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A mathematical analysis of a smart-beam which is equipped with piezoelectric actuators

by

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Abstract: First of all, a brief reminder on piezoelectric effect is given. Then it is applied to a beam equipped with such actuators. The influence of the shape and location is discussed. A smart beam model is finally presented and analyzed. The controllability is carefully examined in the framework of the H.U.M. method of Lions (1988). Let us also underline that the asymptotic harmonic behaviour of the structure is widely used.

Keywords: control theory, smart structures, beam theory

1. Introduction

In the early eighties, several people have underlined the interest of piezoelectric effect for reducing structural vibrations. But the possibility of manufacturing a real "smart structure" appeared immediately to be very much connected with existence of an accurate model which could enable precise understanding and design of the control strategy. Two papers were at the origin of this area of mathematical modelling. The first one is by Bailey and Hubbard (1985), who gave a mechanical discussion of the piezoelectric effect on a beam, and then Hanagud, Obal and Calise (1987), who suggested an optimal control strategy based on a stabilization technique. One of the first mathematical modelling exercises giving a functional interpretation of the piezo-actuators in terms of Dirac distributions was detailed in Destuynder, Legrain, Castel and Richard (1988) in an ONERA report. One of the important drawbacks was pointed out, specially for acoustic applications. It is the spillover phenomenon which is particularly important in the piezo-actuator technology. The reason is that it acts as a pointwise force or a concentrated moment and therefore many eigenmodes are excited by this kind of actuators. Thus, a part of the mechanical energy is transferred from lower to upper eigenmodes through the piezoelectric devices. But there are

Balas has suggested to use Kalman filtering (see Balas, 1982). One interesting advantage of the so-called H.U.M. method introduced and developed by Lions (1988) is that it can considerably decrease the spillover phenomenon. Furthermore, for harmonic or quasi-harmonic structures the natural H.U.M. control is: a) easy to determine and to apply for a real-time procedure and b) it avoids spillover if the control period chosen is the one of the fundamental frequency of the structure. But, obviously, this analysis rests upon a mathematical analysis of the H.U.M. method in the case of a smart structure model with piezodevices. The goal of this paper is to give an overview of what has been studied recently on this problem. Many contributions used in this paper are included in the references listed at the end.

2. Modelling of the smart structure

Let us consider a thin plate, its medium surface being denoted by ω and its boundary by γ. The normal component of the displacement field is $u(x,t)$ where $x \in \omega$ and $t \in [0,T]$.

The thickness of the plate is assumed to be constant and equal to 2ϵ. On the upper or the lower (or both) face(s) of the plate, we set thin layers (few microns) of a piezoelectric material. The additional stiffness due to these actuators is negligible. If we denote by $\sigma_{\alpha\beta}$; $\alpha, \beta \in \{1,2\}$ the in-plane stresses in the piezo-layer, and by V the voltage between the upper and the lower faces, one has the following constitutive relationship ($\sigma_{\alpha\beta}$ is assumed to be constant through the thickness of the layers because of their very small thickness):

$$\sigma_{\alpha\beta} = h_{\alpha\beta}\frac{V}{c}$$

where c is dielectric constant and $h_{\alpha\beta}$ a symmetrical tensor which charaterizes the piezoelectic effect. The bending moment of these stresses estimated at the center of the plate (i.e. on ω) is:

$$M_{\alpha\beta} = \frac{\epsilon + a^e}{c} h_{\alpha\beta} V,$$

where a^e is half the thickness of the piezo-layer. Let us underline that the mechanical effect is not restricted to a concentrated torque, excepted if another piezo-layer is set symmetrically with respect to ω and with an opposite voltage. But nevertheless, the membrane strains are usually negligible because of the high membrane stiffness of the plate. Therefore they will be omitted in the following.

Let us now denote by ρ the mass density of the plate and by D the bending modulus. Then the model (see Destuynder, Legrain, Castel and Richard, 1988) is:

$$\left\{\begin{array}{ll} 2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\Delta^2 u = \left(\frac{\epsilon+a}{c}\right) V\partial_{\alpha\beta}h_{\alpha\beta} & \text{in } \omega\times]0,T[\\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in } \omega, \\ u(x,t) = \left((1-\nu)\frac{\partial^2 u}{\partial n^2} + \nu\Delta u\right)(x,t) = 0 & \text{on } \gamma\times]0,T[, \end{array}\right. \tag{1}$$

where ν is the Poisson coefficient of the plate and n the unit normal along γ outwards of ω. Furthermore, the implicit summation convention over repeated indices has been used (from 1 to 2 for $\alpha, \beta, \ldots$). It is worth noticing that $h_{\alpha\beta}$ is constant on an open set ω_p which – in ω – corresponds to the projection of the position of the piezoelectric layer and is equal to zero outside of ω_p. Hence the right hand side of (1) contains Dirac distributions (and their derivatives!). In order to focus on the physical meaning of such terms let us use a variational formulation. First of all we introduce the functional space

$$V(\omega) = \left\{v \in H^2(\omega) \quad v = 0 \text{ sur } \partial\omega\right\}. \tag{2}$$

Then, multiplying (1) by an arbitrary element v in $V(\omega)$ and integrating by parts we obtain the following variational formulation:

Find $u(x,t)$ such that:

$$\left\{\begin{array}{l} \forall v \in V(\omega), 2\epsilon\rho\int_\omega \frac{\partial^2 u}{\partial t^2} v dx + \mathcal{D}\int_\omega ((1-\nu)\partial_{\alpha\beta}u\partial_{\alpha\beta}v + \nu\Delta u\Delta v)\, dx \\ = \left(\frac{\epsilon+a}{c}\right) V(t)\int_{\omega_p} h_{\alpha\beta}\partial_{\alpha\beta}v \end{array}\right. \tag{3}$$

The open set ω_p corresponds to the piezo device, assuming, for instance, that there is only one of them. But (3) is formal because the functional space for u is not explicited. We just assume that there exists a solution which is sufficiently smooth in order to make sense to (3). Unfortunately, it will be shown in the following that it is not realistic. Nevertheless, the mechanical interpretation of the piezo effect will remain true. From Stokes formula we deduce that:

$$\int_{\omega_p} h_{\alpha\beta}\partial_{\alpha\beta}v = \int_{\partial\omega_p} h_{\alpha\beta}n_\alpha\partial_\alpha v \tag{4}$$

($n = (n_\alpha)$, and $\partial\omega_p$ is the boundary of ω_p). Thus, two kinds of mechanical loading appear clearly in this expression. One is a distributed torque around $\partial\omega_p$ and the other is a normal torque which is also distributed along $\partial\omega_p$. Both are represented in Fig. 1.

If the plate is reduced to a beam as shown in Fig. 2, it is then possible to simplify the expression of the plate model because one can assume that the deflection u does not depend of the coordinate x_2. This is due to the small width of the beam (see Fig. 2). Then a simplified expression of the right hand side of (3) can be derived by choosing virtual functions v which only depend on x_1. Thus we obtain first:

$$-1 \quad \int h_{\alpha\beta}\partial_{\alpha\beta}v = \int h_{11}\partial_{11}v = h_{11}\int \partial_{11}v$$

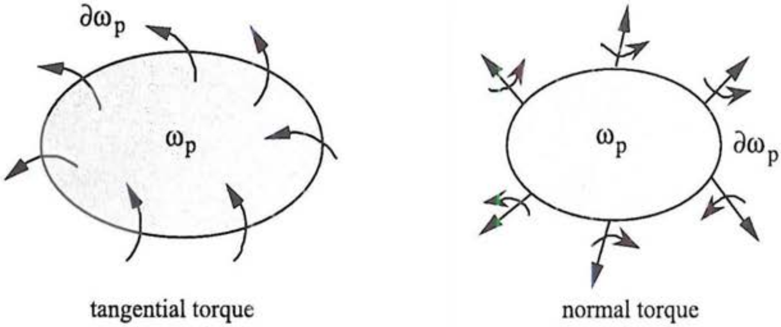

Figure 1. Mechanical interpretation of the piezoelectric effects

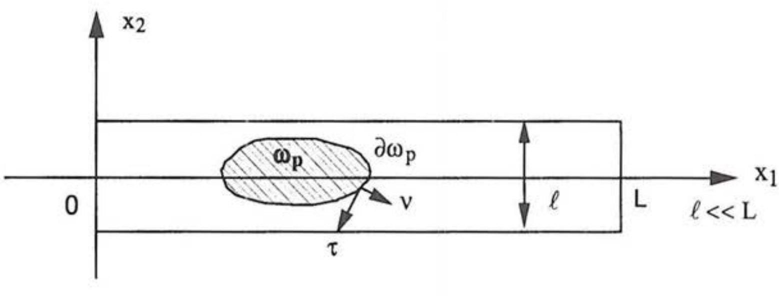

Figure 2. A piezo-wafer sticked on a beam

Then, let us denote by $x_2 = f(x_1)$ the equation of the curve representing the piezoelectric device as indicated in Fig. 2. Furthermore, the shape is assumed to be symmetrical with respect to the axis x_1. The extremities being localized at the points, with coordinates $x_1 = a$ and $x_1 = b$ we deduce that:

$$-2 \quad \int_{\omega_p} h_{\alpha\beta}\partial_{\alpha\beta}v = 2h_{11}\int_a^b f(x_1)\frac{d^2v}{dx_1^2}(x_1)dx_1$$

or else:

$$\int_{\omega_p} h_{\alpha\beta}\partial_{\alpha\beta}v =$$
$$2h_{11}\left(\left[f(x_1)\frac{dv}{dx_1}(x_1)\right]_a^b - \left[\frac{df}{dx_1}(x_1)v(x_1)\right]_a^b + \int_a^b \frac{d^2f}{dx_1^2}(x_1)v(x_1)dx_1\right). \quad (5)$$

From this formula one can derive several particular expressions for the right hand side of the smart beam model. Furthermore, from now on only the coordinate x_1 will appear. Therefore for sake of brevity in the notations, we shall just write x, omitting the subscript 1.

