Control and Cybernetics

vol.**28** (1999) No. 3

Uniform stability in structural acoustic systems with thermal effects and nonlinear boundary damping[1]

by

Irena Lasiecka and Catherine Lebiedzik

Department of Mathematics
Kerchof Hall, University of Virginia, Charlottesville, Virginia 22901
e-mail: `il2v@virginia.edu`

Abstract: The stabilization problem for a structural acoustic model is considered. The model under consideration is that of acoustic cavity (wave equation) coupled at the interface with a flexible wall (plate equation) which accounts for thermal effects. It is shown that frictional, nonlinear damping applied at the boundary of the acoustic chamber provides the uniform decay rates for the energy function of the overall structure. The main novelty of this result, with respect to the literature, is that the uniform stability for the model is established *without* assuming any mechanical damping the wall.

Keywords: structural acoustic model, uniform stabilization, thermoelastic plates, nonlinear boundary damping.

1. Introduction

1.1. Description of the problem

This paper deals with stability/stabilizability analysis of structural acoustic problems. Physical motivation for studying these kinds of problems comes from a variety of engineering applications arising in the context of controlling the pressure in a helicopter's cabin or in reducing the noise in an acoustic cavity which is generated by an exterior field. There is a substantial amount of engineering literature dealing with practical aspects of structural acoustic problems, see Crawley and de Luis (1987), Fuller, Gibbs and Silcox (1990). The formulation of structural acoustic models as a wave and plate/beam equation coupled at the interface goes back to Morse and Ingard (1968), Beale (1976), and references therein. More recently, structural acoustic models have attracted

[1]Research partially supported by a NSF Grant DMS-9504822 and Army Research Grant ARO: DAAH04-96-1-0059.

considerable attention, particularly in the context of numerical computations and experimental studies, Banks, Silcox and Smith (1993), Banks, Smith and Wang (1995). The mathematical control theory of these problems has started to develop quickly in response to the great degree of interest.

A structural acoustic interaction is typically modeled by a coupled system of equations which describes (i) the acoustic medium in a given three dimensional chamber (wave equation), and (ii) the structure (plate equation) representing a flexible (vibrating) wall of the chamber. The coupling between the wave equation (acoustic chamber) and plate equation (wall) provides the essential mechanism for control of the system. The properties of the system depend on the type of coupling and of the type of the model used for a wall. For example, if the wall is assumed *structurally* damped (e.g. Kelvin-Voight damping), the equation governing its behavior is parabolic in nature, Avalos and Lasiecka (1996). The coupling of the parabolic (wall) and hyperbolic (wave) equations yields a system whose dynamics are related to an analytic semigroup, Avalos and Lasiecka (1996, 1997a). In this case, the plate equation alone is exponentially stable and the entire coupled structure is *strongly* stable, but not *uniformly* stable, Avalos and Lasiecka (1998b). In order to secure uniform stability, an additional damping mechanism must be introduced on the wave component. In fact, it was shown in Avalos (1996) and Fahroo and Wang (1999) that the *structurally damped* walls of an acoustic chamber with viscous boundary damping acting on the boundary of the acoustic medium yields exponential stability for the overall system. This result was later extended to include a nonlinear boundary damping by Avalos and Lasiecka (1997a). A much more difficult situation is when there is *no structural damping on the wall* (the situation most often met in practical applications). In this case, there is no natural "damping" mechanism either on the wall or in the acoustic chamber. In order to obtain uniform decay rates, one needs to introduce damping in both components of the structure. The most challenging case is that of *boundary* damping (rather than the interior damping). In Camurdan and Triggiani (1997) and Camurdan (1999), it was shown that a "hyperbolic" structural acoustic model with *boundary* damping in the acoustic medium and *boundary* damping via full hinged boundary conditions applied to the plate model is uniformly stable. In Lasiecka (1999) a similar result was shown to hold with *nonlinear* damping applied to an edge of the plate and affecting the shears only (rather than shears and moments). A common feature of all these results is that, in the case of absence of structural damping, a damping mechanism in the acoustic medium *and* on the wall is needed.

