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Domain decomposition in exact controllability of second order hyperbolic systems on 1-d networks

by

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Abstract: This paper is concerned with domain decomposition in exact controllability of a class of linear second order hyperbolic systems on one-dimensional graphs in $\mathbb{R}^3$ that in particular serve as descriptive models of the dynamics of various multi-link structures consisting of one-dimensional elements, such as networks of Timoshenko beams in $\mathbb{R}^3$. We first consider a standard unconstrained optimal control problem in which the cost functional penalizes the deviation of the final state of the global problem from a given target state. A convergent domain decomposition for the optimality system associated with this problem was recently given by G. Leugering. This decomposition depends on the penalty parameter. On each edge of the graph and at each iteration level the local problem is itself the optimality system associated with an unconstrained optimal control problem in which the cost functional penalizes the deviation of the final state of the particular edge from the target state for that edge. The main purpose of this paper is to show that at each iteration level and on each edge the local optimality system converges as the penalty parameter approaches its limit and that the limit system is a domain decomposition for the problem of norm minimum exact control to the target state.

Keywords: domain decomposition, exact controllability, second order hyperbolic systems, one-dimensional networks.

1. Introduction

This paper is concerned with domain decomposition in exact controllability of a class of linear second order hyperbolic systems on one-dimensional graphs in $\mathbb{R}^3$ that in particular serve as descriptive models of the dynamics of various multi-link structures consisting of one-dimensional elements, such as networks of Timoshenko beams in $\mathbb{R}^3$. Assume that the system is exactly controllable for given target state. The optimality system for this optimal control problem may be constructed by the Hilbert Uniqueness Method, for example. The optimality system lives on the graph and so the components of both the forward running state and the backwards running adjoint state are coupled at the vertices of the graph. Therefore any discretization of the problem will inevitably lead to a very large, highly coupled algebraic systems to be solved.

The purpose of domain decomposition in the context of exact controllability is to approximate the global optimality system by a family of iterative PDE systems, each of which lives on a single edge of the graph and which, in aggregate, converge to the solution of the global optimality system. On each edge of the graph and at each iteration level the corresponding system of equations will be the optimality system associated with some optimal exact controllability problem for that particular edge. Thus, by domain decomposition we approximate the minimum norm control for the global exact controllability problem and the solution of the global optimality system by solving, *in parallel*, a family of local optimality systems on the individual edges of the graph.

To construct an appropriate domain decomposition for the global minimum norm exact controllability problem we proceed as follows. We consider a standard unconstrained optimal control problem in which the cost functional penalizes the deviation of the final state of the global problem from the target state. A convergent domain decomposition for the optimality system associated with this problem was recently given by Leugering (1999). This decomposition depends of course on the penalty parameter. On each edge of the graph and at each iteration level the local problem is itself the optimality system associated with an unconstrained optimal control problem in which the cost functional penalizes the deviation of the final state of the particular edge from the target state for that edge. The main purpose of this paper is to show that at each iteration level and on each edge the local optimality system converges as the penalty parameter approaches its limit and that the limit system is a domain decomposition for the minimum norm exact control problem as described above. Let us mention that the results presented here represent an extension to network problems of a domain decomposition procedure for the computation of the minimum norm exact boundary control in problems of transmission for wave equations, developed in Lagnese and Leugering (2000). The reader may also consult Lagnese and Leugering (2000) and Leugering (1999) and their bibliographies for a discussion of and references to earlier work on domain decomposition in unconstrained, or possibly control constrained, optimal control problems for special systems on 1-d networks and for partial differential equations with constant coefficients in

2. Setting the problem

We consider a simple, connected, oriented graph G in $\mathbb{R}^3$ having n_v vertices and n_e edges, respectively denoted by

$$V = \{v_1, \dots, v_{n_v}\}, \quad E = \{e_1, \dots, e_{n_e}\}.$$

Each edge e_i is parameterized by a smooth, simple path $\pi_i : [0, \ell_i] \mapsto \mathbb{R}^3$. The index set $\mathcal{I}_k$ of a vertex v_k is

$$\mathcal{I}_k = \{i : \pi_i(0) = v_k \text{ or } \pi_i(\ell_i) = v_k\}, \qquad d_k := |\mathcal{I}_k|.$$

If $i \in \mathcal{I}_k$, we define $\varepsilon_{ik} = 1$ if $\pi_i(\ell_i) = v_k$ and $\varepsilon_{ik} = -1$ if $\pi_i(0) = v_k$. We also set $\varepsilon_{ik} = 0$ if $i \notin \mathcal{I}_k$. Set

$$V_M = \{v_k : d_k > 1\}, \quad V_S = V \backslash V_M.$$

Thus V_S is the set of *simple vertices* and V_M contains the *multiple vertices* of the graph. The vertices V_S are further separated into disjoint subsets V_N, V_D, V_C, where the subscripts stand for "Neumann," "Dirichlet," and "controlled." For a vertex $v_k \in V_S$ we write $\mathcal{I}_k = \{i_k\}$.

Let $p \geq 1$. For a function $r : G \mapsto \mathbb{R}^p$, r_i will denote the restriction of r to e_i, that is $r_i = r \circ \pi_i$. If $i \in \mathcal{I}_k$ we write $r_i(v_k)$ for the evaluation of r_i at v_k, that is, $r_i(v_k)$ equals $r_i(0)$ or $r_i(\ell_i)$ depending on whether $\varepsilon_{ik} = -1$ of $+1$. Let $K_i(x)$, $R_i(x)$, $S_i(x)$, $0 \leq x \leq \ell_i$, be $p \times p$ matrices with smooth entries such that K_i and S_i are symmetric on $[0, \ell_i]$, $S_i(x) \geq 0$ and K_i is uniformly positive definite, and define a bilinear form

$$B(r, \phi) = \sum_{i=1}^{n_e} \int_0^{\ell_i} [K_i(r_i' + R_i r_i) \cdot (\phi_i' + R_i \phi_i) + S_i r_i \cdot \phi_i] dx,$$

where $' = \partial/\partial x$.

We set

$$\mathcal{H}^s(0, \ell_i) = \prod_{j=1}^{p} H^s(0, \ell_i),$$

with the product norm, where $H^s(0, \ell_i)$ is the standard Sobolev space of order s on scalar valued functions defined on $(0, \ell_i)$, and introduce Hilbert spaces $(\mathcal{H}, \|\cdot\|)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ as follows.

$$\mathcal{H} = \{\phi : G \mapsto \mathbb{R}^p | \, \phi_i \in \mathcal{H}^0(0, \ell_i)\},$$

$$\|\phi\|^2 = \sum_{i=1}^{n_e} \int_0^{\ell_i} M_i \phi_i \cdot \phi_i \, dx,$$

where $M_i(x)$, $0 \leq x \leq \ell_i$, is symmetric and uniformly positive definite with smooth entries,

$$\mathcal{V} = \{\phi \in \mathcal{H} : \phi_i \in \mathcal{H}^1(0, \ell_i), \ \phi_{i_k}(v_k) = 0 \text{ if } v_k \in V_D,$$

$$\|\phi\|_{\mathcal{V}}^2 = B(\phi, \phi),$$

where C_{ik} is a real, nontrivial $q_k \times p$ matrix of rank q_k, with necessarily $q_k \leq p$. (Note that $C_{ik}\phi_i(v_k) = C_{jk}\phi_j(v_k)$ is a condition only on the components $\Pi_{ik}^{\perp}\phi_i(v_k)$, $i \in \mathcal{I}_k$, where $\Pi_{ik}^{\perp}$ is the orthogonal projection onto the orthogonal complement in $\mathbb{R}^p$ of the kernel of C_{ik}.) For the moment, $\sqrt{B(\phi,\phi)}$ defines only a seminorm on $\mathcal{V}$. However, assumptions that guarantee that this is a norm equivalent to the $\prod_{i=1}^{n_e} \mathcal{H}^1(0, \ell_i)$ norm will be specified shortly. The space $\mathcal{V}$ is dense and compactly embedded in $\mathcal{H}$. The dual space of $\mathcal{V}$ with respect to $\mathcal{H}$ is denoted by $\mathcal{V}^*$.

Fix an interval $\{t : 0 < t < T\}$. We shall consider the variational initial value problem

$$r(t) \in \mathcal{V}, \quad \langle \ddot{r}, \phi \rangle_{\mathcal{V}} + B(r, \phi) = \sum_{k: v_k \in V_C} f_k \cdot \phi_{i_k}(v_k), \qquad (1)$$

$$\forall \phi \in \mathcal{V},\, 0 < t < T,$$

$$r(0) = r_0, \quad \dot{r}(0) = r_1, \qquad (2)$$

where $\dot{} = \partial/\partial t$, $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ denotes the scalar produce in the $\mathcal{V}^* - \mathcal{V}$ duality, and $f_k \in L^2(0, T; \mathbb{R}^p)$. Let Π_{ik} denote the orthogonal projection onto the kernel of C_{ik} and C_{ik}^+ denote the generalized inverse of C_{ik}, that is C_{ik}^+ is a $p \times q_k$ matrix such that

$$C_{ik}C_{ik}^+ = I_k, \quad C_{ik}^+ C_{ik} = \Pi_{ik}^{\perp},$$

where I_k is the $q_k \times q_k$ identity matrix. Set $Q_i = (0, \ell_i) \times (0, T)$. It may be seen that (1) is the variational form of the boundary value problem

$$\begin{array}{l}
M_i \ddot{r}_i = [K_i(r_i' + R_i r_i)]' - R_i^T K_i(r_i' + R_i r_i) - S_i r_i, \ (x,t) \in Q_i, \\
r_i(v_k, t) = 0, \quad v_k \in V_D, \ i = i_k, \\
C_{ik} r_i(v_k, t) = C_{jk} r_j(v_k, t), \quad v_k \in V_M, \ i, j \in \mathcal{I}_k, \\
\varepsilon_{ik}[K_i(r_i' + R_i r_i)](v_k, t) = f_k(t), \quad v_k \in V_C, \ i = i_k, \\
\varepsilon_{ik}[K_i(r_i' + R_i r_i)](v_k, t) = 0, \quad v_k \in V_N, \ i = i_k, \\
\varepsilon_{ik}\Pi_{ik}[K_i(r_i' + R_i r_i)](v_k, t) = 0, \quad v_k \in V_M, \ i \in \mathcal{I}_k, \\
\sum_{i \in \mathcal{I}_k} \varepsilon_{ik} (C_{ik}^+)^T [K_i(r_i' + R_i r_i)](v_k, t) = 0, \quad v_k \in V_M,
\end{array} \qquad (3)$$

where the superscript T on a matrix denotes transpose.

The system (3) serves as a descriptive model of the dynamics of various multi-link structures consisting of one-dimensional elements. Examples include networks of strings in $\mathbb{R}^3$, planar networks of Timoshenko beams with rigid or pinned joints, networks of Timoshenko beams in $\mathbb{R}^3$ with rigid or pinned or ball joints and combinations thereof, networks in $\mathbb{R}^3$ that combine strings with Timoshenko beams, planar networks of precurved Timoshenko beams, among others. The reader is referred to Lagnese, Leugering and Schmidt (1994b) (see also chapter IV, of Lagnese, Leugering and Schmidt, 1994a), where this system

It is proved in Lemma 3.1, Chapter IV, Lagnese, Leugering and Schmidt (1994a), that if $r_0 \in \mathcal{V}$, $r_1 \in \mathcal{H}$, $f_k \in L^2(0,T;\mathbb{R}^p)$, $\forall k : v_k \in \mathcal{V}_C$, and if either

(i) S_i is uniformly positive definite on $[0,\ell_i]$, $i = 1,\dots,n$; or

(ii) $V_D \neq \emptyset$ and for each vertex in V there is a path in G to a vertex in V_D along which the corresponding matrices C_{ik} are invertible,

then $\sqrt{B(\phi,\phi)}$ defines a norm on $\mathcal{V}$ equivalent to the $\prod_{i=1}^{n} \mathcal{H}^1(0,\ell_i)$ norm and (1), (2) has a unique solution with regularity $r \in C([0,T];\mathcal{V})$, $r' \in C([0,T];\mathcal{H})$, $r'' \in C([0,T];\mathcal{V}^*)$. Further, the linear map $(r_0, r_1, \{f_k\}_{k:v_k\in V_C}) \mapsto (r, r', r'')$ is continuous on the indicated spaces. Let us remark that in the case of beam networks, the invertibility of the matrices C_{ik}, $i \in \mathcal{I}_k$, at a vertex v_k is essentially equivalent to the assumption that the beams are rigidly connected at that vertex.