2.1. Rectangular wafers (Fig. 3)

In this case we have:

$$f(x) = F(= \text{ constant on } [a,b]).$$

Thus the beam model becomes:

$$2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + D\frac{\partial^4 u}{\partial x^4} = 2h_{11}(\epsilon + a^e)\frac{F}{c\ell}\left[\delta'_a - \delta'_b\right]V(t), \quad \forall(x,t) \qquad (6)$$

where l is the width of the beam as shown in Fig. 2. Furthermore, δ'_{x_0} denotes the derivative of the Dirac distribution at point x_0. Consequently, the effect of the piezo-device on the beam can be assimilated to two concentrated torques applied respectively at points $x = a$ and $x = b$ and with an opposite sign.

Let us underline that if an eigenmode - say W_n - is such that:

$$\frac{\partial W_n}{\partial x}(a) = \frac{\partial W_n}{\partial x}(b),$$

then the piezoelectric has no effect on it. Since one has:

$$W_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\Pi x}{L}\right),$$

the previous condition is equivalent to:

$$n\Pi a \qquad n\Pi b$$

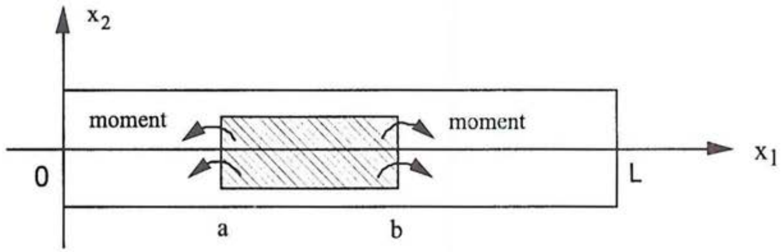

Figure 3. A rectangular wafer

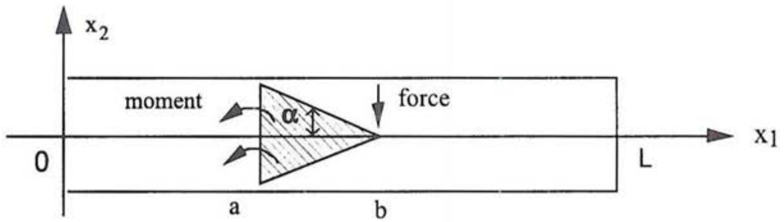

Figure 4. A triangular wafer

or else:

$$\frac{a \pm b}{2L} \text{ is a rational number - say } \frac{k}{n}.$$

Because $0 \leq a \leq b \leq L$, this situation can only occur if $k \leq n$. But even if the position of the piezo-layer can be adjusted for one eigenmode, it appears clearly that it cannot be done for all of them, because the set of rational numbers is dense in the one of real numbers.

2.2. A triangular wafer (Fig. 4)

We set:

$$f(x_1) = \left(\frac{b - x_1}{b - a}\right) F \quad a \leq x_1 \leq b.$$

Then, introducing this expression into the right hand side of (5), we derive the following beam model (let us set, for instance, $a = 0$).

$$2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + D\frac{\partial^4 u}{\partial x^4} = \frac{2h_{11}F\,(\epsilon + a^e)}{c\ell}\left[\delta_0' + \frac{1}{b}\delta_b\right] V(t), \quad \forall(x,t). \tag{7}$$

Thus the mechanical effect can be interpreted by a torque at $x = 0$ and a

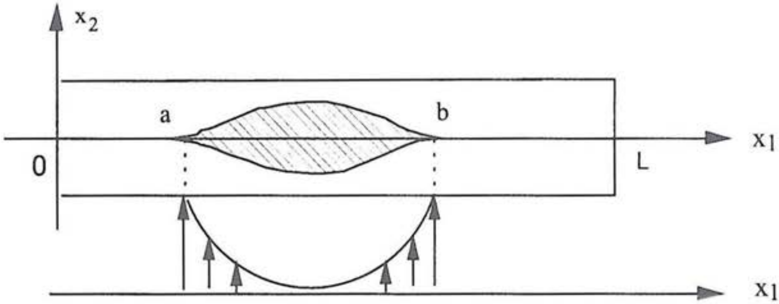

Figure 5. A "Ravioli" wafer

2.3. A "ravioli" wafer (Fig. 5)

We now set (with, for instance, $a = 0$ an $b = x_0$):

$$f(x) = \beta \quad (x_0 - x)^2 \quad 0 < x < x_0$$

where β is a sufficiently small coefficient. Then the smart beam model is:

$$2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\frac{\partial^4 u}{\partial x^4} = \frac{4h_{11}\beta\,(\epsilon + a^e)}{c\ell}\left[(x_0 - x)^2 + 4x\,(x - x_0) + x^2\right] V(t).\,(8)$$

The mechanical effect is the one of a distributed transverse load as shown in Fig. 5. It can be underlined that it induces a local bending effect.

2.4. An almond wafer (Fig. 6)

Let us set:

$$f(x) = \beta(x - a)(x - b), \quad a \leq x \leq b.$$

The beam model is:

$$2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\frac{\partial^4 u}{\partial x^4} = \frac{2h_{11}\beta\,(\epsilon + a^e)}{c\ell}\left[-\,(b - a)\,(\delta_b + \delta_a) + 2\chi_{[a,b]}(x)\right].$$

$\chi_{[a,b]}(x)$ being the characteristic function of the interval $[a, b]$.

2.5. A butterfly wafer (Fig. 7)

In order to prescribe more precisely the forces applied on the beam, one can mix several wafers with different shapes. The first example makes it possible to simulate a pointwise force. Three wafers are used. One is rectangular and the two other are triangular. The electric potentials are denoted by V_1, V_2 and V_3. Then the smart beam model is (we choose $a = b$ and $F = \ell/2$):

$$\quad \partial^2 u \quad\quad \partial^4 u$$

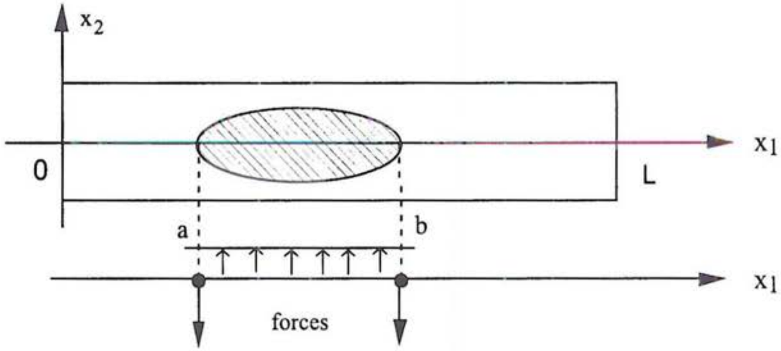

Figure 6. An almond wafer

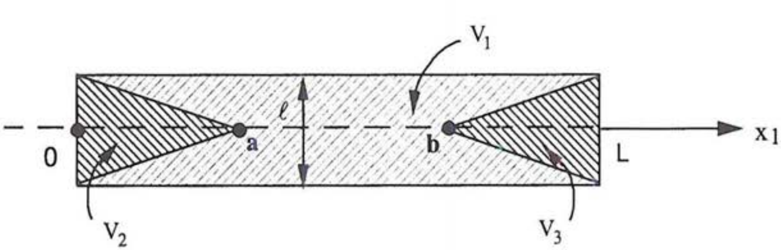

Figure 7. A butterfly wafer

$$= \frac{h_{11}(\epsilon + a^e)}{c} \left[V_1(\delta'_0 - \delta'_L) + V_2 \left(\frac{\delta_a}{a} + \delta'_0 \right) + V_3 \left(\frac{\delta_a}{L-a} - \delta'_L \right) \right].$$

Let us choose V_i, $i = 1, 2, 3$ such that:

$$V_1 + V_3 = 0, \quad V_1 = -V_2.$$

Then we derive the following smart-beam model:

$$2\epsilon\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{D} \frac{\partial^4 u}{\partial x^4} = -2L \frac{h_{11}(\epsilon + a^e)}{ca(L-a)} V_1(t)\delta_a(x). \tag{9}$$

It is worth noting that the system is equivalent to a pointwise force and that the efficiency is very much increased if the position of this pointwise force is at the middle of the beam.