The main novelty of our contribution is that we do not assume any source of structural (e.g. Kelvin-Voight type) damping on the wall nor we require any dissipation on the boundary of the wall. Thus, the wall is mechanically undamped. Our aim is to show that thermal effects and boundary dissipation affecting the acoustic medium only uniformly stabilize the overall system. To accomplish this goal we shall use differential multipliers developed in the context of stability analysis for the wave equation, Lasiecka (1999), together with the

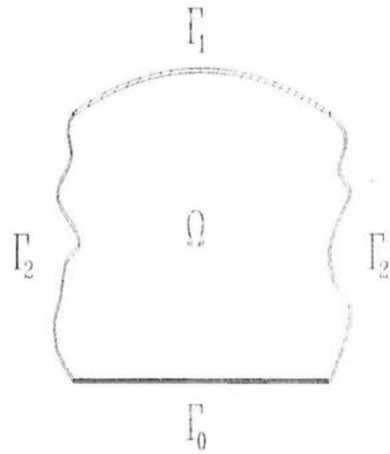

Figure 1. Cross-section of the domain Ω.

operator multiplier method introduced in Avalos and Lasiecka (1997b, 1998a). The newly developed "sharp" trace theory developed for the wave and plate equations plays a critical role in our arguments.

1.2. The PDE model

The structural acoustic interaction considered in this paper is governed by a coupled system of equations which describes (i) the acoustic medium in a given three dimensional chamber (wave equation) and (ii) the thermal structure (plate equation) representing a flexible (vibrating) wall of the chamber. The interaction between the two media takes place on the boundary (interface) between the acoustic chamber and the structure (flexible wall). This leads to a mathematical model of a coupled wave equation with thermoelastic plate equation.

Let $\Omega \subset \mathbf{R}^3$ be an open bounded domain with sufficiently smooth (say C^2) boundary Γ and ν denote an outward unit normal vector to Γ. The boundary Γ consists of three connected regions Γ_0, Γ_1,and Γ_2 - see Fig. (1). The pressure in the chamber (acoustic medium) is defined on a spatial domain Ω, while the displacement of the flexible walls is defined on Γ_0. Γ_1 represents a "hard" wall, which is assumed C^2 and convex (this is to say that the corresponding level set function is C^2 and convex), while Γ_2 allows for the possibility of having "frictional" walls subject to some damping. This damping is represented by absorbing Neumann boundary conditions. We do not assume any geometric restrictions imposed on Γ_2.

In addition to classical notation used for Sobolev's spaces we shall use the following:

with inner product

$$(\omega_1, \omega_2)_{H^1_\gamma(\Gamma_0)} \equiv (\omega_1, \omega_2)_{L^2(\Gamma_0)} + \gamma(\nabla\omega_1, \nabla\omega_2)_{L^2(\Gamma_0)} \quad \forall \omega_1, \omega_2 \in H^1_\gamma(\Gamma_0) \qquad (1)$$

The PDE model considered consists of the wave equation in the variable z (where the quantity ρz_t is the acoustic pressure, and ρ is the density of the fluid)

$$\begin{aligned} & z_{tt} = c^2 \Delta z \quad \text{in} \quad \Omega \times (0,\infty) \\ & \frac{\partial}{\partial \nu} z = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty); \quad \frac{\partial}{\partial \nu} z + dz = -g(z_t) \quad \text{on} \quad \Gamma_2 \times (0,\infty); \\ & \frac{\partial}{\partial \nu} z = -g(z_t) + w_t \quad \text{on} \quad \Gamma_0 \times (0,\infty) \\ & z(0) = z_0 \in H^1(\Omega), \qquad z_t(0) = z_1 \in L^2(\Omega); \end{aligned} \qquad (2)$$

and the elastic equation representing the displacement of the wall w subject to thermal effects (see, e.g., Lagnese, 1989):

$$\left. \begin{aligned} & w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w = -\Delta\theta - \rho z_t \\ & \theta_t - \Delta \theta = \Delta w_t \end{aligned} \right\} \quad \text{on} \quad \Gamma_0 \times (0,\infty) \qquad (3)$$

We shall consider various boundary conditions associated with the plate model. Let $\tilde{\nu}$ denote an outward unit normal to $\partial\Gamma_0$ and $\tilde{\tau}$ denote tangential direction.