To simplify the presentation somewhat, we assume throughout this paper that (ii) above holds, although this is inessential to the main results. In this case, the bilinear form

$$B_i(\phi,\phi) = \int_0^{\ell_i} [K_i(\phi' + R_i\phi)\cdot(\phi' + R_i\phi) + S_i\phi\cdot\phi]dx$$

defines a norm on $\mathcal{V}_i := \mathcal{H}^1(0,\ell_i)$ equivalent to the $\mathcal{H}^1(0,\ell_i)$ norm. Set $\mathcal{H}_i := \mathcal{H}^0(0,\ell_i)$ with norm defined by $\int_0^{\ell_i} M_i\phi\cdot\phi\,dx$, and let $\mathcal{V}_i^*$ be the dual of $\mathcal{V}_i$ with respect to $\mathcal{H}_i$. $\mathcal{V}$ is a closed subspace of $\prod_{i=1}^{n_e} \mathcal{V}_i$ and if $\mathcal{A}$ (resp., $\mathcal{A}_i$) is the Riesz isomorphism of $\mathcal{V}$ onto $\mathcal{V}^*$ (resp., of $\mathcal{V}_i$ onto $\mathcal{V}_i^*$), we have

$$\mathcal{A}\phi = (\mathcal{A}_1\phi_1,\dots,\mathcal{A}_{n_e}\phi_{n_e}), \quad \forall\phi = (\phi_1,\dots,\phi_{n_e}) \in \mathcal{V}.$$

Henceforth we shall assume (without loss of generality) that $r_0 = r_1 = 0$. Set

$$\mathcal{U} = \prod_{k:v_k\in V_C} L^2(0,T;\mathbb{R}^p).$$

Let $f \in \mathcal{U}$, $(z_0,z_1) \in \mathcal{V}\times\mathcal{H}$, let r be the solution of (1), (2), and define

$$J^\kappa(f) = \frac{1}{2}\sum_{k:v_k\in V_C}\int_0^T |f_k|^2 dt + \frac{\kappa}{2}(\|r(T) - z_0\|_{\mathcal{V}}^2 + \|\dot r(T) - z_1\|^2), \tag{4}$$

where $\kappa > 0$. Consider the optimal control problem

$$\inf_{f\in\mathcal{U}} J^\kappa(f) \tag{5}$$

subject to (1), (2). According to standard arguments, (5) admits a unique optimal control f^{opt} which is given by where $p = \{p_i\}_1^{n_e}$ is the solution of the final value problem for the adjoint system

$$\begin{array}{l} M_i\ddot{p}_i = [K_i(p_i' + R_ip_i)]' - R_i^T K_i(p_i' + R_ip_i) - S_ip_i, \ (x,t) \in Q_i, \\ p_i(v_k,t) = 0, \quad v_k \in V_D, \ i = i_k, \\ C_{ik}p_i(v_k,t) = C_{jk}p_j(v_k,t), \quad v_k \in V_M, \ i,j \in \mathcal{I}_k, \\ \varepsilon_{ik}[K_i(p_i' + R_ip_i)](v_k,t) = 0, \quad v_k \in V_N \cup V_C, \ i = i_k, \\ \varepsilon_{ik}\Pi_{ik}[K_i(p_i' + R_ip_i)](v_k,t) = 0, \quad v_k \in V_M, \ i \in \mathcal{I}_k, \\ \sum_{i\in\mathcal{I}_k} \varepsilon_{ik}(C_{ik}^+)^T[K_i(p_i' + R_ip_i)](v_k,t) = 0, \quad v_k \in V_M, \end{array} \tag{7}$$

$$p(T) = \kappa(\dot{r}(T) - z_1) \in \mathcal{H}, \quad \dot{p}(T) = -\kappa\mathcal{A}(r(T) - z_0) \in \mathcal{V}^*, \tag{8}$$

The solution of (7) is to be understood in the sense of transposition:

$$\begin{aligned} &\langle(-\dot{p}(t), p(t)), (\phi(t), \dot{\phi}(t))\rangle_{\mathcal{V}\times\mathcal{H}} + \sum_{k:v_k\in V_C} \int_t^T g_k(s)p_{i_k}(v_k,s)\,ds. \\ &= (\kappa(\dot{r}(T) - z_1), \phi_1) + \langle\kappa\mathcal{A}(r(T) - z_0), \phi_0\rangle_{\mathcal{V}}, \\ &\forall(\phi_0, \phi_1, g) \in \mathcal{V}\times\mathcal{H}\times\mathcal{U}, \ t \le T, \end{aligned} \tag{9}$$

where ϕ is the unique solution of

$$\begin{array}{l} \langle\ddot{\phi}, \psi\rangle_{\mathcal{V}} + B(\phi,\psi) = \sum_{k:v_k\in V_C} g_k \cdot \psi_{i_k}(v_k), \quad \forall\psi \in \mathcal{V}, \ 0 < t < T, \\ \phi(T) = \phi_0, \quad \dot{\phi}(T) = \phi_1, \end{array} \tag{10}$$

Since the map $(\phi_0, \phi_1, g) \mapsto (\phi(t), \dot{\phi}(t), g)$ is an isomorphism of $\mathcal{V}\times\mathcal{H}\times\mathcal{U}$ onto itself, standard arguments give that (9) has a unique solution with regularity

$$p \in C([0,T];\mathcal{H}) \times C^1([0,T];\mathcal{V}^*), \ p_{i_k}(v_k,\cdot) \in L^2(0,T;\mathbb{R}^p), \ \forall v_k \in V_C. \tag{11}$$

The *optimality system* is therefore

$$r(t) \in \mathcal{V}, \quad \langle\ddot{r}, \phi\rangle_{\mathcal{V}} + B(r,\phi) + \sum_{k:v_k\in V_C} p_{i_k}(v_k,\cdot)\cdot\phi_{i_k}(v_k) = 0,$$

$$\forall\phi \in \mathcal{V}, \ 0 < t < T, \tag{12}$$

$$r(0) = \dot{r}(0) = 0, \tag{13}$$

together with (9) (or (7), (8)).

Now suppose that for each $(z_0, z_1) \in \mathcal{V}\times\mathcal{H}$ the system (1), (2) (with $r_0 = r_1 = 0$) is exactly controllable to (z_0, z_1) for $T > T_0$. This is of course equivalent to the observability assumption

$$\|(\phi_0, \phi_1)\|^2_{\mathcal{H}\times\mathcal{V}^*} \le C_T \sum_{k:v_k\in V_C} \|\phi_{i_k}(v_k,\cdot)\|^2_{L^2(0,T;\mathbb{R}^p)}, \quad T > T_0, \tag{14}$$

where ϕ is the solution of (10) with $g_k = 0$. (We remark that for every $T > 0$ one has the reverse inequality

$$\sum \|\phi_{i_k}(v_k,\cdot)\|^2_{L^2(0,T;\mathbb{R}^p)} \le C_T\|(\phi_0,\phi_1)\|^2_{\mathcal{H}\times\mathcal{V}^*}$$

see Chapter IV, Lemma 3.2, Lagnese, Leugering and Schmidt, 1994a). Then as $\kappa \to \infty$ the solution $r(\cdot\,;\kappa)$, $p(\cdot\,;\kappa)$ of the above optimality system converges in appropriate spaces to the solution of (12), (13), (7), and

$$p(T) = p_0, \quad \dot{p}(T) = p_1, \tag{15}$$

where $(p_0, p_1) \in \mathcal{H} \times \mathcal{V}^*$ is the solution of

$$\langle (p_0, p_1), (-z_1, z_0) \rangle_{\mathcal{H}\times\mathcal{V}} = \sum_{k: v_k \in V_C} \|p_{i_k}(v_k, \cdot)\|^2_{L^2(0,T;\mathbb{R}^p)}, \tag{16}$$

$\langle \cdot, \cdot \rangle_{\mathcal{H}\times\mathcal{V}}$ denoting the inner product in the $\mathcal{H} \times \mathcal{V}^* - \mathcal{H} \times \mathcal{V}$ duality. It is well known that this system is the optimality system for the state constrained optimal control problem

$$\inf_{f \in \mathcal{U}_{\text{ad}}} \sum_{k: v_k \in V_C} \int_0^T |f_k|^2 dt \tag{17}$$

subject to (12), (13), where

$$\mathcal{U}_{\text{ad}} = \{ f \in \mathcal{U} : r(T) = z_0, \; \dot{r}(T) = z_1 \}. \tag{18}$$

The main purpose of this paper is to develop a domain decomposition method for uncoupling this optimality system. Let us note that although what follows is presented in a general framework, the observability estimate (14) is known to hold only for tree graphs in which the root node is in V_D, all of the terminal nodes are in V_C, and $C_{ik} = I_p$, the p-dimensional identity matrix, at each vertex $v_k \in V_M$.

The remainder of the paper is organized as follows. In the next section a domain decomposition due to Leugering (1999) for the optimality system (7), (8), (12), (13), associated with the penalized cost functional (4), is introduced. This decomposition is an iterative procedure in which the coupling conditions at the multiple vertices are replaced by Robin type boundary conditions, thereby replacing the global optimality system by a sequence of *local problems* on the individual elements of the graph. It is proved in Leugering (1999) that the solution of the global optimality system is recovered in the limit of the local solutions as the number of iterations goes to infinity. For each value of the iteration parameter, each local problem is itself the optimality system for a certain local optimal control problem on an edge of the graph. The main contribution of this paper is in Sections 4 and 5, where we investigate the limit of the local iterations (for each fixed iteration parameter) as $\kappa \to \infty$ and prove that the limit iteration represents a decomposition for the optimality system associated with the cost functional (17), that is to say, the local problems obtained in the limit as $\kappa \to \infty$ are themselves optimality systems for certain state constrained local optimal control problems and the solution of the global optimality system is the limit of the solutions of the local problems as the iteration parameter

3. The local optimal control problems and domain decomposition

Consider an edge e_i of the graph G joining vertices v_j and v_k. At least one of these vertices, say v_k, belongs to V_M. For v_j there are four possibilities: $v_j \in V_C$, $v_j \in V_M$, $v_j \in V_N$, $v_j \in V_D$. The form of the local optimal control problem will differ slightly depending on the particular case considered, but the analysis is the same in all cases. We will therefore consider in detail only the first possibility, $v_j \in V_C$, and leave the minor changes needed to treat the others to the reader.

Let λ_{ik}, μ_{ik} be arbitrary $L^2(0,T;\mathbb{R}^p)$ functions. We introduce the cost functional

$$J_i^\kappa(f_1,f_2) = \frac{1}{2}\int_0^T (|f_1|^2 + \frac{1}{\beta}|f_2|^2)dt + \frac{1}{2\beta}\int_0^T |\beta C_{ik} r_i(v_k,\cdot) + \mu_{ik}|^2 dt \qquad (19)$$
$$+\frac{\kappa}{2}(\|r_i(T) - z_{i0}\|^2_{\mathcal{V}_i} + \|\dot r_i(T) - z_{i1}\|^2_{\mathcal{H}_i}),$$

where $\mathcal{H}_i = \mathcal{H}^0(0,\ell_i)$ with norm $\|\rho_i\|_{\mathcal{H}_i} = (\int_0^{\ell_i} M_i\rho_i\rho_i dx)^{1/2}$, $\mathcal{V}_i = \mathcal{H}^1(0,\ell_i)$, $\beta > 0$, and z_{i0}, z_{i1} are the ith components of z_0, z_1. Consider the optimal control problem

$$\inf_{f_1,f_2\in L^2(0,T;\mathbb{R}^p)} J_i^\kappa(f_1,f_2) \qquad (20)$$

subject to the variational problem

$$\langle \ddot r_i, \phi\rangle_{\mathcal{V}_i} + B_i(r_i,\phi) = f_1\cdot\phi(v_j) + (f_2 + \lambda_{ik})\cdot C_{ik}\phi(v_k),\ \forall\phi\in\mathcal{V}_i, \qquad (21)$$

$$r_i(0) = \dot r_i(0) = 0, \qquad (22)$$

where

$$B_i(r_i,\phi) = \int_0^{\ell_i} [K_i(r_i' + R_i r_i)\cdot(\phi' + R_i\phi) + S_i r_i\cdot\phi]dx.$$