2.6. A candy wafer (Fig. 8)

Let us use four piezo-devices: one is almond shaped, one is rectangular and the two last are triangular. The smart beam model is:

$$2\epsilon\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{D} \frac{\partial^4 u}{\partial x^4} = \frac{2h_{11}(\epsilon + a^e)}{c\ell} \left[-\beta(b-a)(\delta_a + \delta_b) V_4(t) \right.$$
$$+ 2\beta\chi_{[a,b]}(x)V_4(t) + FV_1(t)(\delta'_0 - \delta'_L) + FV_2(t)\left(\delta'_0 + \frac{\delta_a}{a}\right)$$
$$\left. + FV_3(t)\left(\frac{\delta_b}{L-b} - \delta'_L\right)\right].$$

Then, setting

$$V_1 + V_3 = 0, \quad V_2 + V_1 = 0 \text{ (hence } V_2 = V_3 = -V_1).$$

and

$$-\beta(b-a)V_4 + \frac{F}{a}V_2 = 0 \, (a + b = L!),$$

we deduce that:

$$2\epsilon\rho \frac{\partial^2 u}{\partial t^2} + D \frac{\partial^4 u}{\partial x^4} = \frac{4h_{11}(\epsilon + a^e)}{c\ell} \beta\chi_{[a,b]}(x)V_4(t) \tag{10}$$

with:

$$V_2 = V_3 = \frac{a\beta(b-a)}{F} V_4 = -V_1.$$

REMARK 2.1 *From a practical point of view, it should be underlined that the wafers must be disconnected in order to avoid a short cut. Therefore they could be set in different parallel plans through the thickness of the beam.*

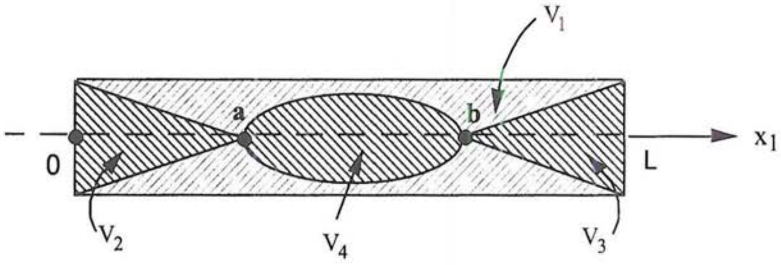

Figure 8. A candy wafer

3. Mathematical analysis of a smart beam

Let us consider a smart beam model which corresponds to one of those described in Section 2:

find $u(x,t)$ such that:

$$\left\{\begin{array}{l} \left(2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\frac{\partial^4 u}{\partial x^4}\right)(x_1,t) = AV_1(t)\delta_a(t) + BV_2(t)\delta'_a(t) \\ \quad \forall (x,t) \in]0,L[\times]0,T[\\ u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \quad \forall t \in]0,T[. \end{array}\right. \tag{11}$$

$$u(x,0) = u_0(x), \frac{\partial u}{\partial t}(x,0) = u_1(x), \forall x \in \left]O,L\right[. \tag{12}$$

Functions V_1 and V_2 are electric potentials. Let us assume that they are both in the space $L^2(]0,T[)$. The initial values u_0 and u_1 are chosen in functional spaces on $]O,L[$ which are described in the following. But first of all, it is necessary to introduce several notations. Let us consider the eigenvectors W_n (and the eigenvalues λ_n) of the structural model:

$$\left\{\begin{array}{l} 2\epsilon\rho\lambda_n W_n(x) = \mathcal{D}\frac{d^4 W_n}{dx^4}(x), \quad \forall x \in \left]0,L\right[, \\ W_n(0) = \frac{d^2 W_n}{dx^2}(0) = W_n(L) = \frac{d^2 W_n}{dx^2}(L) = 0, \end{array}\right.$$

and the normalization condition that we have chosen is:

$$2\epsilon\rho\int_0^L W_n^2(x)dx = 1.$$

The solutions can be explicited by:

$$\left\{\begin{array}{l} W_n(x) = \sqrt{\frac{1}{\epsilon\rho L}}\sin\left(\frac{n\Pi x}{L}\right), \end{array}\right. \tag{13}$$

Then, for any $s \geq -1$ we define the functional spaces (Lions, Magenes, 1968):

$$D_s\left(]0,L[\right) = \left\{ v \in V', \sum_{n\geq 1} \langle v, \omega_n \rangle^2 \lambda_n^s < \infty \right\} \tag{14}$$

where $<,>$ is the duality product between $V = H_0^1(]0,L[) \cap H^2(]0,L[)$ and its dual – say V'. One has the following classical identities:

$$\begin{cases} \text{i)} & \mathcal{D}_0\left(]0,L[\right) = L^2\left(]0,L[\right), \\ \text{ii)} & \mathcal{D}_1\left(]0,L[\right) = V = H_0^1\left(]0,L[\right) \cap H^2\left(]0,L[\right), \\ \text{iii)} & \mathcal{D}_{-1}\left(]0,L[\right) = V', \\ \text{iv)} & \mathcal{D}_{\frac{1}{2}}\left(]0,L[\right) = H_0^1\left(]0,L[\right), \\ \text{v)} & \mathcal{D}_{-\frac{1}{2}}\left(]0,L[\right) = H^{-1}\left(]0,L[\right). \end{cases} \tag{15}$$

The assumed solution of (11)-(12) can be expressed in the basis $\{W_n\}$. One has the following expression:

$$\begin{cases} u(x,t) = \sum_{n\geq 1} \alpha_n(t) W_n(x), \\ \alpha_n(t) = \langle u_0, W_1 \rangle \cos(\sqrt{\lambda_n} t) + \frac{\langle u_1, W_n \rangle}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} t) \\ + \frac{1}{\sqrt{\lambda_n}} \left[A W_n(a) \int_0^t V_1(s) \sin(\sqrt{\lambda_n}(t-s)) ds - \right. \\ \left. B \frac{\partial W_n}{\partial x}(a) \int_0^t V_2(s) \sin(\sqrt{\lambda_n}(t-s)) ds \right]. \end{cases} \tag{16}$$

The question we are dealing with is to characterize the space in which the convergence of the series (16) occurs. One has quite immediately the well known result:

THEOREM 3.1

1. *Let* $(u_0, u_1) \in L^2\left(]O,L[\right) \times V'$, $\quad V_i \in L^2\left(]O,T[\right), i = 1,2$ *then:* $u \in C^0\left([O,T]; L^2\left(]O,L[\right)\right) \cap C^1\left([O,T]; V'\right)$
2. *Let* $(u_0, u_1) \in H_0^1\left(]O,L[\right) \times H^{-1}\left(]O,L[\right)$, $\quad V_i \in L^2\left(]O,T[\right), i = 1,2,$ *then if* $B = 0$*:* $u \in \mathcal{C}^0\left([0,T]; H_0^1\left(]0,L[\right)\right) \cap \mathcal{C}^1\left([0,T]; H^{-1}\left(]0,L[\right)\right)$.
3. *Let* $(u_0, u_1) \in V \times L^2\left(]O,L[\right)$, $\quad V_i \in L^2\left(]O,T[\right), i = 1,2,$ *then if* $A = B = 0$*:* $u \in \mathcal{C}^0\left([0,T]; V\right) \cap \mathcal{C}^1\left([0,T]; L^2\left(]0,L[\right)\right)$.

Sketch of the proof.