Case 1 – hinged BC

$$w = \Delta w = 0; \theta = 0; \text{ on } \partial\Gamma_0 \times (0,\infty) \qquad (4)$$

Case 2 – clamped BC

$$w = \frac{\partial}{\partial \tilde{\nu}} w = 0; \theta = 0; \text{ on } \partial\Gamma_0 \times (0,\infty) \qquad (5)$$

Case 3 – free BC

$$\left. \begin{aligned} & \Delta w + (1-\mu) B_1 w + \theta = 0 \\ & \tfrac{\partial}{\partial \tilde{\nu}} \Delta w + (1-\mu) B_2 w - \gamma \tfrac{\partial}{\partial \tilde{\nu}} w_{tt} + \tfrac{\partial}{\partial \tilde{\nu}} \theta = 0 \\ & \tfrac{\partial}{\partial \tilde{\nu}} \theta + \lambda\theta = 0 \end{aligned} \right\} \quad \text{on} \quad \partial\Gamma_0 \times (0,\infty) \quad (6)$$

The constant λ is assumed positive. The boundary conditions given for a free plate involve the following boundary operators B_1, B_2

$$B_1 \equiv 2\nu_1\nu_2 D^2_{x,y} - \nu_1^2 D^2_{y,y} - \nu_2^2 D^2_{x,x}$$

$$B_2 \equiv \frac{\partial}{\partial \tau} \left[(\nu_1^2 - \nu_2^2) D^2_{x,y} + \nu_1\nu_2 (D^2_{y,y} - D^2_{x,x}) \right] + lw$$

where $\tilde{\nu} = (\nu_1, \nu_2)$. With the model (2), (3) we associate appropriate initial conditions where in the clamped and hinged case the initial conditions are subject to appropriate compatibility conditions on the boundary. This is to say that $w_0 = 0, w_1 = 0$ on $\partial\Gamma_0$. In the clamped case we also require that $\frac{\partial}{\partial\nu}w_0 = 0$; on $\partial\Gamma_0$.

Here θ is the temperature, c^2 is the speed of sound as usual and ρ represents back pressure acting upon the wall. The function g, representing a potential damping (friction) is assumed continuous, monotone increasing and zero at the origin. The constant γ accounts for rotational forces and here is taken to be small and non-negative.

It should be noted that the presence of the parameter γ in equation (3) changes the character of the dynamics. Indeed, the "uncoupled" thermoelastic plate is of hyperbolic type when $\gamma > 0$, Lasiecka and Triggiani (1999) and of analytic type when $\gamma = 0$ (Liu and Renardy, 1995, for the clamped-hinged case, and Lasiecka and Triggiani, 1999, for the free case).

Our goal is to show that the energy of the entire system given by (2), (3), decays to zero at a *uniform* rate. For convenience, and without loss of generality, we choose $c = \rho = 1$. Also, in what follows, we assume that the constants l, d are positive. This assumption is not essential to mathematical analysis of the problem, but it guarantees that the constants functions are properly damped. In the absence of this restriction, one needs either to formulate the result on an appropriate quotient space or to account for other (lower order) terms in the equation which prevent the value zero from being a member of the spectrum of the corresponding linear elliptic operator.

1.3. Main results

We begin with a preliminary result stating that the system is well-posed.

THEOREM 1.1 *(Well-posedness) Let Ω be a bounded open domain in $\mathbf{R}^3$ with boundary Γ as previously described. For all initial data $y_0 = [z_0, z_1, w_0, w_1, \theta_0] \in Y$, subject to compatibility conditions on the boundary, where*

$$Y \equiv H^1(\Omega) \times L^2(\Omega) \times H^2(\Gamma_0) \times H^1_\gamma(\Gamma_0) \times L^2(\Gamma_0)$$

the solution $y(t) = [z, z_t, w, w_t, \theta]$ of the model (2), (3) exists in $C([0,\infty); Y)$ and is unique.