The problem (21), (22) has a unique solution with regularity $C([0,T];\mathcal{V}_i)\cap C^1([0,T];\mathcal{H}_i)\cap C^2([0,T];\mathcal{V}_i^*)$. The boundary value problem corresponding to (21) is

$$\begin{aligned}
&M_i\ddot r_i = [K_i(r_i' + R_i r_i)]' - R_i^T K_i(r_i' + R_i r_i) - S_i r_i,\ (x,t)\in Q_i,\\
&\varepsilon_{ij}[K_i(r_i' + R_i r_i)](v_j,t) = f_1(t),\ \varepsilon_{ij}\neq 0,\ v_j\in V_C,\\
&\varepsilon_{ik}\Pi_{ik}[K_i(r_i' + R_i r_i)](v_k,t) = 0,\\
&\varepsilon_{ik}(C_{ik}^+)^T[K_i(r_i' + R_i r_i)](v_k,t) = f_2(t) + \lambda_{ik}(t),\ v_k\in V_M.
\end{aligned} \qquad (23)$$

The local optimal control problem has a unique solution and the optimal controls are given by where p_i is the solution of

$$\begin{aligned} &M_i\ddot{p}_i = [K_i(p_i' + R_ip_i)]' - R_i^T K_i(p_i' + R_ip_i) - S_ip_i, \ (x,t) \in Q_i, \\ &\varepsilon_{ij}[K_i(p_i' + R_ip_i)](v_j,t) = 0, \ \varepsilon_{ij} \neq 0, \ v_j \in V_C, \\ &\varepsilon_{ik}\Pi_{ik}[K_i(p_i' + R_ip_i)](v_k,t) = 0, \\ &\varepsilon_{ik}(C_{ik}^+)^T[K_i(p_i' + R_ip_i)](v_k,t) - \beta C_{ik}r_i(v_k,t) = \mu_{ik}(t), \ v_k \in V_M, \end{aligned} \tag{25}$$

$$p_i(T) = \kappa(\dot{r}_i(T) - z_{i1}), \quad \dot{p}_i(T) = -\kappa\mathcal{A}_i(r_i(T) - z_{i0}), \tag{26}$$

where $\mathcal{A}_i$ is the Riesz isomorphism of $\mathcal{V}_i$ onto $\mathcal{V}_i^*$. The solution p_i of (25), (26) may be taken in the transposition sense:

$$\begin{aligned} &\langle(-\dot{p}_i(t), p_i(t)), (\phi(t), \dot{\phi}(t))\rangle_{\mathcal{V}_i\times\mathcal{H}_i} = \langle(-\dot{p}_i(T), p_i(T)), (\phi^0, \phi^1)\rangle_{\mathcal{V}_i\times\mathcal{H}_i} \\ &\quad - \int_t^T [f_1(t)\cdot p_i(v_j,t) + f_2(t)\cdot C_{ik}p_i(v_k,t)]dt \\ &\quad + \int_t^T (\beta C_{ik}r_i(v_k,t) + \mu_{ik}^\infty)\cdot C_{ik}\phi(v_k,t)dt, \\ &\forall f_1, f_2 \in L^2(0,T;\mathbb{R}^p), \ (\phi^0,\phi^1) \in \mathcal{V}_i\times\mathcal{H}_i, \ 0 \le t \le T, \end{aligned}$$

where ϕ is the solution of

$$\begin{aligned} &\langle\ddot{\phi}, \psi\rangle_{\mathcal{V}_i} + B_i(\phi,\psi) = f_1\cdot\psi(v_j) + f_2\cdot C_{ik}\psi(v_k), \ \forall\psi \in \mathcal{V}_i, \ t < T, \\ &\phi(T) = \phi^0, \quad \dot{\phi}(T) = \phi^1. \end{aligned}$$

To allow for the possibility that v_j belongs to V_D, V_N or to V_M rather than to V_C, we adjoin to (23), (25) the conditions

$$\begin{aligned} &r_i(v_j,t) = p_i(v_j,t) = 0, \ \varepsilon_{ij} \neq 0, \ v_j \in V_D, \\ &\begin{cases} \varepsilon_{ij}[K_i(r_i' + R_ir_i)](v_j,t) = 0, \\ \varepsilon_{ij}[K_i(p_i' + R_ip_i)](v_j,t) = 0, \quad \varepsilon_{ij} \neq 0, \ v_j \in V_N, \end{cases} \\ &\begin{cases} \varepsilon_{ij}(C_{ij}^+)^T[K_i(r_i' + R_ir_i)](v_j,t) + \beta C_{ij}p_i(v_j,t) = \lambda_{ij}(t), \\ \varepsilon_{ij}(C_{ij}^+)^T[K_i(p_i' + R_ip_i)](v_j,t) - \beta C_{ij}r_i(v_j,t) = \mu_{ij}(t), \quad v_j \in V_M. \end{cases} \end{aligned} \tag{27}$$

The local optimality system is therefore (23) - (27).

3.1. Domain decomposition

The idea of domain decomposition is to decouple the global optimality system (7), (8), (12), (13) associated with the penalized cost functional (4) through an iterative procedure in which the coupling conditions at the multiple vertices (the third and last conditions in (3) and (7)) are replaced by the following Robin-type boundary conditions:

$$\begin{aligned} &\varepsilon_{ik}(C_{ik}^+)^T[K_i((r_i^{n+1})' + R_ir_i^{n+1})](v_k,t) + \beta C_{ik}p_i^{n+1}(v_k,t) = \lambda_{ik}^n, \\ &\varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^{n+1})' + R_ip_i^{n+1})](v_k,t) - \beta C_{ik}r_i^{n+1}(v_k,t) = \mu_{ik}^n, \end{aligned}$$

where $\beta > 0$, λ_{ik}^0, μ_{ik}^0 are arbitrarily chosen $L^2(0,T;\mathbb{R}^p)$ functions and for $n \geq 1$

$$\begin{aligned}\lambda_{ik}^n := \frac{2\beta}{d_k}\sum_{j\in\mathcal{I}_k} C_{jk}p_j^n(v_k,t) - \beta C_{ik}p_i^n(v_k,t)\\ -\frac{2}{d_k}\sum_{j\in\mathcal{I}_k}\varepsilon_{jk}(C_{jk}^+)^T[K_j((r_j^n)' + R_j r_j^n)](v_k,t)\\ +\varepsilon_{ik}(C_{ik}^+)^T[K_i((r_i^n)' + R_i r_i^n)](v_k,t),\end{aligned} \tag{29}$$

$$\begin{aligned}\mu_{ik}^n := -\frac{2\beta}{d_k}\sum_{j\in\mathcal{I}_k} C_{jk}r_j^n(v_k,t) + \beta C_{ik}r_i^n(v_k,t)\\ -\frac{2}{d_k}\sum_{j\in\mathcal{I}_k}\varepsilon_{jk}(C_{jk}^+)^T[K_j((p_j^n)' + R_j p_j^n)](v_k,t)\\ +\varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^n)' + R_i p_i^n)](v_k,t).\end{aligned} \tag{30}$$

The decomposition expressed in (28) was introduced by Leugering (1999), who arrived at it through an augmented Lagrangian approach combined with a saddle point iteration. Thus for $i = 1,\ldots,n_e$, one considers the sequence of local problems

$$\begin{aligned}&M_i\ddot{r}_i^{n+1} = [K_i((r_i^{n+1})' + R_i r_i^{n+1})]' - R_i^T K_i((r_i^{n+1})' + R_i r_i^{n+1}) - S_i r_i^{n+1},\\ &M_i\ddot{p}_i^{n+1} = [K_i((p_i^{n+1})' + R_i p_i^{n+1})]' - R_i^T K_i((p_i^{n+1})' + R_i p_i^{n+1}) - S_i p_i^{n+1},\\ &(x,t)\in Q_i,\end{aligned} \tag{31}$$

$$r_i^{n+1}(v_j,t) = p_i^{n+1}(v_j,t) = 0, \quad v_j \in V_D, \tag{32}$$

$$\begin{aligned}&\varepsilon_{ik}[K_i((r_i^{n+1})' + R_i r_i^{n+1})](v_j,t) = -p_i^{n+1}(v_j,t),\\ &\varepsilon_{ik}[K_i((p_i^{n+1})' + R_i p_i^{n+1})](v_j,t) = 0, \quad v_j \in V_C,\end{aligned} \tag{33}$$

$$\begin{aligned}&\varepsilon_{ik}[K_i((r_i^{n+1})' + R_i r_i^{n+1})](v_j,t) = 0,\\ &\varepsilon_{ik}[K_i((p_i^{n+1})' + R_i p_i^{n+1})](v_j,t) = 0,\ v_j \in V_N,\end{aligned} \tag{34}$$

$$\begin{aligned}&\varepsilon_{ik}\Pi_{ik}[K_i((r_i^{n+1})' + R_i r_i^{n+1})](v_k,t) = 0,\\ &\varepsilon_{ik}\Pi_{ik}[K_i((p_i^{n+1})' + R_i p_i^{n+1})](v_k,t) = 0,\ v_k \in V_M,\end{aligned} \tag{35}$$

$$\begin{aligned}&\varepsilon_{ik}(C_{ik}^+)^T[K_i((r_i^{n+1})' + R_i r_i^{n+1})](v_k,t) + \beta C_{ik}p_i^{n+1}(v_k,t) = \lambda_{ik}^n,\\ &\varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^{n+1})' + R_i p_i^{n+1})](v_k,t) - \beta C_{ik}r_i^{n+1}(v_k,t) = \mu_{ik}^n,\\ &v_k \in V_M,\end{aligned} \tag{36}$$

$$r_i^{n+1}(0) = \dot{r}_i^{n+1}(0) = 0.$$

where $z_{i0} = z_0 \circ \pi_i$, $z_{i1} = z_1 \circ \pi_i$.

From the discussion in the previous subsection it can be seen that the above system is the optimality system for the local optimal control problem

$$\inf_{f_1, f_2 \in L^2(0,T;\mathbb{R}^p)} J_i^{n,\kappa}(f_1, f_2) \tag{38}$$

where (when $v_j \in V_C$)

$$J_i^{n,\kappa}(f_1, f_2) = \frac{1}{2}\int_0^T (|f_1|^2 + \frac{1}{\beta}|f_2|^2)dt + \frac{1}{2\beta}\int_0^T |\beta C_{ik} r_i^{n+1}(v_k, \cdot) + \mu_{ik}^n|^2 dt$$
$$+\frac{\kappa}{2}(\|r_i^{n+1}(T) - z_{i0}\|^2_{\mathcal{V}_i} + \|\dot{r}_i^{n+1}(T) - z_{i1}\|^2_{\mathcal{H}_i}),$$

subject to the variational problem

$$\langle \ddot{r}_i^{n+1}, \phi\rangle_{\mathcal{V}_i} + B_i(r_i^{n+1}, \phi) = f_1 \cdot \phi(v_j) + (f_2 + \lambda_{ik}^n) \cdot C_{ik}\phi(v_k),$$
$$\forall \phi \in \mathcal{V}_i, \tag{39}$$

$$r_i^{n+1}(0) = \dot{r}_i^{n+1}(0) = 0, \tag{40}$$

The following convergence result is proved in Leugering (1999)

THEOREM 3.1 *Let $\{r_i, p_i\}_{i=1}^{n_e}$ be the solution of the global optimality system (7), (8), (12), (13), and $\{r_i^{n+1}, p_i^{n+1}\}_{i=1}^{n_e}$ be the solutions of the local optimality systems (31) - (37). Then as $n \to \infty$,*

$$(r_i^n, \dot{r}_i^n) \to (r_i, \dot{r}_i) \text{ in } C([0,T]; \mathcal{V}_i \times \mathcal{H}_i),\ i = 1, \dots, n_e,$$
$$(p_i^n, \dot{p}_i^n) \to (p_i, \dot{p}_i) \text{ in } C([0,T]; \mathcal{H}_i \times \mathcal{V}_i^*),\ i = 1, \dots, n_e,$$
$$p_{i_k}^n(v_k, \cdot) \to p_{i_k}(v_k, \cdot) \text{ strongly in } L^2(0,T;\mathbb{R}^p),\ \forall v_k \in V_C.$$

4. The limit as $\kappa \to \infty$

In this section we study the limit as $\kappa \to \infty$ of the optimality system (31) - (37) *for each fixed value* of the iteration parameter n. We therefore omit the iteration index and begin by considering the limit as $\kappa \to \infty$ of the optimality system (23) - (27) in which it is assumed that the inputs $\lambda_{ik} := \lambda_{ik}(\cdot\,;\kappa)$, $\mu_{ik} := \mu_{ik}(\cdot\,;\kappa)$ depend on κ. We consider in detail only the case $v_j \in V_C$, that is, the system (23) - (26), and comment briefly on the other possibilities $v_j \in V_N$, $v_j \in V_D$, $v_j \in V_M$. The solution of the optimality system is denoted by $r_i(\cdot\,;\kappa)$, $p_i(\cdot\,;\kappa)$. The optimal controls are

$$f_1^\kappa(t) := -p_i^\kappa(v_j, t; \kappa), \quad f_2^\kappa(t) := -\beta C_{ik} p_i^\kappa(v_k, t; \kappa).$$