Let us introduce two elements z_1 and z_2 of V, such that:

$$\begin{cases} D \frac{d^4 z_1}{dx^4} = \delta_a, \\ \mathcal{D} \frac{d^4 z_2}{dx^4} = \delta_a', \\ z_i(0) = \frac{d^2 z_i}{dx^2}(o) = z_i(L) = \frac{d^2 z_i}{dx^2}(L) = 0, \quad i = 1,2. \end{cases}$$

Then one has:

$$\begin{cases} z_1 = \sum_{n\geq 1} \frac{1}{\lambda_n} W_n(a) W_n(x) \in H_0^1(]0,L[) \cap H^2(]0,L[), \end{cases}$$

and:

$$\begin{cases} \|z_1\|_{2,0L}^2 = \sum_{n\geq 1} \frac{|W_n(a)|^2}{\lambda_n}, \\ \|z_2\|_{2,0L}^2 = \sum_{n\geq 1} \frac{1}{\lambda_n} \left|\frac{\partial W_n(a)}{\partial x}\right|^2. \end{cases}$$

Finally, from (16), we deduce that:

$$\sum_{n\geq 1} |\alpha_n(t)|^2 \leq c \left[\sum_{n\geq 1} \left(\langle u_0, W_n\rangle^2 \cos^2(\sqrt{\lambda_n} t) + \frac{\langle u_1, W_n\rangle^2}{\lambda_n} \sin^2(\sqrt{\lambda_n} t) \right.\right.$$
$$\left. + \left(A^2\|z_1\|_{2,0L}^2 + B^2\|z_2\|_{2,0L}^2\right) \int_0^t (V_1^2 + V_2^2)(s)ds \right]$$

The first result given in the Theorem is a direct consequence of the previous inequality. But one can also notice that on the one hand:

$$z_1 \in H^3(]0, L[)$$

and that on the other hand:

$$\frac{d^3 z_1}{dx^3}(x) = -\sum_{n\geq 1} \frac{W_n(a)}{\lambda_n} \frac{n^3 \Pi^3}{L^3} \cos\left(\frac{n\Pi x}{L}\right).$$

Then from:

$$\lambda_n = \frac{\mathcal{D}}{2\epsilon\rho} \frac{n^4 \Pi^4}{L^4},$$

we deduce that:

$$\|z_1\|_{3,0L}^2 c \sum_{n\geq 1} \frac{|W_n(a)|^2}{\lambda_n^{\frac{1}{2}}}.$$

Finally, we obtain that (if for instance $B = 0$), $\forall t \in [0, T]$:

$$\sum_{n\geq 1} \alpha_n^2(t) \lambda_n^{\frac{1}{2}} \leq c \left[\|u_0\|_{1,0L}^2 + \|u_1\|_{-1,0L}^2 + \sum_{n\geq 1} \frac{|W_n(a)|^2}{\lambda n^{\frac{1}{2}}} \int_0^t V_1^2(s)ds \right]$$
$$\leq c \left[\|u_0\|_{1,0L}^2 + \|u_1\|_{-1,0L}^2 + \|V_1\|_{L^2(]0,L[)}^2 \right].$$

The continuity with respect to time is deduced directly from the uniform

4. The optimal control problem

Let us consider a smart beam model like the one studied in Section 3. The electrical potentials are the control variables. The solution $u(x,t)$ is the state variable. For (u_0, u_1) and $V = (V_1, V_2)$ given, we set (Section 3):

$$u(x,t) = \sum_{n \geq 1} \alpha_n(t) W_n(x).$$

In order to minimize $u(x,T)$ and $\frac{\partial u}{\partial t}(x,T)$ we define the following criterion with respect to the control variable: $V = (V_1, V_2)$

$$J(V) = \frac{1}{2} \sum_{n \geq 1} \left[\alpha_n(T)^2 + \frac{\dot{\alpha}_n^2(T)}{\lambda_n} \right] + \frac{\epsilon}{2} \int_0^T \left(V_1^2(s) + V_2^2(s) \right) ds \tag{17}$$

where ϵ is an arbitrary (small) real and positive parameter. Then we introduce the control problem:

$$\begin{cases} \text{minimize } J(V), \\ V \in \left(L^2\left( ]O, T[\right) \right)^2. \end{cases} \tag{18}$$

The solution of (18) is such that $u(x,T)$ and $\frac{\partial u}{\partial t}(x,T)$ are smaller and smaller when ϵ tends to zero. But the existence and uniqueness of a solution to (18) is only obvious for $\epsilon > 0$.

It can be obtained, for instance, through a classical theorem, because:
i) $(L^2(]0,T[))^2$ is a Hilbert space,
ii) J is continuous and strictly convex,
iii) J is coercive i.e. $\lim J(V) = +\infty$ when $\|V\|_{(L^2(]0,T[))^2} \to \infty$.

The optimality equation can be easily formulated using an adjoint state function. Because this has been a quite standard method since Pontryagin (1974), we just give the results.

The optimal function $V = (V_1, V_2)$ and the corresponding state function $u(x,t)$ are solution of (it is implicitely assumed that $(x,t) \in]0,L[\times]0,T[$):

$$\begin{cases} 2\epsilon\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{D} \frac{\partial^4 u}{\partial x^4} = A V_1(t) \delta_a(x) + B V_2(t) \delta_a'(x), \forall (x,t), \\ u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \forall t, \\ u(x,0) = u_0(u), \frac{\partial u}{\partial t}(x,0) = u_1(x), \forall x, \\ Ap(a,t) + \epsilon V_1(t) = 0, \forall t, \\ -B \frac{\partial p}{\partial x}(a,t) + \epsilon V_2(t) = 0, \forall t, \end{cases} \tag{19}$$

where $p(x,t)$ is the adjoint state function which is defined as the solution of:

$$\begin{cases} \left(2\epsilon\rho \frac{\partial^2 p}{\partial t^2} + \mathcal{D} \frac{\partial^4 p}{\partial x^4} \right)(x,t) = 0, \forall (x,t), \\ p(0,t) = \frac{\partial^2 p}{\partial x^2}(0,t) = p(L,t) = \frac{\partial^2 p}{\partial x^2}(L,t) = 0, \forall t, \end{cases} \tag{20}$$

The basic point in our analysis is to study the asymptotic behaviour of the previous solution when ϵ goes to zero. Therefore we set:

$$\left\{\begin{array}{l} u = u^0 + \epsilon u^1 + \text{etc.} \\ p = p^0 + \epsilon p^1 + \text{etc.} \\ V = V^0 + \epsilon V^1 + \text{etc.} \end{array}\right. \tag{21}$$

By substituting the expression (21) in (19)-(20) and by equating the terms with the same power in ϵ in the resulting expression, we obtain the following set of relations. It enables characterization of various terms in the asymptotic expansion.

Terms of order zero:

$$\left\{\begin{array}{l} \left(2\epsilon\rho\frac{\partial^2 u^0}{\partial t^2} + \mathcal{D}\frac{\partial^4 u^0}{\partial x^4}\right)(x,t) = AV_1^0(t)\delta_a(x) + BV_2^0(t)\delta_a'(x), \forall (x,t) \\ u^0(0,t) = \frac{\partial^2 u^0}{\partial x^2}(0,t) = u^0(L,t) = \frac{\partial^2 u^0}{\partial x^2}(L,t) = 0, \forall t, \\ u^0(x,0) = u_0(x), \frac{\partial u^0}{\partial t}(x,0) = u_1(x). \end{array}\right. \tag{22}$$

$$Ap^0(a,t) = 0, B\frac{\partial p^0}{\partial x}(a,t) = 0, \forall t, \tag{23}$$

$$\left\{\begin{array}{l} \left(2\epsilon\rho\frac{\partial^2 p^0}{\partial t^2} + \mathcal{D}\frac{\partial^4 p^0}{\partial x^4}\right)(x,t) = 0, \forall (x,t) \\ p^0(x,t) = \sum_{n\geq 1}\frac{\dot{\alpha}_n^0(T)}{\lambda_n}W_n(x), \frac{\partial^0 p}{\partial t}(x,T) = -\sum_{n\geq 1}\alpha_n^0(T)W_n(x), \forall x \\ p^0(0,t) = \frac{\partial^2 p^0}{\partial x^2}(o,t) = p^0(L,t) = \frac{\partial^2 p^0}{\partial x^2}(L,t) = 0, \forall t, \\ \left(u^0(x,t) = \sum_{n\geq 1}\alpha_n^0(t)W_n(x)\right). \end{array}\right. \tag{24}$$

Terms of order 1:

$$\left\{\begin{array}{l} \left(2\epsilon\rho\frac{\partial^2 u^1}{\partial t^2} + D\frac{\partial^4 u^1}{\partial x^4}\right)(x,t) = AV_1^1(t)\delta_a(x) + BV_2^1(t)\delta_a'(x), \forall (x,t), \\ u^1(o,t) = \frac{\partial^2 u^1}{\partial x^2}(0,t) = u^1(L,t) = \frac{\partial^2 u^1}{\partial x^2}(L,t) = 0, \forall t, \\ u^1(x,0) = 0, \frac{\partial u^1}{\partial x}(x,0) = 0, \forall x. \end{array}\right. \tag{25}$$

$$p^1(a,t) + V_1^0(t) = 0, -B\frac{\partial p^1}{\partial x}(a,t) + V_2^0(t) = 0, \forall t, \tag{26}$$

$$\left\{\begin{array}{l} \left(2\epsilon\rho\frac{\partial^2 p^1}{\partial t^2} + \mathcal{D}\frac{\partial^4 p^1}{\partial x^4}\right)(x,t) = 0, \forall (x,t), \\ p^1(0,t) = \frac{\partial^2 p^1}{\partial x^2}(0,t) = p^1(L,t) = \frac{\partial^2 p^1}{\partial x^2}(L,t) = 0, \forall t, \\ p^1(x,T) = \sum_{n\geq 1}\frac{\dot{\alpha}_n^1(T)}{\lambda_n}W_n(x), \\ \frac{\partial p^1}{\partial t}(x,T) = -\sum_{n\geq 1}\alpha_n^1(T)W_n(x), \forall x, \\ \left(u^1(x,t) = \sum_{n\geq 1}\alpha_n^1(t)W_n(x)\right). \end{array}\right. \tag{27}$$

Let us sketch the solution method for this set of equations. As a matter of fact it leads to the well known H.U.M. method suggested and developed by Lions (1988).

