$

where

$$W^{1,p}_{\Sigma_h(\gamma_0)}(S_h(\omega)) = \{F \in W^{1,p}(S_h(\omega)) : F|_{\Sigma_h(\gamma_0)} = 0\}$$
$$\Sigma_h(\gamma_0) = \{x : |b(x)| < h \text{ and } p(x) \in \gamma_0\}.$$

In that framework it is possible to give a direct proof of Korn's inequality for a $C^{1,1}$ midsurface ω (see, for instance, Delfour and Zolésio, 1997).

THEOREM 4.1 *Let ω be a bounded open subset of Γ satisfying the assumptions of Definition 4.1. Further assume that ω is Lipschitzian when ω has a non-empty boundary γ. There exists a constant $c(\omega) > 0$ such that*

$$\forall v \in E_\Gamma(\omega) \overset{def}{=} \{v \in L^2(\omega)^N : v_n = v \cdot n = 0 \text{ and } \varepsilon^P_\Gamma(v_\Gamma) \in L^2(\omega)^{N \times N}\} \tag{37}$$

$$\int_\omega \|D_\Gamma(v)\|^2 \, d\Gamma \leq c(\omega)^2 \int_\omega |v|^2 + \|\varepsilon^P_\Gamma(v)\|^2 \, d\Gamma \tag{38}$$

$$\int_\omega \|D^P_\Gamma(v)\|^2 \, d\Gamma \leq c(\omega)^2 \int_\omega |v|^2 + \|\varepsilon^P_\Gamma(v)\|^2 \, d\Gamma \tag{39}$$

and $E_\Gamma(\omega) = \{v \in H^1(\omega)^N : v_n = v \cdot n = 0 \text{ on } \omega\}$. In particular

$$\left\{\|v\|^2_{L^2(\omega)} + \|\varepsilon^P_\Gamma(v)\|^2_{L^2(\omega)}\right\}^{1/2} \text{ and } \left\{\|v\|^2_{L^2(\omega)} + \|D^P_\Gamma(v)\|^2_{L^2(\omega)}\right\}^{1/2} \tag{40}$$

are equivalent norms on the space

LEMMA 4.2 *Let V and H be two real Hilbert spaces with continuous compact injection of V into H and identify the elements of the dual H' of H with those of H. Assume that $A : V \to V'$ is a linear continuous symmetrical operator which is V-H coercive, that is – there exist λ and $\alpha > 0$ such that*

$$\forall v \in V, \quad \lambda\,|v|_H^2 + \langle Av, v\rangle_{V' \times V} \geq \alpha \|v\|_V^2.$$

Then

$$\ker A \stackrel{def}{=} \{v \in V \,:\, Av = 0\}$$

is a finite dimensional subspace of V and

$$\exists c > 0, \forall v \in \ker A, \quad \|v\|_V \leq c\,|v|_H. \tag{42}$$

Proof. Clearly we can always pick $\lambda > 0$ and $\lambda I + A$ has a continuous symmetrical inverse $[\lambda I + A]^{-1} : V' \to V$. In particular, from the compactness of the injection of V into H, the operator $[\lambda I + A]^{-1}$ is continuous linear and compact from H to H. Hence the multiplicity of all its non-zero eigenvalues is finite, that is, the eigenspace associated with an eigenvalue $\mu \neq 0$ is finite dimensional (see, e.g., Riesz and Nagy, 1965, Chap. VI, §93, p. 201). For any $0 \neq v \in \ker A$, $Av = 0$ and

$$[\lambda I + A]v = \lambda\, v \quad \Rightarrow \frac{1}{\lambda}\, v = [\lambda I + A]^{-1}\, v$$

and v belongs to the eigensubspace of $[\lambda I + A]^{-1}$ for $\mu = 1/\lambda \neq 0$. Therefore, $\ker A$ is finite dimensional. Moreover, by V-H coercivity

$$\alpha\, \|v\|^2 \leq \lambda |v|^2 + \langle Av, v\rangle = \lambda |v|^2$$

and we get the equivalence of norms on $\ker A$. ■