It is assumed that $\lambda_{ik}(\cdot\,;\kappa)$, $\mu_{ik}(\cdot\,;\kappa)$ satisfy and that the local problem

$$\begin{aligned}&\langle \ddot{r}_i, \psi\rangle_{\mathcal{V}_i} + B_i(r_i,\psi) = f_1\cdot\psi(v_j) + f_2\cdot C_{ik}\psi(v_k),\ \forall\psi\in\mathcal{V}_i,\\ &r_i(0)=\dot{r}_i(0)=0,\end{aligned} \tag{42}$$

$f_1, f_2 \in L^2(0,T;\mathbb{R}^p)$, is exactly controllable to $\mathcal{V}_i\times\mathcal{H}_i$ for $T > T_0$, which is equivalent to the observability assumption

$$\|(\phi_{i0},\phi_{i1})\|^2_{\mathcal{H}_i\times\mathcal{V}_i^*} \le C_T\int_0^T (|\phi_i(v_j,t)|^2 + |C_{ik}\phi_i(v_k,t)|^2)dt \tag{43}$$

for $T > T_0$, where ϕ_i is the solution of

$$\begin{aligned}&\langle \ddot{\phi}_i,\psi\rangle_{\mathcal{V}_i} + B_i(\phi_i,\psi) = 0, \quad \forall\psi\in\mathcal{V}_i,\\ &\phi_i(0)=\phi_{i0}\in\mathcal{V}_i, \quad \dot{\phi}_i(0)=\phi_{i1}\in\mathcal{H}_i.\end{aligned}$$

It is known that for every $T > 0$ the reverse inequality is true:

$$\int_0^T (|\phi_i(v_j,t)|^2 + |C_{ik}\phi_i(v_k,t)|^2)dt \le C_T\|(\phi_{i0},\phi_{i1})\|^2_{\mathcal{H}_i\times\mathcal{V}_i^*}. \tag{44}$$

We remark that if $v_j\in V_N$ the first term on the right hand side of (42) is omitted (i.e., we set $f_1 = 0$), as is the first term in the integral in (43). If $v_j\in V_M$, the first term on the right hand side of (42) is replaced by $f_1\cdot C_{ij}\psi(v_j)$ and the first term in the integral in (43) by $|C_{ij}\phi_i(v_j,t)|^2$. If $v_j\in V_D$ the first term on the right hand side of (42) and the first term in the integral in (43) are dropped and the space $\mathcal{V}_i$ is modified to include the requirement that $\psi(v_j)=0$.

Let $\hat{f}_1$, $\hat{f}_2$ be such that the corresponding solution $\hat{r}_i$ of (42) satisfies

$$\hat{r}_i(T) = z_{i0}, \quad \frac{d\hat{r}_i}{dt}(T) = z_{i1},$$

and set $\hat{f}_2^\kappa(\cdot) := \hat{f}_2(\cdot) - \lambda_{ik}(\cdot\,;\kappa)$. Then

$$\begin{aligned}J_i^\kappa(f_1^\kappa, f_2^\kappa) &\le J_i^\kappa(\hat{f}_1,\hat{f}_2^\kappa) = \frac{1}{2}\int_0^T (|\hat{f}_1(t)|^2 + \frac{1}{\beta}|\hat{f}_2^\kappa(t)|^2)dt\\ &+\frac{1}{2\beta}\int_0^T |\beta C_{ik}\hat{r}_i(v_k,t) + \mu_{ik}(t;\kappa)|^2 dt.\end{aligned}$$

It follows that

$$\begin{aligned}&f_1^\kappa,\ f_2^\kappa,\ C_{ik}r_i(v_k,\cdot;\kappa) \text{ are bounded in } L^2(0,T;\mathbb{R}^p),\\ &\sqrt{\kappa}(r_i(T;\kappa) - z_{i0}) \text{ is bounded in } \mathcal{V}_i,\\ &\sqrt{\kappa}(\dot{r}_i(T;\kappa) - z_{i1}) \text{ is bounded in } \mathcal{H}_i.\end{aligned}$$

From (41) and the boundedness of f_1^κ, f_2^κ we also have

Therefore

$$(r_i(T;\kappa), \dot r_i(T;\kappa)) \to (z_{i0}, z_{i1}) \text{ strongly in } \mathcal{V}_i \times \mathcal{H}_i, \tag{45}$$

and, as $\kappa \to \infty$ through some subnet of $\kappa > 0$,

$$\begin{aligned} & f_1^\kappa \to f_1^\infty, \quad f_2^\kappa \to f_2^\infty \text{ weakly in } L^2(0,T;\mathbb{R}^p), \\ & (r_i(\cdot\,;\kappa), \dot r_i(\cdot\,;\kappa)) \to (r_i(\cdot), \dot r_i(\cdot)) \text{ weakly}^* \text{ in } L^\infty(0,T;\mathcal{V}_i \times \mathcal{H}_i) \\ & \qquad\qquad \text{and strongly in } C([0,T];\mathcal{H}^{1-\epsilon}(0,\ell_i) \times (\mathcal{H}^\epsilon(0,\ell_i))^*), \\ & C_{ik} r_i(v_k,\cdot;\kappa) \to C_{ik} r_i(v_k,\cdot) \text{ strongly in } L^2(0,T;\mathbb{R}^p), \end{aligned} \tag{46}$$

where

$$\langle \ddot r_i, \phi\rangle_{\mathcal{V}_i} + B_i(r_i,\phi) = f_1^\infty \cdot \phi(v_j) + (f_2^\infty + \lambda_{ik}^\infty)\cdot C_{ik}\phi(v_k), \ \forall \phi \in \mathcal{V}_i, \tag{47}$$

$$r_i(0) = \dot r_i(0) = 0, \tag{48}$$

$$r_i(T) = z_{i0}, \quad \dot r_i(T) = z_{i1}. \tag{49}$$

Set

$$\begin{aligned} \mathcal{U}^i_{\text{ad}} = \{ & f_1, f_2 \in L^2(0,T;\mathbb{R}^p) : \text{the solution of (21), (22)} \\ & \text{with } \lambda_{ik} = \lambda_{ik}^\infty \text{ satisfying (49)}\}. \end{aligned}$$

Then $f_1^\infty, f_2^\infty \in \mathcal{U}^i_{\text{ad}}$. For any $f_1, f_2 \in \mathcal{U}^i_{\text{ad}}$ we have

$$\begin{aligned} J_i^\kappa(f_1^\kappa, f_2^\kappa) \le J_i^\kappa(f_1,f_2) = \frac{1}{2}\int_0^T (|f_1(t)|^2 + \frac{1}{\beta}|f_2(t)|^2)dt \\ + \frac{1}{2\beta}\int_0^T |\beta C_{ik}\rho_i(v_k,t;\kappa) + \mu_{ik}(t;\kappa)|^2 dt, \end{aligned}$$

where $\rho_i(\cdot\,;\kappa)$ is the solution of (21), (22) with $\lambda_{ik} = \lambda_{ik}(\cdot\,;\kappa)$. Since $\lambda_{ik}(\cdot\,;\kappa) \to \lambda_{ik}^\infty$ strongly in $L^2(0,T;\mathbb{R}^p)$, it follows that $(\rho_i(\cdot\,;\kappa), \dot\rho_i(\cdot\,;\kappa)) \to (\rho_i(\cdot), \dot\rho_i(\cdot))$ strongly in $C([0,T];\mathcal{V}_i \times \mathcal{H}_i)$, where ρ_i is the solution of

$$\langle \ddot\rho_i, \phi\rangle_{\mathcal{V}_i} + B_i(\rho_i,\phi) = f_1 \cdot \phi(v_j) + (f_2 + \lambda_{ik}^\infty)\cdot C_{ik}\phi(v_k), \ \forall \phi \in \mathcal{V}_i, \tag{50}$$

$$\rho_i(0) = \dot\rho_i(0) = 0, \quad \rho_i(T) = z_{i0}, \ \dot\rho_i(T) = z_{i1}. \tag{51}$$

Therefore

$$\begin{aligned} \limsup_{\kappa\to\infty} J_i^\kappa(f_1^\kappa, f_2^\kappa) \le \frac{1}{2}\int_0^T (|f_1(t)|^2 + \frac{1}{\beta}|f_2(t)|^2)dt \\ + \frac{1}{2\beta}\int_0^T |\beta C_{ik}\rho_i(v_k,t) + \mu_{ik}^\infty|^2 dt, \quad \forall (f_1, f_2) \in \mathcal{U}^i_{ad} \end{aligned} \tag{52}$$

On the other hand, as $\kappa \to \infty$ through a subnet we have

$$
\begin{aligned}
&\int_0^T (|f_1^\infty(t)|^2 + \frac{1}{\beta}|f_2^\infty(t)|^2)dt + \frac{1}{\beta}\int_0^T |\beta C_{ik} r_i(v_k,t) + \mu_{ik}^\infty|^2 dt \\
&\le \liminf \bigg(\textstyle\int_0^T (|f_1^\kappa(t)|^2 + \frac{1}{\beta}|f_2^\kappa(t)|^2)dt \\
&\qquad\qquad + \textstyle\frac{1}{\beta}\int_0^T |\beta C_{ik} r_i(v_k,t;\kappa) + \mu_{ik}(t;\kappa)|^2 dt \bigg) \\
&\le \liminf J_i^\kappa(f_1^\kappa, f_2^\kappa).
\end{aligned} \tag{53}
$$

Therefore

$$
J_i(f_1^\infty, f_2^\infty) \le J_i(f_1, f_2), \quad \forall (f_1,f_2) \in \mathcal{U}_{\mathrm{ad}}^i, \tag{54}
$$

where

$$
\begin{aligned}
J_i(f_1,f_2) &= \frac{1}{2}\int_0^T (|f_1(t)|^2 + \frac{1}{\beta}|f_2(t)|^2)dt \\
&\quad + \frac{1}{2\beta}\int_0^T |\beta C_{ik}\rho_i(v_k,t) + \mu_{ik}^\infty(t)|^2 dt,
\end{aligned} \tag{55}
$$

and where ρ_i is the solution of (50), (51). It follows that (f_1^∞, f_2^∞) is the element in $\mathcal{U}_{\mathrm{ad}}^i$ that minimizes $J_i(f_1,f_2)$ subject to (50), (51). Therefore, the limits in (46) hold as $\kappa \to \infty$ through the entire net $\kappa > 0$, and then (52), (53) imply that $f_1^\kappa \to f_1^\infty$ and $f_2^\kappa \to f_2^\infty$ *strongly* in $L^2(0,T;\mathbb{R}^p)$ as $\kappa \to \infty$. As a consequence $(r_i(\cdot\,;\kappa), \dot r_i(\cdot\,;\kappa)) \to (r_i(\cdot), \dot r_i(\cdot))$ strongly in $C([0,T];\mathcal{V}_i \times \mathcal{H}_i)$ where r_i is the solution of (47)-(49).