A basic point is to prove that for T large enough, the equations (23) and (24) lead to $p^0(x,t) = 0$ and therefore $u^0(x,T) = \frac{\partial u^0}{\partial t}(x,T) = 0$, $\forall x$. But unfortunately the multiplier method cannot be applied directly as it was the case for the problems treated by Lions (1988) (see also Komornik, 1994, for the multiplier method). This difficulty is due to a second order derivative term at the point $x = a$.

4.1. Characterisation of the terms p^0 and u^0

Assuming, for instance, that $u^0(x,T)$ and $\frac{\partial u^0}{\partial t}(x,T)$ are elements of the space $L^2(]0,L[) \times V'$, the solution $p^0(x,t)$ of (24) is also given by:

$$p^0(x,t) = \sum_{n\geq 1} \beta_n(t) W_n(x)$$

where:

$$\beta_n(t) = \frac{\dot{\alpha}_n^0(T)}{\lambda_n} \cos\left[\sqrt{\lambda_n}(t-T)\right] - \frac{\alpha_n^0(T)}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}(t-T)\right)$$

($\alpha_n^0(t)$ are the coefficients of the expansion of u^0 in the basis W_n).

But the structure is harmonic (i.e. the eigenfrequencies are integer multiples of a fundamental one). Hence, setting:

$$T = T_1 = \frac{2\Pi}{\sqrt{\lambda_1}}, \left(\lambda_n = \frac{n^2\Pi^2 \mathcal{D}}{L^2 2\epsilon\rho}\right),$$

and from (23), multiplying successively by $\cos\sqrt{\lambda_n}(t-T)$, $\sin\sqrt{\lambda_n}(t-T)$ and by integrating from 0 to T_1, we deduce that (A and B are assumed to be different from zero):

$$\dot{\alpha}_n^0(T_1) = \alpha_n^0(T_1) = 0.$$

and therefore:

$$p^0(x,t) = 0 \quad \forall (x,t) \in\,]0,L[\, \times\,]0,T_1[\,.$$

A similar property will be discussed in details in the following. Furthermore it is obviously true for any $T \geq T_1$.

4.2. Characterization of terms of order 1 and the H.U.M. algorithm

Our goal is to characterize a solution of (25)-(26) and (27). This is where the

Step 1. Let (ϕ_0, ϕ_1) be two functions in, respectively, the space $H^2(]0, L[) \cap H_0^1(]0, L[)$ and $L^2(]0, L[)$. We define a function – say $\varphi^1(x,t)$ – solution of (analogous to (27)):

$$\left\{ \begin{array}{ll} 2\epsilon\rho\frac{\partial^2\varphi}{\partial t^2} + \mathcal{D}\frac{\partial^2\varphi}{\partial x^4}(x,t) = 0, & \forall(x,t) \\ \varphi(0,t) = \frac{\partial^2\varphi}{\partial x^2}(0,t) = \varphi(L,t) = \frac{\partial^2\varphi}{\partial x^2}(L,t) = 0, & \forall t, \\ \varphi(x,0) = \phi_1(x), \frac{\partial\varphi}{\partial t}(x,0) = \phi_2(x), & \forall x. \end{array} \right. \tag{28}$$

We know from Theorem 1 that:

$$\varphi \in \mathcal{C}^0\left([0,T]; H^2\left(]0,L[\right) \cap H_0^1\left(]0,L[\right)\right) \cap \mathcal{C}^1\left([0,T]; L^2\left(]0,T[\right)\right)$$

Step 2. Let φ^1 and φ^2 be the two solutions of (28) but for two initial conditions – say $\phi^1 = (\phi_1^1, \phi_2^1)$ and $\phi^2 = (\phi_1^2, \phi_2^2)$. Then we define the bilinear form which is obviously symmetrical and positive:

$$\lambda^T(\phi^1, \phi^2) = A^2 \int_0^T \varphi^1(a,t)\varphi^2(a,t)dt + B^2 \int_0^T \frac{\partial\varphi^1}{\partial x}(a,t)\frac{\partial\varphi^2}{\partial x}(a,t)dt. \tag{29}$$

By multiplying (22) by $\varphi^2(x,t)$ and integrating over $]0, L[\times]0, T[$, we obtain:

$$\begin{aligned} &2\epsilon\rho \int_0^T \int_0^L \left(\frac{\partial^2 u^0}{\partial t^2} + \mathcal{D}\frac{\partial^4 u^0}{\partial x^4}\right)\varphi^2(x,t)dxdt \\ &= A\int_0^T V_1^0(t)\varphi^2(a,t)dt - B\int_0^T V_2^0(t)\frac{\partial\varphi^2}{\partial x}(a,t)dt \\ &= -A^2\int_0^T p^1(x,t)\varphi^2(a,t)dt - B^2\int_0^T \frac{\partial p^1}{\partial x}(a,t)\frac{\partial\varphi^2}{\partial x}(a,t)dt. \end{aligned}$$

Finally, using an integration by part:

$$\begin{aligned} &2\epsilon\rho\left[\int_0^L \frac{\partial u^0}{\partial t}(x,0)\varphi^2(x,0)dx - \int_0^L u^0(x,0)\frac{\partial\varphi^2}{\partial t}(x,0)dx\right] \\ &= A^2\int_0^T p^1(x,t)\varphi^2(x,t) + B^2\int_0^T \frac{\partial p^1}{\partial x}(a,t)\frac{\partial\varphi^2}{\partial x}(a,t)dt \end{aligned} \tag{30}$$

Thus we introduce a variational problem:

find $\phi \in \mathcal{V}^*$ such that:

$$\forall\delta\phi \in \mathcal{V}^* \quad \lambda^T(\phi, \delta\phi) = \ell(\delta\phi) \tag{31}$$

where:

$$\ell(\delta\phi) = 2\epsilon\rho\left[\int_0^L u_1(x)\delta\phi_1(x)dx - \int_0^L u_0(x)\delta\phi_2(x)dx\right] \tag{32}$$

and:

$$\delta\phi = (\delta\phi_1, \delta\phi_2).$$

REMARK 4.1 *If (31) has a solution – say ϕ – then the control V^0 defined by:*

$$V_1^0 = -A\varphi(a,t), V_2^0 = B\frac{\partial\varphi}{\partial x}(a,t), \forall t \tag{33}$$

(φ being solution of (28) with the initial condition ϕ), is such that $u^0(x,t)$ satisfies:

$$u^0(x,T) = \frac{\partial u^0}{\partial t}(x,T) = 0.$$

Hence the H.U.M. algorithm consists in solving (31) and then setting V^0 defined by (33). But the analysis of the bilinear form λ^T is not so easy. This is the goal of the next step.

Step 3. Analysis of λ^T and characterization of $\mathcal{V}^*$. Let us first choose for T the value of the fundamental period; i.e. $T = \frac{2\Pi}{\sqrt{\lambda_1}} = T_1$ for which we have proved that $p^0 = 0$.

Furthermore, we assume that $\min(A^2, B^2) > 0$. Then let φ be the solution of (28) for $\phi \in V \times L^2(]0,L[)$. From:

$$\varphi(x,t) = \sum_{n\geq 1}\left[\left(\int_0^L \phi_1(x)W_n(x)dx\right)\cos(\sqrt{\lambda_n}t)\right.$$
$$\left.+\frac{\left(\int_0^L \phi_2(x)W_n(x)dx\right)}{\sqrt{\lambda_n}t}\sin(\sqrt{\lambda_n}t)\right]W_n(x)$$

and because of the orthogonality in $L^2(]0,T[)$ of the harmonic functions, we deduce that:

$$\lambda^{T_1}(\phi,\phi) = \sum_{n\geq 1}\frac{T_1}{2}\left(A^2\sin^2\left(\frac{n\Pi a}{L}\right) + \frac{n^2\Pi^2}{L^2}B^2\cos^2\left(\frac{n\Pi a}{L}\right)\right) \tag{34}$$
$$\left[\left(\int_0^L \phi_1(x)W_n(x)dx\right)^2 + \frac{1}{\lambda_n}\left(\int_0^L \phi_2(x)W_n(x)dx\right)^2\right]$$

Thus, one has in this particular case (see the definition (15)),

$$\begin{cases} \lambda^{T_1}(\phi,\phi) \geq cT_1\min(A^2,B^2)\left(\|\phi_1\|_{0,0L}^2 + \|\phi_2\|_{V^*}^2\right) \\ \lambda^{T_1}(\phi,\phi) \leq cT_1\max(A^2,B^2)\left(\|\phi_1\|_{1,0L}^2 + \|\phi_2\|_{-1,0L}^2\right)\end{cases} \tag{35}$$

where c is a constant (equal to $\epsilon\rho$).