Now consider the adjoint state $p_i(\cdot\,;\kappa)$, which we write as $p_i(\cdot\,;\kappa) = q_i(\cdot\,;\kappa) + s_i(\cdot\,;\kappa)$, where $q_i(\cdot\,;\kappa)$ differs from $p_i(\cdot\,;\kappa)$ in that

$$
q_i(T;\kappa) = \dot q_i(T;\kappa) = 0,
$$

instead of (26), and $s_i(\cdot\,;\kappa)$ differs in that

$$
\varepsilon_{ik}(C_{ik}^+)^T[K_i(s_i' + R_i s_i)](v_k,t;\kappa) = 0, \ v_k \in V_M
$$

instead of the fourth equation in (25). For every $T > 0$ we have the estimate

$$
\|(q_i(\cdot\,;\kappa), \dot q_i(\cdot\,;\kappa))\|^2_{C([0,T];\mathcal{V}_i\times\mathcal{H}_i)} \le C_T \int_0^T |\beta C_{ik} r_i(v_k,t;\kappa) + \mu_{ik}(t;\kappa)|^2 dt,
$$

and from the observability assumption (43) we have for $T > T_0$

Therefore

$$\|(p_i(T;\kappa),\dot p_i(T;\kappa))\|^2_{\mathcal{H}_i\times\mathcal{V}_i^*}\le C_T\int_0^T[|(p_i-q_i)(v_j,t;\kappa)|^2$$
$$+|C_{ik}(p_i-q_i)(v_k,t;\kappa)|^2]dt$$
$$\le C_T\int_0^T(|f_1^\kappa(t)|^2+|f_2^\kappa(t)|^2$$
$$+|\beta C_{ik}r_i(v_k,t;\kappa)+\mu_{ik}(t;\kappa)|^2)dt.$$

Applying this inequality to the difference of solutions corresponding to different values of κ, we obtain

$$(p_i(T;\kappa),\dot p_i(T;\kappa))\to(p_{i0},p_{i1})\text{ strongly in }\mathcal{H}_i\times\mathcal{V}_i^*$$

for some $(p_{i0},p_{i1})\in\mathcal{H}_i\times\mathcal{V}_i^*$. Since also

$$\beta C_{ik}r_i(v_k,\cdot;\kappa)+\mu_{ik}(\cdot\,;\kappa)\to\beta C_{ik}r_i(v_k,\cdot)+\mu_{ik}^\infty(\cdot)\text{ strongly in }L^2(0,T;\mathrm{IR}^p),$$

it follows that

$$(p_i(\cdot\,;\kappa),\dot p_i(\cdot\,;\kappa))\to(p_i(\cdot),\dot p_i(\cdot))\text{ strongly in }C([0,T];\mathcal{H}_i\times\mathcal{V}_i^*),$$

where p_i is the solution of

$$\begin{array}{l}M_i\ddot p_i=[K_i(p_i'+R_ip_i)]'-R_i^TK_i(p_i'+R_ip_i)-S_ip_i,\ (x,t)\in Q_i,\\ \varepsilon_{ij}[K_i(p_i'+R_ip_i)](v_j,t)=0,\ \varepsilon_{ij}\ne0,\ v_j\in V_C,\\ \varepsilon_{ik}\Pi_{ik}[K_i(p_i'+R_ip_i)](v_k,t)=0,\\ \varepsilon_{ik}(C_{ik}^+)^T[K_i(p_i'+R_ip_i)](v_k,t)-\beta C_{ik}r_i(v_k,t)=\mu_{ik}^\infty,\ v_k\in V_M,\end{array}\tag{56}$$

$$p_i(T)=p_{i0},\quad\dot p_i(T)=p_{i1}.\tag{57}$$

This solution has the additional regularity

$$p_i(v_j,\cdot),\ C_{ik}p_i(v_k,\cdot)\in L^2(0,T;\mathrm{IR}^p),$$

and we have

$$\begin{array}{l}f_1^\infty(\cdot)=-\lim_{\kappa\to\infty}p_i(v_j,\cdot;\kappa)=-p_i(v_j,\cdot),\\ f_2^\infty(\cdot)=-\lim_{\kappa\to\infty}\beta C_{ik}p_i(v_k,\cdot;\kappa)=-\beta C_{ik}p_i(v_k,\cdot)\end{array}\tag{58}$$

strongly in $L^2(0,T;\mathrm{IR}^p)$. The solution p_i of (56), (57) may be taken in the transposition sense:

$$\langle(-\dot p_i(t),p_i(t)),(\phi(t),\dot\phi(t))\rangle_{\mathcal{V}_i\times\mathcal{H}_i}=\langle(-p_{i1},p_{i0}),(\phi_0,\phi_1)\rangle_{\mathcal{V}_i\times\mathcal{H}_i}$$
$$-\int_t^T[f_1(t)\cdot p_i(v_j,t)+f_2(t)\cdot C_{ik}p_i(v_k,t)]dt$$
$$+\int_t^T(\beta C_{ik}r_i(v_k,t)+\mu_{ik}^\infty)\cdot C_{ik}\phi(v_k,t)dt,\tag{59}$$

where ϕ is the solution of

$$\langle\ddot{\phi},\psi\rangle_{\mathcal{V}_i}+B_i(\phi,\psi)=f_1\cdot\psi(v_j)+f_2\cdot C_{ik}\psi(v_k),\ \forall\psi\in\mathcal{V}_i,\ t<T,$$
$$\phi(T)=\phi_0,\quad \dot{\phi}(T)=\phi_1.$$

If we choose $f_1(\cdot)=-r_i(v_j,\cdot)$, $f_2(\cdot)=-\beta C_{ik}r_i(v_k,\cdot)+\lambda_{ik}^{\infty}$, $\phi_0=z_{i0}$, $\phi_1=z_{i1}$, then $\phi=r_i$ and (59) implies

$$\begin{aligned}\langle(p_{i1},-p_{i0}),(z_{i0},z_{i1})\rangle_{\mathcal{V}_i\times\mathcal{H}_i}=\int_0^T&[|p_i(v_j,t)|^2+\beta(|C_{ik}p_i(v_k,t)|^2\\ &+|C_{ik}r_i(v_k,t)|^2)-\lambda_{ik}^{\infty}\cdot p_i(v_k,t)+\mu_{ik}^{\infty}\cdot r_i(v_k,t)]dt.\end{aligned}\tag{60}$$

We have proven the following result:

THEOREM 4.1 *Suppose that $T>T_0$, that (43) holds and $\lambda_{ik}(\cdot\,;\kappa)\to\lambda_{ik}^{\infty}$, $\mu_{ik}(\cdot\,;\kappa)\to\mu_{ik}^{\infty}$ strongly in $L^2(0,T;I\!R^p)$ as $\kappa\to\infty$. Let $r_i(\cdot\,;\kappa)$, $p_i(\cdot\,;\kappa)$ be the solution of the optimality system (23) - (27) with inputs $\lambda_{ik}(\cdot\,;\kappa)$, $\mu_{ik}(\cdot\,;\kappa)$. Then*

$$\begin{aligned}&(r_i(\cdot\,;\kappa),\dot{r}_i(\cdot\,;\kappa))\to(r_i(\cdot),\dot{r}_i(\cdot))\ \text{in } C([0,T];\mathcal{V}_i\times\mathcal{H}_i),\\ &(p_i(\cdot\,;\kappa),\dot{p}_i(\cdot\,;\kappa))\to(p_i(\cdot),\dot{p}_i(\cdot))\ \text{in } C([0,T];\mathcal{H}_i\times\mathcal{V}_i^*),\\ &p_i(v_j,\cdot;\kappa)\to p_i(v_j,\cdot)\ \text{strongly in } L^2(0,T;I\!R^p)\ \text{if } v_j\in V_C,\\ &C_{ik}p_i(v_k,\cdot;\kappa)\to C_{ik}p_i(v_k,\cdot)\ \text{strongly in } L^2(0,T;I\!R^p),\ v_k\in V_M,\end{aligned}$$

where r_i, p_i satisfy (47) - (49), (56) - (58) with $(p_{i0},p_{i1})\in\mathcal{H}_i\times\mathcal{V}_i$ the unique solution of (60). This latter system is the optimality system for the state constrained optimal control problem

$$\min_{\mathcal{U}^i_{ad}} J_i(f_1,f_2)$$

where J_i is given by (55).

REMARK 4.1 If $v_j\in V_D\cup V_N$, the first term in the integral in (60) is not present and one sets $f_1=0$ in the definition of the cost functional J_i. If $v_j\in V_M$, the first term in the integral in (60) is replaced by

$$\beta(|C_{ij}p_i(v_j,t)|^2+|C_{ij}r_i(v_j,t)|^2)-\lambda_{ij}^{\infty}\cdot p_i(v_j,t)+\mu_{ij}^{\infty}\cdot r_i(v_j,t),$$

and the cost functional J_i is given by

$$\begin{aligned}J_i(f_1,f_2)=\frac{1}{2\beta}\int_0^T&\{|f_1|^2+|f_2|^2+|\beta C_{ik}\rho_i(v_k,t)+\mu_{ik}(t)|^2\\ &+|\beta C_{ij}\rho_i(v_j,t)+\mu_{ij}(t)|^2\}\,dt,\end{aligned}$$

where ρ_i is the solution of

$$\rho_i(0) = \dot\rho_i(0) = 0,$$
$$\rho_i(T) = z_{i0}, \quad \dot\rho_i(T) = z_{i1}.$$

REMARK 4.2 Assume that $v_j \in V_C$ (the other possibilities are handled similarly) and let us comment further on equation (60) for (p_{i0}, p_{i1}), which in fact may be constructed in a manner similar to the Hilbert Uniqueness Method. This pair is chosen so that the solution of the problem

$$\begin{cases} M_i\ddot r_i = [K_i(r_i' + R_i r_i)]' - R_i^T K_i(r_i' + R_i r_i) - S_i r_i, \\ M_i\ddot p_i = [K_i(p_i' + R_i p_i)]' - R_i^T K_i(p_i' + R_i p_i) - S_i p_i, \ (x,t) \in Q_i, \end{cases}$$
$$\begin{cases} \varepsilon_{ij}[K_i(r_i' + R_i r_i)](v_j,t) + p_i(v_j,t) = 0, \\ \varepsilon_{ij}[K_i(p_i' + R_i p_i)](v_j,t) = 0, \ \varepsilon_{ij} \neq 0, \ v_j \in V_C, \end{cases}$$
$$\begin{cases} \varepsilon_{ik}\Pi_{ik}[K_i(r_i' + R_i r_i)](v_k,t) = 0, \\ \varepsilon_{ik}\Pi_{ik}[K_i(p_i' + R_i p_i)](v_k,t) = 0, \ v_k \in V_M, \end{cases} \tag{61}$$
$$\begin{cases} \varepsilon_{ik}(C_{ik}^+)^T[K_i(r_i' + R_i r_i)](v_k,t) + \beta C_{ik} p_i(v_k,t) = \lambda_{ik}^\infty, \\ \varepsilon_{ik}(C_{ik}^+)^T[K_i(p_i' + R_i p_i)](v_k,t) - \beta C_{ik} r_i(v_k,t) = \mu_{ik}^\infty, \ v_k \in V_M, \end{cases}$$
$$r_i(0) = \dot r_i(0) = 0, \quad p_i(T) = p_{i0}, \ \dot p_i(T) = p_{i1}$$

satisfies

$$r_i(T) = z_{i0}, \quad \dot r_i(T) = z_{i1}. \tag{62}$$

For arbitrary $(p_{i0}, p_{i1}) \in \mathcal{V}_i \times \mathcal{H}_i$, (61) has a unique solution (it is the optimality system for an unconstrained LQR problem). This solution may be written $r_i = r_i^0 + r_i^\infty$, $p_i = p_i^0 + p_i^\infty$, where

r_i^0, p_i^0 is the solution corresponding to $\lambda_{ik}^\infty = \mu_{ik}^\infty = 0$,
r_i^∞, p_i^∞ is the solution corresponding to $p_{i0} = p_{i1} = 0$.

For $T > T_0$ we may define a space F_i which is the completion of $\mathcal{V}_i \times \mathcal{H}_i$ with respect to the norm

$$\|(p_{i0}, p_{i1})\|_{F_i} = \left[\int_0^T (|p_i^0(v_j,t)|^2 + |C_{ik}p_i^0(v_k,t)|^2 + |C_{ik}r_i^0(v_k,t)|^2)\, dt\right]^{1/2}.$$

Indeed, if $\|(p_{i0}, p_{i1})\|_{F_i} = 0$ then clearly $r_i^0 = 0$ and then the observability assumption (43) gives $p_{i0} = p_{i1} = 0$. In fact, it may be shown using (43) and (44) that F_i is the same as the space $\mathcal{H}_i \times \mathcal{V}_i^*$ with equivalent norms. In addition, one verifies that

$$\langle (p_{i0}, p_{i1}), (-\dot r_i^0(T), r_i^0(T))\rangle_{\mathcal{H}_i \times \mathcal{V}_i} = \|(p_{i0}, p_{i1})\|_{F_i}^2.$$

Let Λ_i denote the canonical isomorphism $F_i \mapsto F_i^* = \mathcal{H}_i \times \mathcal{V}_i$. Then if $(-\hat z_{i1}, \hat z_{i0}) \in F_i^*$ and if we set we will have

$$r_i^0(T) = \hat{z}_{i0}, \quad \dot{r}_i^0(T) = \hat{z}_{i1}.$$

It follows that (62) will hold if we choose

$$(p_{i0}, p_{i1}) = \Lambda_i^{-1}((-z_{i1}, z_{i0}) - (-\dot{r}_i^\infty(T), r_i^\infty(T))). \tag{63}$$

It may be checked that (63) is the same as (60).