Therefore, λ^{T_1} is a norm in the space $H_0^1(]0,L[) \times H^{-1}(]0,L[)$. Furthermore, it is bilinear and continuous. The completed space with respect to the norm induced by λ^{T_1} is contained in $L^2(]0,L[) \times V'$. Furthermore, it can be character-

Then, depending on the ratio a/L we can define precisely the completed space $\mathcal{V}^*$. It appears, for instance, that if $a/L = r/q$ where q is odd and obviously $0 \leq r \leq q$, then one can choose $A = 0$ and $B \neq 0$ in order to have the coercivity of the bilinear form λ^{T_1} on the space: $L^2(]0, L[) \times V'$. Let us also point out that for $a = 0$ or $a = L$ one has directly:

$$\mathcal{V}^* = H_0^1\left(]O, L[\right) \times H^{-1}\left(]O, L[\right). \tag{36}$$

In this particular situation ($a = 0$ or L), only one control is necessary (the Dirac derivative).

Finally the linear form $\ell(\cdot)$ is continuous on $\mathcal{V}^*$ as far as the initial data satisfy:

$$(u_0, u_1) \in H_0^1\left(]O, L[\right) \times H^{-1}\left(]O, L[\right).$$

Hence we can formulate our conclusions as follows: "if $\min(A^2, B^2) > 0$, the variational equation (31) has a unique solution (and this is even true with $A = 0$ and $a = 0$ or L), as soon as the initial data satisfy: $(u_0, u_1) \in H_0^1\left(]O, L[\right) \times H^{-1}\left(]O, L[\right)$".

Step 4. The controls defined by:

$$\left\{ \begin{array}{l} V_1(t) = -A\varphi(a,t), \forall t \in \left]O, T_1\right[, \\ V_2(t) = B\frac{\partial \varphi}{\partial x}(a,t), \forall t \in \left]O, T_1\right[, \end{array} \right. \tag{37}$$

are such that:

$$u(x, T_1) = \frac{\partial u}{\partial x}(x, T_1) = 0, \forall x \in \left]O, L\right[$$

Because of the definition of $\mathcal{V}^*$ one has $V_1, V_2 \in L^2(]0, T_1[)$.

REMARK 4.2 *For any time larger than T_1 there also exists an exact control.*

Let us summarize the previous results.

THEOREM 4.1 *Let us assume that $T \geq T_1$ and that $\min(A^2, B^2) > 0$. Then for any initial data (u_0, u_1) in the space $H_0^1(]0, L[) \times H^{-1}(]0, L[)$ there exists an exact control – say $V_1(t)$ and $V_2(t)$ – such that:*

$$V_1, V_2 \in L^2(]0, T[), \text{ and } u(x, T) = \frac{\partial u}{\partial t}(x, T) = 0, \quad \forall x \in]0, L[,$$

where u is the solution of:

$$\left\{ \begin{array}{ll} 2\epsilon\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\frac{\partial^4 u}{\partial x^4} = AV_1(t)\delta_a(x) + BV_2(t)\delta_a'(x), & \forall (x,t) \in]0, L[\times]0, T[\\ u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, & \forall t \in]0, T[\end{array} \right.$$

4.3. Convergence of $(u^\epsilon, p^\epsilon, V^\epsilon)$ to $(u^0, 0, V^0)$ when ϵ goes to zero

As a matter of fact, the asymptotic expansion of $(u^\epsilon, p^\epsilon, V^\epsilon)$ with respect to ϵ that we have defined, is formal. A convergence result can nevertheless be proved (and even under more suitable assumptions). This is summarized hereafter.

THEOREM 4.2 *The assumptions are those of Theorem 4.1. Then:*

$$\begin{cases} V^\epsilon \to_{\epsilon\to 0} V^0 & \text{in } (L^2(]0,T[))^2 \\ u^\epsilon \to_{\epsilon\to 0} u^0 & \text{in } \mathcal{C}^1([0,T];H^{-1}(]0,L[)) \cap \mathcal{C}^0([0,T];H_0^1(]0,L[)) \end{cases}$$

($p^\epsilon \to_{\epsilon\to 0} 0$ in the same space as u^ϵ).

Sketch of the proof. First of all let us recall that the H.U.M. control is such that (relation (26) ensures that this exact control is the one which is minimum in $L^2(]0,T[)$ norm):

$$J^\epsilon(V^\epsilon) \le J^\epsilon(V^0) = \frac{\epsilon}{2}\left\|V^0\right\|^2_{(L^2(]0,T[))^2} = \inf_{V\in\mathcal{U}^T}\frac{\epsilon}{2}\left\|V\right\|^2_{(L^2(]O,T[))^2}$$

where $\mathcal{U}^T$ is the subspace of $L^2(]0,T[)$ denoting the controls which are exact at time T (i.e. the solutions u satisfy: $u(x,T) = \frac{\partial u}{\partial t}(x,T) = 0$). Then from (26) one has:

$$p^1(a,t) + AV_1^0(t) = 0, \frac{\partial p^1}{\partial x}(a,t) - BV_2^0(t) = 0, \forall t \in]O,T[\,.$$

It is then clear that V^ϵ is uniformly bounded in $(L^2(]0,T[))^2$ with respect to ϵ. Hence, there is a subsequence – say $V^{\epsilon'}$ – which converges for each component in $L^2(]0,T[)$ weakly to an element V^* of the same space. Then, the corresponding sequence of solutions $u^{\epsilon'}$ to the beam model is a Cauchy sequence in $C^0([0,T];L^2(]0,L[)) \cap \mathcal{C}^1([0,T];V')$ and therefore, it is convergent to u^*. From the inequality $J^\epsilon(V^\epsilon) \le J^\epsilon(V^0)$ we deduce that:

$$u^*(x,T) = \frac{\partial u^*}{\partial t}(x,T) = 0.$$

Thus, $V^* \in \mathcal{U}^\perp$ and because:

$$\|V^*\|_{L^2(]0,T[)^2} \le \lim\inf_{\epsilon'\to 0}\left\|V^{\epsilon'}\right\|_{L^2(]0,T[)^2},$$

we can conclude that $V^* = V^0$ and $u^* = u^0$.

The strong convergence of V^ϵ (the whole sequence because of the uniqueness of V^0!), is finally a direct consequence of:

$$\left\|V^\epsilon - V^0\right\|^2_{L^2(]0,T[)^2} = \|V^\epsilon\|^2_{L^2(]0,T[)^2} + \left\|V^0\right\|^2_{L^2(]0,T[)^2}$$
$$-2\int_0^T V^\epsilon(t)\bullet V^0(t)ds \le 2\left[\left\|V^0\right\|^2_{L^2(]O,T[)^2} - \int_0^T V^\epsilon(t)\bullet V^0(t)ds\right]$$

5. Non harmonic structures and arbitrary boundary conditions

Let us consider now the case of a clamped beam. First of all, the eigenmodes of the structure are now the solution of:

$$\left\{ \begin{array}{l} 2\epsilon\rho\lambda_n W_n(x) = \frac{d^4 W_n}{dx^4}(x), 0 < x < L, \\ W_n(0) = \frac{dW_n}{dx}(0) = W_n(L) = \frac{dW_n}{dx}(L) = 0. \end{array} \right. \tag{38}$$

The normalization condition is, for instance:

$$2\epsilon\rho \int_0^L |W_n(x)|^2 \, dx = 1.$$

Then a simple calculation gives:

$$\left\{ \begin{array}{l} \lambda_n = \frac{\mathcal{D}}{\epsilon\epsilon\rho} \left(\frac{\mu_n}{L}\right)^4 \\ W_n(x) = \Lambda_n \left[\text{ch } \left(\frac{\mu_n x}{L}\right) - \cos\left(\frac{\mu_n x}{L}\right) + \right. \\ \left. \text{cotg } \left(\frac{\mu_n}{2}\right)\left(\sin\left(\frac{\mu_n x}{L}\right) - \text{sh } \left(\mu_n \frac{x}{L}\right)\right)\right] \end{array} \right. \tag{39}$$

where μ_n are the solutions of the equation:

$$\text{ch } \mu_n \cos \mu_n = 1;$$

for a cantilever beam we would have obtained -1 instead of 1 and the expression of W_n would be slightly different.

It is easy to check that:

$$\sqrt{\lambda_n} = K\left[k(n) + 0\left(\frac{1}{n}\right)\right] \quad K \in \mathbf{R}^{+*}, k(n) \in \mathbf{N}^*, \tag{40}$$

where $0\left(\frac{1}{n}\right)$ is equivalent to $\frac{1}{n}$ when n tends to infinity.