4.1. Application of the Theorem to the optimality system

We assume that

$$\lambda_{ik}^0,\ \mu_{ik}^0 \text{ are independent of } \kappa,\ v_k \in V_M,\ i \in \mathcal{I}_k.$$

Then according to Theorem 4.1 the solution $r_i^1(\cdot\,;\kappa)$, $p_i^1(\cdot\,;\kappa)$ of (31) - (37) corresponding to $n = 0$ converges as $\kappa \to \infty$ in the manner described in Theorem 4.1 to r_i^1, p_i^1, where r_i^1, p_i^1 satisfy (31) - (36) with $n = 0$ and

$$r_i^1(0) = \dot{r}_i^1(0) = 0, \quad r_i^1(T) = z_{i0},\ \dot{r}_i^1(T) = z_{i1},$$
$$p_i^1(T) = p_{i0}^1, \quad \dot{p}_i^1(T) = p_{i1}^1,$$

and where $(p_{i0}^1, p_{i1}^1) \in \mathcal{H}_i \times \mathcal{V}_i^*$ is the solution of (when $v_j \in V_C$)

$$\langle (p_{i1}^1, -p_{i0}^1), (z_{i0}, z_{i1}) \rangle_{\mathcal{V}_i \times \mathcal{H}_i} = \int_0^T [|p_i^1(v_j, t)|^2 + \beta(|C_{ik} p_i^1(v_k, t)|^2$$
$$+ |C_{ik} r_i^1(v_k, t)|^2) - \lambda_{ik}^0 \cdot p_i^1(v_k, t) + \mu_{ik}^0 \cdot r_i^1(v_k, t)]dt.$$

In particular,

$$\begin{aligned} &C_{ik} r_i^1(v_k, \cdot; \kappa) \to C_{ik} r_i^1(\cdot\,;\kappa), \quad C_{ik} p_i^1(v_k, \cdot; \kappa) \to C_{ik} p_i^1(\cdot\,;\kappa), \\ &\quad v_k \in V_M, \end{aligned} \tag{64}$$

strongly in $L^2(0, T; \mathbb{R}^p)$, hence

$$\begin{aligned} &\varepsilon_{ik}(C_{ik}^+)^T[K_i((r_i^1)' + R_i r_i^1)](v_k, \cdot; \kappa) \to \lambda_{ik}^0 - C_{ik} p_i^1(v_k, \cdot) \\ &:= \varepsilon_{ik}(C_{ik}^+)^T[K_i((r_i^1)' + R_i r_i^1)](v_k, \cdot), \\ &\varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^1)' + R_i p_i^1)](v_k, \cdot; \kappa) \to \mu_{ik}^0 + C_{ik} r_i^1(v_k, \cdot) \\ &:= \varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^1)' + R_i p_i^1)](v_k, \cdot) \end{aligned} \tag{65}$$

strongly in $L^2(0, T; \mathbb{R}^p)$. The convergence expressed in (64) and (65) holds for every $v_k \in V_M$ and $i \in \mathcal{I}_k$. It follows from the definitions (29), (30) of λ_{ik}^n, μ_{ik}^n that as $\kappa \to \infty$

$$\lambda_{ik}^1(\cdot\,;\kappa) \to \frac{2}{d_k} \sum_{j \in \mathcal{I}_k} C_{jk} p_j^1(v_k, \cdot) - C_{ik} p_i^1(v_k, \cdot)$$
$$-\frac{2}{d_k} \sum_{j \in \mathcal{I}_k} \varepsilon_{jk}(C_{jk}^+)^T[K_j((r_j^1)' + R_j r_j^1)](v_k, \cdot)$$

$$\mu_{ik}^1(\cdot\,;\kappa) \to -\frac{2}{d_k}\sum_{j\in\mathcal{I}_k} C_{jk}r_j^1(v_k,\cdot) - C_{ik}r_i^1(v_k,\cdot)$$
$$-\frac{2}{d_k}\sum_{j\in\mathcal{I}_k} \varepsilon_{jk}(C_{jk}^+)^T[K_j((p_j^1)' + R_j p_j^1)](v_k,\cdot)$$
$$-\varepsilon_{ik}(C_{ik}^+)^T[K_i((p_i^1)' + R_i p_i^1)](v_k,\cdot) := \mu_{ik}^1(\cdot)$$

strongly in $L^2(0,T;\mathbb{R}^p)$. It follows from Theorem 4.1 that as $\kappa\to\infty$

$$\begin{aligned}
&(r_i^2(\cdot\,;\kappa),\dot r_i^2(\cdot\,;\kappa)) \to (r_i^2(\cdot),\dot r_i^2(\cdot)) \text{ in } C([0,T];\mathcal{V}_i\times\mathcal{H}_i),\\
&(p_i^2(\cdot\,;\kappa),\dot p_i^2(\cdot\,;\kappa)) \to (p_i^2(\cdot),\dot p_i^2(\cdot)) \text{ in } C([0,T];\mathcal{H}_i\times\mathcal{V}_i^*),\\
&p_i^2(v_j,\cdot;\kappa) \to p_i^2(v_j,\cdot) \text{ strongly in } L^2(0,T;\mathbb{R}^p),\ v_j\in V_C,\\
&C_{ik}p_i^2(v_k,\cdot;\kappa) \to C_{ik}p_i^2(v_k,\cdot) \text{ strongly in } L^2(0,T;\mathbb{R}^p),\ v_k\in V_M,
\end{aligned}$$

where r_i^2, p_i^2 satisfies (31) - (36) with $n=1$ and

$$\begin{aligned}
&r_i^2(0) = \dot r_i^2(0) = 0, \quad r_i^2(T) = z_{i0},\ \dot r_i^2(T) = z_{i1},\\
&p_i^2(T) = p_{i0}^2, \quad \dot p_i^2(T) = p_{i1}^2,
\end{aligned}$$

and where $(p_{i0}^2,p_{i1}^2)\in\mathcal{H}_i\times\mathcal{V}_i^*$ is the solution of (when $v_j\in V_C$)

$$\langle(p_{i1}^2,-p_{i0}^2),(z_{i0},z_{i1})\rangle_{\mathcal{V}_i\times\mathcal{H}_i} = \int_0^T [|p_i^2(v_j,t)|^2 + \beta(|C_{ik}p_i^2(v_k,t)|^2$$
$$+|C_{ik}r_i^2(v_k,t)|^2) - \lambda_{ik}^1\cdot p_i^2(v_k,t) + \mu_{ik}^1\cdot r_i^2(v_k,t)]dt.$$

One may now proceed inductively to obtain the following result.

THEOREM 4.2 *Assume that μ_{ik}^0, $\lambda_{ik}^0\in L^2(0,T;\mathbb{R}^p)$ are independent of κ, $\forall k$: $v_k\in V_M$, $i\in\mathcal{I}_k$, and let $r_i^{n+1}(\cdot;\kappa)$, $p_i^{n+1}(\cdot;\kappa)$, $n=0,1,\ldots,$ be the solution of the optimality system (31) - (37). Then as $\kappa\to\infty$,*

$$\begin{aligned}
&(r_i^{n+1}(\cdot\,;\kappa),\dot r_i^{n+1}(\cdot\,;\kappa)) \to (r_i^{n+1}(\cdot),\dot r_i^{n+1}(\cdot)) \textit{ in } C([0,T];\mathcal{V}_i\times\mathcal{H}_i),\\
&(p_i^{n+1}(\cdot\,;\kappa),\dot p_i^{n+1}(\cdot\,;\kappa)) \to (p_i^{n+1}(\cdot),\dot p_i^{n+1}(\cdot)) \textit{ in } C([0,T];\mathcal{H}_i\times\mathcal{V}_i^*),\\
&p_i^{n+1}(v_j,\cdot;\kappa) \to p_i^{n+1}(v_j,\cdot) \textit{ strongly in } L^2(0,T;\mathbb{R}^p),\ v_j\in V_C,\\
&C_{ik}p_i^{n+1}(v_k,\cdot;\kappa) \to C_{ik}p_i^{n+1}(v_k,\cdot) \textit{ strongly in } L^2(0,T;\mathbb{R}^p),\ v_k\in V_M,
\end{aligned}$$

where r_i^{n+1}, p_i^{n+1} satisfies (31) - (36) with λ_{ik}^n, μ_{ik}^n given by (29) and (30), and

$$\begin{aligned}
&r_i^{n+1}(0) = \dot r_i^{n+1}(0) = 0, \quad r_i^{n+1}(T) = z_{i0},\ \dot r_i^{n+1}(T) = z_{i1},\\
&p_i^{n+1}(T) = p_{i0}^{n+1},\ \dot p_i^{n+1}(T) = p_{i1}^{n+1},
\end{aligned} \tag{66}$$

and where $(p_{i0}^{n+1},p_{i1}^{n+1})\in\mathcal{H}_i\times\mathcal{V}_i^*$ is the solution of (when $v_j\in V_C$)

$$\langle(p_{i1}^{n+1},-p_{i0}^{n+1}),(z_{i0},z_{i1})\rangle_{\mathcal{V}_i\times\mathcal{H}_i} = \int_0^T [|p_i^{n+1}(v_j,t)|^2$$
$$+\beta(|C_{ik}p_i^{n+1}(v_k,t)|^2 + |C_{ik}r_i^{n+1}(v_k,t)|^2)$$

5. Convergence

In this section it is proved that the aggregate of local solutions $\{r_i^{n+1}, p_i^{n+1}\}_{i=1}^{n_e}$ of (31) - (36), (66), (67), where λ_{ik}^n, μ_{ik}^n are given by (29) and (30), converges as $n \to \infty$ to the solution $\{r_i, p_i\}_{i=1}^{n_e}$ of the global optimality system (9), (12) - (16) for the global, state constrained optimal control problem (17) subject to (12), (13). The proof follows along the lines of the proof of Theorem 5.1 of Leugering (1999). We first observe that $\{r_i, p_i\}_{i=1}^{n_e}$ satisfy the local boundary conditions (36) at the multiple nodes and therefore

$$\tilde{r}_i^{n+1} := r_i^{n+1} - r_i, \quad \tilde{p}_i^{n+1} := p_i^{n+1} - p_i, \quad n \geq 0,$$

satisfy the local problems (31) - (36) with $\lambda_{ik}^n, \mu_{ik}^n$ replaced by

$$\tilde{\lambda}_{ik}^n = \lambda_{ik}^n - \lambda_{ik}, \quad \tilde{\mu}_{ik}^n = \mu_{ik}^n - \mu_{ik},$$

where

$$\begin{aligned}
\lambda_{ik} &:= \frac{2\beta}{d_k} \sum_{j \in \mathcal{I}_k} C_{jk} p_j(v_k, t) - \beta C_{ik} p_i(v_k, t) \\
&- \frac{2}{d_k} \sum_{j \in \mathcal{I}_k} \varepsilon_{jk} (C_{jk}^+)^T [K_j(r_j' + R_j r_j)](v_k, t) + \varepsilon_{ik} (C_{ik}^+)^T [K_i(r_i' + R_i r_i)](v_k, t),
\end{aligned}$$

$$\begin{aligned}
\mu_{ik} &:= -\frac{2\beta}{d_k} \sum_{j \in \mathcal{I}_k} C_{jk} r_j(v_k, t) + \beta C_{ik} r_i(v_k, t) \\
&- \frac{2}{d_k} \sum_{j \in \mathcal{I}_k} \varepsilon_{jk} (C_{jk}^+)^T [K_j(p_j' + R_j p_j)](v_k, t) + \varepsilon_{ik} (C_{ik}^+)^T [K_i(p_i' + R_i p_i)](v_k, t).
\end{aligned}$$

In addition,

$$\begin{aligned}
&\tilde{r}_i^{n+1}(0) = \tilde{r}_{i,t}^{n+1}(0) = \tilde{r}_i^{n+1}(T) = \tilde{r}_{i,t}^{n+1}(T) = 0, \\
&\tilde{p}_i^{n+1}(T) = p_{i0}^{n+1} - p_{i0} := \tilde{p}_{i0}^{n+1}, \quad \tilde{p}_{i,t}^{n+1}(T) = p_{i1}^{n+1} - p_{i1} := \tilde{p}_{i1}^{n+1},
\end{aligned}$$

where $\phi_{,t} := \dot{\phi}$, (p_0, p_1) is the solution of (16),

$$p_{i0} := p_0 \circ \pi_i \in \mathcal{H}_i, \quad p_{i1} = \mathcal{A}_i[(\mathcal{A}^{-1} p_1) \circ \pi_i] \in \mathcal{V}_i^*.$$

The solution $\tilde{r}_i^{n+1}, \tilde{p}_i^{n+1}$ has the regularity

$$\begin{aligned}
&(\tilde{r}_i^{n+1}, \tilde{r}_{i,t}^{n+1}) \in C([0,T]; \mathcal{V}_i \times \mathcal{H}_i), \\
&(\tilde{p}_i^{n+1}, \tilde{p}_{i,t}^{n+1}) \in C([0,T]; \mathcal{H}_i \times \mathcal{V}_i^*), \\
&p_i^{n+1}(v_j, \cdot),\, C_{ik} \tilde{p}_i^{n+1}(v_k, \cdot) \in L^2(0, T; \mathbb{R}^p),\ v_j \in V_C,\ v_k \in V_M,
\end{aligned}$$

and satisfies (31) - (36) in the following sense:

where

$$\langle \tilde{p}_i^{n+1}, \phi \rangle (v_j, v_k, t) = \begin{cases} -\tilde{p}_i^{n+1}(v_j, t) \cdot \phi(v_j) \\ \qquad -(\beta C_{ik}\tilde{p}_i^{n+1}(v_k, t) - \lambda_{ik}^n(t)) \cdot \phi(v_k), \\ -(\beta C_{ij}\tilde{p}_i^{n+1}(v_j, t) - \lambda_{ij}^n(t)) \cdot \phi(v_j) \\ \qquad -(\beta C_{ik}\tilde{p}_i^{n+1}(v_k, t) - \lambda_{ik}^n(t)) \cdot \phi(v_k), \\ -(\beta C_{ik}\tilde{p}_i^{n+1}(v_k, t) - \tilde{\lambda}_{ik}^n(t)) \cdot \phi(v_k), \end{cases}$$

depending on whether ($v_j \in V_C$, $v_k \in V_M$), ($v_j, v_k \in V_M$), or ($v_j \in V_D \cup V_N$, $v_k \in V_M$), respectively,

$$\begin{aligned} &\langle (-\tilde{p}_{i,t}^{n+1}(t), \tilde{p}_i^{n+1}(t)), (\phi(t), \phi_{,t}(t)) \rangle_{\mathcal{V}_i \times \mathcal{H}_i} = \langle (-\tilde{p}_{i1}^{n+1}, \tilde{p}_{i0}^{n+1}), (\phi_0, \phi_1) \rangle_{\mathcal{V}_i \times \mathcal{H}_i} \\ &- \int_t^T \langle F, \tilde{p}_i^{n+1} \rangle (v_j, v_k, t) dt + \int_t^T (\beta C_{ik}\tilde{r}_i^{n+1}(v_k, t) + \tilde{\mu}_{ik}^n) \cdot C_{ik}\phi(v_k, t)\, dt, \qquad (69) \\ &\quad 0 \le t \le T,\ \forall f_1, f_2 \in L^2(0, T; \mathbb{R}^p),\ (\phi_0, \phi_1) \in \mathcal{V}_i \times \mathcal{H}_i, \end{aligned}$$

where

$$\langle F, \tilde{p}_i^{n+1} \rangle (v_j, v_k, t) = \begin{cases} f_1(t) \cdot \tilde{p}_i^{n+1}(v_j, t) + f_2(t) \cdot C_{ik}\tilde{p}_i^{n+1}(v_k, t), \\ f_1(t) \cdot C_{ij}\tilde{p}_i^{n+1}(v_j, t) + f_2(t) \cdot C_{ik}\tilde{p}_i^{n+1}(v_k, t), \\ f_2(t) \cdot C_{ik}\tilde{p}_i^{n+1}(v_k, t), \end{cases}$$

again depending on whether ($v_j \in V_C$, $v_k \in V_M$), ($v_j, v_k \in V_M$), or ($v_j \in V_D \cup V_N$, $v_k \in V_M$), respectively, and where ϕ is the solution of

$$\begin{aligned} &\langle \ddot{\phi}, \psi \rangle_{\mathcal{V}_i} + B_i(\phi, \psi) = \langle F, \psi \rangle (v_j, v_k, t), \quad \forall \psi \in \mathcal{V}_i \\ &\phi(T) = \phi_0, \quad \dot{\phi}(T) = \phi_1. \end{aligned}$$

The following convergence result will be proven:

THEOREM 5.1 *Assume that $T > T_0$ and the observability assumptions (14) and (43) hold. Then as $n \to \infty$,*

$$\begin{aligned} &(\tilde{r}_i^n, \tilde{r}_{i,t}^n) \to 0 \ \textit{weakly}^* \ \textit{in}\ L^\infty(0, T; \mathcal{V}_i \times \mathcal{H}_i), \\ &(\tilde{p}_i^n, \tilde{p}_{i,t}^n) \to 0 \ \textit{weakly}^* \ \textit{in}\ L^\infty(0, T; \mathcal{H}_i \times \mathcal{V}_i^*), \\ &(\tilde{p}_{i0}^n, \tilde{p}_{i1}^n) \to 0 \ \textit{weakly in}\ \mathcal{H}_i \times \mathcal{V}_i^*, \\ &\tilde{p}_{i_j}^n(v_j, \cdot) \to 0 \ \textit{strongly in}\ L^2(0, T; \mathbb{R}^p). \end{aligned}$$

Proof. We sum (68) and (69) over $i = 1, \dots, n_e$ and obtain

$$\sum_{i=1}^{n_e} [\langle \tilde{r}_{i,tt}^{n+1}, \phi_i \rangle_{\mathcal{V}_i} + B_i(\tilde{r}_i^{n+1}, \phi_i) + \sum_{j: v_j \in V_C} \tilde{p}_{i_j}^{n+1}(v_j, \cdot) \cdot \phi_{i_j}(v_j) \qquad (70)$$

$$+ \sum \sum (\beta C_{ik}\tilde{p}_i^{n+1}(v_k, \cdot) - \tilde{\lambda}_{ik}^n(\cdot)) \cdot C_{ik}\phi_i(v_k) = 0, \ \forall \phi_i \in \mathcal{V}_i$$

$$\sum_{i=1}^{n_e}\langle(-\tilde{p}_{i,t}^{n+1}(t),\tilde{p}_i^{n+1}(t)),(\phi_i(t),\phi_{i,t}(t))\rangle_{\mathcal{V}_i\times\mathcal{H}_i}$$
$$=\sum_{i=1}^{n_e}\langle(-\tilde{p}_{i1}^{n+1},\tilde{p}_{i0}^{n+1}),(\phi_{i0},\phi_{i1})\rangle_{\mathcal{V}_i\times\mathcal{H}_i}$$
$$-\sum_{j:v_j\in V_C}\int_t^T f_{i_j}\cdot\tilde{p}_{i_j}^{n+1}(v_j,t)\,dt-\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_t^T g_{ik}(t)\cdot C_{ik}\tilde{p}_i^{n+1}(v_k,t)\,dt$$
$$+\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_t^T(\beta C_{ik}\tilde{r}_i^{n+1}(v_k,t)+\tilde{\mu}_{ik}^n(t))\cdot C_{ik}\phi_i(v_k,t)\,dt, \tag{71}$$
$$\forall f_{i_j},\,g_{ik}\in L^2(0,T;\mathbb{R}^p),\ (\phi_{i0},\phi_{i1})\in\mathcal{V}_i\times\mathcal{H}_i.$$

Now set $t=0$ in (71) and choose $\phi_{i0}=\phi_{i1}=0$,

$$f_{i_j}(\cdot)=-\tilde{p}_{i_j}^{n+1}(v_j,\cdot),\ v_j\in V_C,$$
$$g_{ik}(\cdot)=-\beta C_{ik}\tilde{p}_i^{n+1}(v_k,\cdot)+\tilde{\lambda}_{ik}^n(\cdot),\ v_k\in V_M.$$

Then $\phi_i=\tilde{r}_i^{n+1}$ and we obtain

$$0=\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^{n+1}(v_j,t)|^2dt+\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_0^T\big[\beta(|C_{ik}\tilde{p}_i^{n+1}(v_k,t)|^2 \tag{72}$$
$$+|C_{ik}\tilde{r}_i^{n+1}(v_k,t)|^2)-\tilde{\lambda}_{ik}^n(t)\cdot C_{ik}\tilde{p}_i^{n+1}(v_k,t)+\tilde{\mu}_{ik}^n(t)\cdot C_{ik}\tilde{r}_i^{n+1}(v_k,t)\big]\,dt.$$

From (36) we have

$$\begin{aligned}&-\tilde{\lambda}_{ik}^n(t)\cdot C_{ik}\tilde{p}_i^{n+1}(v_k,t)=\tfrac{1}{2\beta}|(C_{ik}^+)^T[K_i((\tilde{r}_i^{n+1})'+R_i\tilde{r}_i^{n+1})](v_k,t)|^2\\&\quad-\tfrac{1}{2\beta}|\tilde{\lambda}_{ik}^n(t)|^2-\tfrac{\beta}{2}|C_{ik}\tilde{p}_i^{n+1}(v_k,t)|^2,\\&\tilde{\mu}_{ik}^n(t)\cdot C_{ik}\tilde{r}_i^{n+1}(v_k,t)=\tfrac{1}{2\beta}|(C_{ik}^+)^T[K_i((\tilde{p}_i^{n+1})'+R_i\tilde{p}_i^{n+1})](v_k,t)|^2\\&\quad-\tfrac{1}{2\beta}|\tilde{\mu}_{ik}^n(t)|^2-\tfrac{\beta}{2}|C_{ik}\tilde{r}_i^{n+1}(v_k,t)|^2.\end{aligned}$$

Therefore (72) implies

$$0=2\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^{n+1}(v_j,t)|^2dt$$
$$+\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_0^T\big[\beta(|C_{ik}\tilde{p}_i^{n+1}(v_k,t)|^2+|C_{ik}\tilde{r}_i^{n+1}(v_k,t)|^2)$$
$$+\frac{1}{\beta}\big(|(C_{ik}^+)^T[K_i((\tilde{p}_i^{n+1})'+R_i\tilde{p}_i^{n+1})](v_k,t)|^2$$
$$+|(C_{ik}^+)^T[K_i((\tilde{r}_i^{n+1})'+R_i\tilde{r}_i^{n+1})](v_k,t)|^2\big)-\frac{1}{\beta}(|\tilde{\lambda}_{ik}^n(t)|^2+|\tilde{\mu}_{ik}^n(t)|^2)\big]\,dt,$$

which we write as

$$E^{n+1} = -2 \sum_{j:v_j \in V_C} \int_0^T |\tilde{p}_{i_j}^{n+1}(v_j,t)|^2 dt$$
$$+\frac{1}{\beta} \sum_{k:v_k\in V_M} \sum_{i\in\mathcal{I}_k} \int_0^T (|\tilde{\lambda}_{ik}^n(t)|^2 + |\tilde{\mu}_{ik}^n(t)|^2)\, dt, \tag{73}$$

where

$$E^{n+1} := \sum_{k:v_k\in V_M} \sum_{i\in\mathcal{I}_k} \int_0^T \big[\beta(|C_{ik}\tilde{p}_i^{n+1}(v_k,t)|^2 + |C_{ik}\tilde{r}_i^{n+1}(v_k,t)|^2)$$
$$+\frac{1}{\beta}\big(|(C_{ik}^+)^T[K_i((\tilde{p}_i^{n+1})' + R_i\tilde{p}_i^{n+1})](v_k,t)|^2$$
$$+|(C_{ik}^+)^T[K_i((\tilde{r}_i^{n+1})' + R_i\tilde{r}_i^{n+1})](v_k,t)|^2\big)\big]\, dt.$$

We proceed to calculate the double sum on the right hand side of (73). To simplify the notation we set

$$D_{ik}r_i := \varepsilon_{ik}(C_{ik}^+)^T K_i(r_i' + R_i r_i).$$

Then

$$\tilde{\lambda}_{ik}^n = \frac{2\beta}{d_k} \sum_{j\in\mathcal{I}_k} C_{jk}\tilde{p}_j^n(v_k,t) - \beta C_{ik}\tilde{p}_i^n(v_k,t)$$
$$-\frac{2}{d_k} \sum_{j\in\mathcal{I}_k} D_{jk}\tilde{r}_j^n(v_k,t) + D_{ik}\tilde{r}_i^n(v_k,t).$$

We have

$$|\tilde{\lambda}_{ik}^n(t)|^2 = \frac{4\beta^2}{d_k^2}|\sum_{j\in\mathcal{I}_k} C_{jk}\tilde{p}_j^n(v_k,t)|^2 + \beta^2|C_{ik}\tilde{p}_i^n(v_k,t)|^2$$
$$-\frac{4\beta^2}{d_k} C_{ik}\tilde{p}_i^n(v_k,t)\cdot \sum_{j\in\mathcal{I}_k} C_{jk}\tilde{p}_j^n(v_k,t)$$
$$+\frac{4}{d_k^2}|\sum_{j\in\mathcal{I}_k} D_{jk}\tilde{r}_j^n(v_k,t)|^2 + |D_{ik}\tilde{r}_i^n(v_k,t)|^2 - \frac{4}{d_k} D_{ik}\tilde{r}_i^n(v_k,t)\cdot \sum_{j\in\mathcal{I}_k} D_{jk}\tilde{r}_j^n(v_k,t)$$
$$-\frac{8\beta}{d_k^2} \sum_{j\in\mathcal{I}_k} C_{jk}\tilde{p}_j^n(v_k,t)\cdot \sum_{j\in\mathcal{I}_k} D_{jk}\tilde{r}_j^n(v_k,t) + \frac{4\beta}{d_k} D_{ik}\tilde{r}_i^n(v_k,t)\cdot \sum_{j\in\mathcal{I}_k} C_{jk}\tilde{p}_j^n(v_k,t)$$
$$+\frac{4\beta}{d_k} C_{ik}\tilde{p}_i^n(v_k,t)\cdot \sum_{j\in\mathcal{I}_k} C_{jk}\tilde{r}_j^n(v_k,t) - 2\beta C_{ik}\tilde{p}_i^n(v_k,t)\cdot D_{ik}\tilde{r}_i^n(v_k,t).$$