A mechanical structure such that (40) is satisfied is called: "Asymptotically harmonic". It is worth noticing that (40) allows multiple eigenvalues. But the beam does not admit multiple eigenvalues. Hence, the following will not care about this possibility, even if the results can be extended to operators such that the multiplicity is upper bounded. Then, the basic property which will be used in the following is summarized in the next lemma. In the formulation given hereafter eigenvalues are implicitely assumed to be simple. Furthermore, such results are very much connected to those of Ingham (1936). Nevertheless, they are explicited briefly in order to simplify the reading of the paper.

Lemma 5.1 *Let λ_n be a sequence of positive real numbers satisfying*

$$\sqrt{\lambda_n} = K\left[k(n) + 0\left(\frac{1}{n}\right)\right]$$

where:

[illegible]

and let us set:

$$T_1 = \frac{2\Pi}{K}.$$

Then there is a constant c which is independent of n and such that for any n and m:

$$\left| \int_0^{T_1} \sin^2\left(\sqrt{\lambda_n} t\right) dt - \frac{T_1}{2} \right| \leq \frac{c}{n},$$

$$\left| \int_0^{T} \cos^2\left(\sqrt{\lambda_n} t\right) dt - \frac{T_1}{2} \right| \leq \frac{c}{n},$$

$$\left| \int_0^{T_1} \cos\left(\sqrt{\lambda_n} t\right) \sin\left(\sqrt{\lambda_m} t\right) dt \right| \leq \frac{c}{\inf(n,m)}$$

$$n \neq m: \quad \left| \int_0^{T_1} \cos\left(\sqrt{\lambda_n} t\right) \cos\left(\sqrt{\lambda_m} t\right) dt \right| \leq \frac{c}{\inf(n,m)}$$

$$n \neq m, \quad \left| \int_0^{T_1} \sin\left(\sqrt{\lambda_n} t\right) \sin\left(\sqrt{\lambda_m} t\right) dt \right| \leq \frac{c}{\inf(n,m)}.$$

Proof. Because the proof is a simple calculation we just give the main lines. Thus let us set:

$$s = Kt, \text{ where } K = \frac{2\Pi}{T_1}.$$

Then from:

$$A = \int_0^{T_1} \sin^2\left(\sqrt{\lambda_n} t\right) dt = \frac{T_1}{2} - \frac{1}{2}\int_0^{2\Pi} \cos\left[2\left(k(n) + 0\left(\frac{1}{n}\right)\right) s\right]$$

we deduce that:

$$A = \frac{T_1}{2} - \frac{1}{2}\int_0^{2\Pi} \cos\left(2k(n)s\right) \cos\left(2 \cdot o\left(\frac{1}{n}\right) s\right) ds$$

$$+\frac{1}{2}\int_0^{2\Pi} \sin\left(2k(n)s\right) \sin\left(2 \cdot o\left(\frac{1}{n}\right) s\right) ds$$

and from standard inequalities:

$$\left| \cos\left(2 \cdot o\left(\frac{1}{n}\right) s\right) - 1 \right| \leq \frac{c_1}{n^2}, \forall s;$$

$$\left| \sin\left(2 \cdot o\left(\frac{1}{n}\right) s\right) \right| \leq \frac{c_2}{n}, \forall s,$$

we obtain the first estimate. The other ones are obtained by a similar method.

5.1. Analysis of the H.U.M. algorithm

Because of the similarity with the previous analysis, the only new point is the coercivity of the bilinear form λ^T. Its definition is the same:

$$\lambda^T(\phi,\phi)=\int_0^T\left[A^2\left|\varphi(a,t)\right|^2(t)+B^2\left|\frac{\partial\varphi}{\partial x}(a,t)\right|^2(t)\right]dt$$

where φ is now the solution of:

$$\left\{\begin{array}{ll} 2\epsilon\rho\frac{\partial^2\varphi}{\partial t^2}+\mathcal{D}\frac{\partial^4\varphi}{\partial x^4}=0, & \forall(x,t)\in]0,L[\times]0,T[,\\ \varphi(0,t)=\frac{\partial\varphi}{\partial x}(0,t)=\varphi(L,t)=\frac{\partial\varphi}{\partial x}(L,t), & \forall t\in]0,T[,\\ \varphi(x,0)=\phi_1(x),\frac{\partial\varphi}{\partial t}(x,0)=\phi_2(x), & \forall x\in]0,L[. \end{array}\right. \tag{41}$$

Let us assume for instance that (see Section 4):

$$\phi=(\phi_1,\phi_2)\in H_0^1(]O,L[)\times H^{-1}(]O,L[).$$

Then the solution $\varphi(x,t)$ can be written:

$$\varphi(x,t)=\sum_{n\geq1}\alpha_n(t)W_n(x)$$

with

$$\alpha_n(t)=\left(\int_0^L\phi_1(x)W_n(x)dx\right)\cos\left(\sqrt{\lambda_n}t\right)+$$

$$\frac{\left(\int_0^L\phi_2(x)W_n(x)dx\right)}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}t\right).$$

Let n_0 be an integer number which will be specified later on.
Then we split φ (and ϕ) into two contributions setting:

$$\phi=\phi_{n_0}+\phi_{cn_0}, \tag{42}$$

and

$$\left\{\begin{array}{l} \phi_{n_0}(x)=\sum_{1\leq n\leq n_0}\left(\int_0^L\phi_1(x)W_n(x)dx,\int_0^L\phi_2(x)W_n(x)dx\right)W_n(x),\\ \varphi_{n_0}(x,t)=\sum_{1\leq n\leq n_0}\alpha_n(t)W_n(x),\\ \varphi(x,t)=\varphi_{n_0}(x,t)+\varphi_{cn_0}(x,t). \end{array}\right. \tag{43}$$

Let us now consider the expression of λ^{T_1} $(T=T_1!)$, and Lemma 5.1:

$$\lambda^{T_1}(\phi \quad \phi)\geq\frac{1}{\;}\left(T-\frac{c}{\;}\right)\sum\left(A^2(W(a))^2+B^2\left(\frac{dW_n}{\;}(a)\right)^2\right) \tag{44}$$

$$\left[\left(\int_0^L \phi_1(x)W_n(x)dx\right)^2 + \frac{1}{\lambda_n}\left(\int_0^L \phi_2(x)W_n(x)dx\right)^2\right].$$

Hence, for $n_0 > \frac{c}{T_1} - 1$, one has ($c_2 > 0$):

$$\lambda^{T_1}(\phi_{cn_0}, \phi_{cn_0}) \geq c_2 \sum_{n>n_0} \left(A^2(W_n(a))^2 + B^2\left(\frac{dW_n}{dx}\right)(a)^2\right) \tag{45}$$

$$\left[\left(\int_0^L \phi_1(x)W_n(x)dx\right)^2 + \frac{1}{\lambda_n}\left(\int_0^L \phi_2(x)W_n(x)dx\right)^2\right].$$

Let us now assume that the eigenvectors W_n satisfy the following property which can be checked directly using the explicit expression of W_n given in (39):

$$(W_n(a))^2 + L^2\left(\frac{dW_n}{dx}(a)\right)^2 \geq c_3 > 0, \forall n. \tag{46}$$

As a matter of fact, the details of this proof are in Destuynder, Santi (1999). Thus, from (43)-(44) we finally obtain (where: $c_4 > 0$):

$$\lambda^{T_1}(\phi_{cn_0}, \phi_{cn_0}) \geq c_4\left(\|\phi_{1cn_0}\|^2_{L^2(]0,L[)} + \|\phi_{2cn_0}\|^2_{H^{-2}(]0,L[)}\right). \tag{47}$$

The inequality (45) proves the coercivity of λ^{T_1} on the space of functions of $L^2(]0,L[) \times H^{-2}(]0,L[)$, but restricted to those which are orthogonal (for each component) to the n_o first eigenvectors.

Let us now use the Cauchy-Schwartz inequality with the bilinear form λ^{T_1} (which is obviously symmetrical and positive). With the terms ϕ_{n_0} and ϕ_{cn_0} as defined in (40) and (41), one has:

$$\lambda^{T_1}(\phi_{n_0}, \phi_{cn_0}) \leq \frac{c}{n_0+1}\left[\lambda^{T_1}(\phi_{n_0}, \phi_{n_0}) + \lambda^{T_1}(\phi_{cn_0}, \phi_{cn_0})\right].$$

Thus, we deduce that:

$$\lambda^{T_1}(\phi, \phi) = \lambda^{T_1}(\phi_{n_0}, \phi_{n_0}) + \lambda^{T_1}(\phi_{cn_0}, \phi_{cn_0}) + 2\lambda^{T_1}(\phi_{n_0}, \phi_{cn_0}) \tag{48}$$

$$\geq \left(1 - \frac{c}{n_0+1}\right)\left[\lambda^{T_1}(\phi_{n_0}, \phi_{n_0}) + \lambda^{T_1}(\phi_{cn_0}, \phi_{cn_0})\right].$$

Finally, the coercivity of λ^{T_1} is proved as soon as it can be done for the finite dimensional space spanned by the n_0-first eigenmodes for each component of ϕ. The norm is of no importance because we consider a finite dimensional space. Thus, it is sufficient to prove that $\sqrt{\lambda^{T_1}(\cdot,\cdot)}$ is a norm on this space.

Castel and Richard (1988). We used Bellman Theorem (see for instance Faurre, Robin, 1984). The method can be presented as follows.

Let $\phi_{n_0} = (\phi_{1n_0}, \phi_{2n_0})$ be a couple of functions in the space spanned by the n_0-first eigenvectors. We set:

$$\varphi_{n_0}(x,t) = \sum_{n=1,n_0} \left[\beta_n^0 \cos(\sqrt{\lambda_n}t) + \frac{\beta_n^1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}t)\right] W_n(x)$$

where:

$$\phi_{n_0} = \sum_{n=1,n_0} (\beta_n^0, \beta_n^1) W_n(x).$$

Then the condition:

$$\lambda^{T_1}(\phi_{n_0}, \phi_{n_0}) = 0$$

is equivalent to:

$$\left\{ \begin{array}{ll} \varphi_{n_0}(a,t) = 0, & \forall t \in [O, T_1], \\ \frac{\partial \varphi_{n_0}}{\partial x}(a,t) = 0, & \forall t \in [O, T_1]. \end{array} \right. \tag{49}$$

Expliciting these relations and their derivatives at $t = 0$, one has:

$$\left\{ \begin{array}{ll} \sum_{n=1,n_0} \beta_n^0 \lambda_n^k W_n(a) = 0, & \forall k = 0, 1, 2, \ldots \text{etc.} \\ \sum_{n=1,n_0} \beta_n^0 \lambda_n^k \frac{dW_n}{dx}(a) = 0, & \forall k = 0, 1, 2, \ldots \text{etc.} \end{array} \right. \tag{50}$$

$$\left\{ \begin{array}{ll} \sum_{n=1,n_0} \beta_n^1 \lambda_n^k W_n(a) = 0, & \forall k = 0, 1, 2, \ldots \text{etc.} \\ \sum_{n=1,n_0} \beta_n^1 \lambda_n^k \frac{dW_n}{dx}(a) = 0, & \forall k = 0, 1, 2, \ldots \text{etc.} \end{array} \right. \tag{51}$$

Because we already mentioned that:

$$(W_n(a))^2 + L^2 \left(\frac{dW_n}{dx}(a)\right)^2 > 0, \forall n,$$

we can conclude from (48)-(49) that ($\lambda_n \neq \lambda_m$ if $n \neq m$):

$$\beta_n^0 = \beta_n^1 = 0 \quad \forall n = 1, n_0. \tag{52}$$

Hence, $\sqrt{\lambda^{T_1}}(\cdot,\cdot)$ is a norm, and finally we proved that the bilinear form is coercive on the space $L^2(]0,L[) \times H^{-2}(]0,L[)$. Because it is also continuous on $H_0^1(]0,L[) \times H^{-1}(]0,L[)$, one can say that the completed space $\mathcal{V}^*$ (see (31)) is between these two. As we did this for the simply supported beam it can be characterized using the zero of the sine and cosine functions.

REMARK 5.1 *Many various examples can be treated by the strategy used in this section. If the eigenvalues have a multiplicity larger or equal to two, it becomes necessary to add another control point at, for instance, $x = b \neq a$. Then, the coordinate b should be chosen such that (if the multiplicity is two):*

$$\det \left| \begin{array}{cc} W_n(a), & W_{n+1}(b) \\ \ldots & \ldots \end{array} \right| \neq 0$$

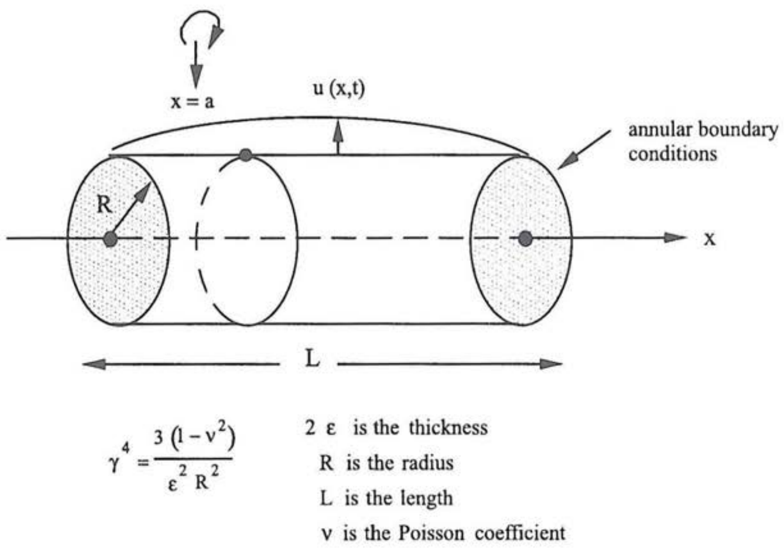

Figure 9. An axisymmetrical and cylindrical shell

REMARK 5.2 *There exists an exact control for $T = T^1$ but also for any time larger than T^1.*

6. Case of an axisymmerical shell

Let us consider the case of a cylindrical shell as shown in Fig. 9. The transverse displacement is denoted by $u(x,t)$ and is solution of:

$$\left\{\begin{array}{l} 2\epsilon\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{D}\left(\frac{\partial^4 u}{\partial x^4} + \gamma^4 u\right) = \\ A\delta_n(x)V_1(t) + B\delta_a'(x)V_2(t) \quad \forall (x,t) \in]0,L[\times]0,T[\\ u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \quad \forall t \in]0,T[\end{array}\right. ,$$

$$u(x,0) = u_0(x), \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad \forall x \in]0,T[.$$

where γ^4 is the so-called Batdorf coefficient of the shell. The eigenvectors are easy to characterize. One has:

$$W_n(x) = \sqrt{\frac{1}{\epsilon\rho L}}\sin\left(\frac{n\Pi x}{L}\right)$$

and the eigenvalues are:

$$\lambda_n = \frac{\mathcal{D}}{2\rho\epsilon}\left(\frac{n^4\Pi^4}{L^4} + \gamma^4\right)$$

The hypothesis of Lemma 5.1 are clearly satisfied and therefore the H.U.M.

7. Conclusion

The exact controllability of several smart beam models has been studied in this paper. The differences between them reside in the boundary conditions. The first point is that we showed how combinations of several piezo-devices make it possible to simulate a pointwise force or a pointwise bending moment. Even a uniformly distributed loading can be generated. In the experimental manufacturing the smart beam will be realized by setting the various piezo wafers at different positions through the thickness of the beam. This will enable avoiding shortcuts. But a Faraday protection is certainly recommended in order to suppress electromagnetic effect between two different devices.

Then, the harmonic behaviour of the spectrum of the beam leads to a simple way of analysis of the H.U.M. method suggested and developed by Lions (1988). It is also shown that the important feature is that we only used the asymptotic behaviour of the spectrum. In the present situation the exact H.U.M. control has also two main advantages. First of all this control is the one which has the minimum $L^2(]0,T[)$ norm. Secondly, certainly a decisive argument in favor of the method of J.L. Lions, is that the "Spillover" is completely avoided for a harmonic structure and very much reduced for an asymptotically harmonic structure. Let us shortly explain why. Let us assume, for instance, that the initial condition of the smart beam model is proportional to the eigenvector w_n. If an optimal control is computed in the one dimensional space spanned by the single mode w_n, then a part of the initial energy is spilled over the other modes even if they had no energy at the initial time. Let us recall that this optimal control is the solution of (18) but the functional J is given by (17) where only the term with the index n is considered in the summation. Nothing guarantees that this optimal control will tend to the H.U.M. control when ϵ tends to zero, but that would be true, as we proved it, if all the terms were taken into account in the definition of J at (17). Besides, because for an harmonic structure, the bilinear form λ^{T_1} is diagonal in the basis of the eigenmodes w_n, all the coefficients of the eigenmodes are equal to zero at time T_1 even if only the contribution of the nth mode is considered (the other ones are zero because the bilinear form λ^{T_1} is diagonal in the basis of the eigen modes). This property is really very important. In the practical implementation of the control procedure, it means that it is not necessary to identify the coefficients of all the other eigenmodes except for the ones (displacement and velocity) of w_n. Furthermore, the computation of this optimal control is analytical (it is a classical exercise). Because only "sine" and "cosine" functions are used in the explicit expression of the H.U.M. control, they can be easily generated using electronic devices in experimental simulations.

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