Therefore

$$\frac{1}{\beta} \sum_{k:v_k\in V_M} \sum_{i\in\mathcal{I}_k} |\tilde{\lambda}_{ik}^n(t)|^2 = \sum_{k:v_k\in V_M} \sum_{i\in\mathcal{I}_k} \Big(\beta|C_{ik}\tilde{p}_i^n(v_k,t)|^2 + \frac{1}{\beta}|D_{ik}\tilde{r}_i^n(v_k,t)|^2 \tag{74}$$

$$-2C_{ik}\tilde{p}_i^n(v_k,t)\cdot D_{ik}\tilde{r}_i^n(v_k,t)\Big).$$

Similarly,

$$\frac{1}{\beta}\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}|\tilde{\mu}_{ik}^n(t)|^2=\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\Big(\beta|C_{ik}\tilde{r}_i^n(v_k,t)|^2+\frac{1}{\beta}|D_{ik}\tilde{p}_i^n(v_k,t)|^2 \quad (75)$$

$$+2C_{ik}\tilde{r}_i^n(v_k,t)\cdot D_{ik}\tilde{p}_i^n(v_k,t)\Big).$$

From (73) - (75) we obtain

$$E^{n+1}=E^n-2\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^{n+1}(v_j,t)|^2dt \tag{76}$$
$$+2\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_0^T(C_{ik}\tilde{r}_i^n\cdot D_{ik}\tilde{p}_i^n-C_{ik}\tilde{p}_i^n\cdot D_{ik}\tilde{r}_i^n)(v_k,t)\,dt.$$

Next, use (71) with n in place of $n+1$, $t=0$ and $\phi_i=\tilde{r}_i^n$. By utilizing the local boundary conditions (36) we obtain

$$0=\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^n(v_j,t)|^2dt$$
$$+\sum_{k:v_k\in V_M}\sum_{i\in\mathcal{I}_k}\int_0^T(C_{ik}\tilde{r}_i^n\cdot D_{ik}\tilde{p}_i^n-C_{ik}\tilde{p}_i^n\cdot D_{ik}\tilde{r}_i^n)(v_k,t)\,dt.$$

It now follows that

$$E^{n+1}=E^n-2\sum_{j:v_j\in V_C}\int_0^T(|\tilde{p}_{i_j}^{n+1}(v_j,t)|^2+|\tilde{p}_{i_j}^n(v_j,t)|^2)\,dt$$

and then, by iteration, that

$$E^{n+1}=E^1-4{\sum_{\ell=1}^{n+1}}'\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^\ell(v_j,t)|^2dt, \tag{77}$$

where ${\sum_{\ell=1}^{n+1}}' a_\ell=(a_1+a_{n+1})/2+\sum_{\ell=2}^n a_\ell$. We deduce from (77) that

$$\sum_{\ell=1}^\infty\sum_{j:v_j\in V_C}\int_0^T|\tilde{p}_{i_j}^\ell(v_j,t)|^2dt<\infty,$$
$\{E^n\}_{n=1}^\infty$ is bounded.

and, on a subsequence,

$$\begin{aligned} &C_{ik}\tilde{r}_i^n(v_k,\cdot) \to \tilde{r}_{ik}^\infty,\ C_{ik}\tilde{p}_i^n(v_k,\cdot) \to \tilde{p}_{ik}^\infty \text{ weakly in } L^2(0,T;\mathbb{R}^p),\\ &D_{ik}\tilde{r}_i^n(v_k,\cdot) \to \tilde{R}_{ik}^\infty,\ D_{ik}\tilde{p}_i^n(v_k,\cdot) \to \tilde{P}_{ik}^\infty \text{ weakly in } L^2(0,T;\mathbb{R}^p) \end{aligned} \tag{79}$$

for some $\tilde{r}_{ik}^\infty, \tilde{p}_{ik}^\infty, \tilde{R}_{ik}^\infty, \tilde{P}_{ik}^\infty \in L^2(0,T;\mathbb{R}^p)$. In fact, it may be proved as in Leugering (1999) that the convergence in (79) is through the entire sequence. Therefore we may pass to the limit in (36) to obtain

$$\tilde{R}_{ik}^\infty + \beta\tilde{p}_{ik}^\infty = \frac{2\beta}{d_k}\sum_{j\in\mathcal{I}_k}\tilde{p}_{jk}^\infty - \beta\tilde{p}_{ik}^\infty - \frac{2}{d_k}\sum_{j\in\mathcal{I}_k}\tilde{R}_{jk}^\infty + \tilde{R}_{ik}^\infty, \tag{80}$$

$$\tilde{P}_{ik}^\infty - \beta\tilde{r}_{ik}^\infty = -\frac{2\beta}{d_k}\sum_{j\in\mathcal{I}_k}\tilde{r}_{jk}^\infty + \beta\tilde{r}_{ik}^\infty - \frac{2}{d_k}\sum_{j\in\mathcal{I}_k}\tilde{P}_{jk}^\infty + \tilde{P}_{ik}^\infty. \tag{81}$$

By summing (80) and (81) over $i \in \mathcal{I}_k$ we find that

$$\begin{aligned} &\textstyle\sum_{i\in\mathcal{I}_k}\tilde{R}_{ik}^\infty = \sum_{i\in\mathcal{I}_k}\tilde{P}_{ik}^\infty = 0,\\ &\tilde{r}_{ik}^\infty = \frac{1}{d_k}\textstyle\sum_{j\in\mathcal{I}_k}\tilde{r}_{jk}^\infty, \quad \tilde{p}_{ik}^\infty = \frac{1}{d_k}\sum_{j\in\mathcal{I}_k}\tilde{p}_{jk}^\infty, \end{aligned} \tag{82}$$

from the third and fourth of which follows that

$$\tilde{r}_{ik}^\infty = \tilde{r}_{jk}^\infty, \quad \tilde{p}_{ik}^\infty = \tilde{p}_{jk}^\infty, \quad \forall i,j \in \mathcal{I}_k. \tag{83}$$

The boundedness of $\{E^n\}$ together with (78) implies that

$$(\tilde{r}_i^n, \tilde{r}_{i,t}^n, \tilde{r}_{i,tt}^n) \text{ is bounded in } C([0,T];\mathcal{V}_i \times \mathcal{H}_i \times \mathcal{V}_i^*).$$

Write $\tilde{p}_i^n = \tilde{q}_i^n + \tilde{s}_i^n$, where $\tilde{q}_i^n$ has homogeneous boundary data and $\tilde{s}_i^n(T) = \tilde{s}_{i,t}^n(T) = 0$. The boundedness of $\{E^n\}$ implies that $(\tilde{s}_i^n, \tilde{s}_{i,t}^n)$ is bounded in $C([0,T];\mathcal{V}_i \times \mathcal{H}_i)$ and, together with the observability assumption (43), that $(\tilde{p}_{i0}^n, \tilde{p}_{i1}^n)$ is bounded in $\mathcal{H}_i \times \mathcal{V}_i^*$. Hence $(\tilde{q}_i^n, \tilde{q}_{i,t}^n)$ is bounded in $C([0,T];\mathcal{H}_i \times \mathcal{V}_i^*)$ and therefore so is $(\tilde{p}_i^n, \tilde{p}_{i,t}^n)$. It follows that, on a subsequence,

$$\begin{aligned} &(\tilde{r}_i^n, \tilde{r}_{i,t}^n) \to (\tilde{r}_i, \tilde{r}_{i,t}) \text{ weakly* in } L^\infty(0,T;\mathcal{V}_i \times \mathcal{H}_i),\\ &(\tilde{p}_i^n, \tilde{p}_{i,t}^n) \to (\tilde{p}_i, \tilde{p}_{i,t}) \text{ weakly* in } L^\infty(0,T;\mathcal{H}_i \times \mathcal{V}_i^*), \end{aligned}$$

Upon passing to the limit in (68) and (69) and utilizing (82) and (83), it is seen that $\tilde{r}_i, \tilde{p}_i$ is a solution (in an appropriate weak sense) of the system

$$\left\{ \begin{array}{l} M_i\tilde{r}_{i,tt} = [K_i(\tilde{r}_i' + R_i\tilde{r}_i)]' - R_i^T K_i(\tilde{r}_i' + R_i\tilde{r}_i) - S_i\tilde{r}_i, \\ M_i\tilde{p}_{i,tt} = [K_i(\tilde{p}_i' + R_i\tilde{p}_i)]' - R_i^T K_i(\tilde{p}_i' + R_i\tilde{p}_i) - S_i\tilde{p}_i, \ (x,t) \in Q_i, \end{array} \right.$$

$$\tilde{r}_i(v_k,t) = \tilde{p}_i(v_k,t) = 0, \quad v_k \in V_D,$$

$$\left\{ \begin{array}{l} \varepsilon_{ij}[K_i(\tilde{r}_i' + R_i\tilde{r}_i)](v_j,t) = 0, \\ \varepsilon_{ij}[K_i(\tilde{p}_i' + R_i\tilde{p}_i)](v_j,t) = 0, \ \varepsilon_{ij} \neq 0, \ v_j \in V_C \cup V_N, \end{array} \right.$$

$$\left\{ \begin{array}{l} \varepsilon_{ik}\Pi_{ik}[K_i(\tilde{r}_i' + R_i\tilde{r}_i)](v_k,t) = 0, \\ \varepsilon_{ik}\Pi_{ik}[K_i(\tilde{p}_i' + R_i\tilde{p}_i)](v_k,t) = 0, \ v_k \in V_M, \end{array} \right.$$

$$\left\{ \begin{array}{l} \varepsilon_{ik}(C_{ik}^+)^T[K_i(\tilde{r}_i' + R_i\tilde{r}_i)](v_k,t) = \tilde{R}_{ik}^\infty, \\ \varepsilon_{ik}(C_{ik}^+)^T[K_i(\tilde{p}_i' + R_i\tilde{p}_i)](v_k,t) = \tilde{P}_{ik}^\infty, \ v_k \in V_M, \end{array} \right.$$

$$\tilde{r}_i(0) = \dot{\tilde{r}}_i(0) = 0, \quad \tilde{p}_i(T) = \tilde{p}_{i0}, \ \dot{\tilde{p}}_i(T) = \tilde{p}_{i1}.$$

From (82) and (83) it follows that $\{\tilde{r}_i\}_{i=1}^{n_e}$ is the solution of the global system (3) with $f_k = 0$ and vanishing initial data. Therefore $\tilde{r}_i \equiv 0$ in Q_i, $i = 1, \dots, n_e$. On the other hand, it is also seen that $\{\tilde{p}_i\}_{i=1}^{n_e}$ is a solution of the global system (7) and satisfies $\tilde{p}_{i_j}(v_j, \cdot) = 0$ for every $j : v_j \in V_C$. The observability assumption (14) then implies that $\tilde{p}_{i0} = \tilde{p}_{i1} = 0$ if $T > T_0$, hence $\tilde{p}_i \equiv 0$ in Q_i, $i = 1, \dots, n_e$.

References

LAGNESE, J. and LEUGERING, G. (2000) Dynamic domain decomposition in approximate and exact boundary control in problems of transmission for wave equations. *SIAM J. Control and Opt.*, to appear.

LAGNESE, J., LEUGERING, G. and SCHMIDT, G. (1994A) *Modeling, Analysis and Control of Dynamic Elastic Mulit-Link Structures.* Birkhäuser, Boston.

LAGNESE, J., LEUGERING, G. and SCHMIDT, G. (1994B) *Analysis and control of hyperbolic systems associated with vibrating networks.* Proc. Royal Acad. Sci. Edinburgh, **124A**, 77-104.

LEUGERING, G. (1999) Dynamic domain decomposition of optimal control problems for networks of strings and beams. *SIAM J. Control and Opt.*, **37**, 1649-1675.