Proof. Since the problem under consideration is maximal dissipative, the result of Theorem 1.1 follows from the general theory of m-dissipative operators, Barbu (1976), and also Lasiecka (1994) where, more specifically, problems with nonlinear monotone boundary conditions are considered. ■

In order to formulate our main result on stability, we introduce some notation. We define the energy functional associated with the model and given by

$$E_p(t) = \int_\Omega |\nabla z|^2 d\Omega + d\int_{\Gamma_2} z^2 d\Gamma_2 + \int_{\Gamma_0} |\theta|^2 d\Gamma_0 + a(w,w) \tag{9}$$

where in the *clamped or hinged* case, $a(w,z) \equiv \int_{\Gamma_0} \Delta w \Delta z d\Gamma_0$ and for the *free* case

$$\begin{aligned} a(w,z) &\equiv \int_{\Gamma_0} [w_{x,x}z_{x,x} + w_{y,y}z_{y,y} + \mu w_{x,x}z_{y,y} + \mu w_{y,y}z_{x,x} \\ &+ 2(1-\mu)w_{x,y}z_{x,y}]d\Gamma_0 + l\int_{\partial\Gamma_0} wzd\partial\Gamma_0 \end{aligned}$$

It is well known that $a(w,w)$ is topologically equivalent to $H^2(\Gamma_0)$ norm, Lagnese (1989). We also introduce the function $h(s)$ which is assumed concave, strictly increasing, zero at the origin and such that the following inequalities are satisfied:

$$h(sg(s)) \geq s^2 + |g(s)|^2 \ , \ \text{for } |s| \leq 1$$

Such a function can be easily constructed in view of the monotonicity assumption imposed on g, Lasiecka and Tataru (1993).

Our main result is the following

THEOREM 1.2 *(Uniform stability) Let Ω be a bounded open domain in $\mathbf{R}^3$ with boundary Γ as previously described. Assume that the nonlinear function g satisfies:*

$$ms^2 \leq g(s)s \leq Ms^2;\ |s| \geq 1 \tag{10}$$

Then, with the constant $\gamma \geq 0$, every weak solution $y(t) = [z, z_t, w, w_t, \theta]^T$ of (2), (3) decays uniformly. This is to say, the following estimate holds

$$E_\gamma(t) \leq C_\gamma s_\gamma(t/T_0 - 1)); \quad t \geq T_0 \tag{11}$$

where the real variable function $s_\gamma(t)$ converges to zero as $t \to \infty$ and satisfies the following ordinary differential equation

$$\frac{d}{dt}s_\gamma(t) + q_\gamma(s_\gamma(t)) = 0, \quad s_\gamma(0) = E_\gamma(0) \tag{12}$$

The (nonlinear), monotone increasing function $q_\gamma(s)$ is determined entirely from the behavior at the origin of the nonlinear function g and it is given by the following algorithm.

$$q_\gamma \equiv I - (I + p_\gamma)^{-1} \tag{13}$$

$$p_\gamma \equiv (I + h_0)^{-1}(\frac{1}{K}) \tag{14}$$

$$h_0 \equiv h(\cdot/mes(0,T) \times \Gamma_0 \cup \Gamma_2) \tag{15}$$

Moreover, in the **clamped and hinged** *case the quantities describing the decay rates can be made independent of $\gamma > 0$. For the* **free** *case, the constants (hence*

REMARK 1.1 *Note that the decay rates established in Theorem 1.2 can be computed explicitly by solving the nonlinear ODE system (12), once the behavior of $g(s)$ at the origin is specified. If the nonlinear function g is bounded from below by a linear function, then it can be shown that the decay rates predicted by Theorem 1.2 are exponential. This is to say that there exist positive constants C, ω, possibly depending on $E(0)$ and such that*

$$E_\gamma(t) \leq Ce^{-\omega t} E_{\gamma_0}, \quad \textit{for } t > 0.$$

If, instead, this function has a polynomial growth (resp. exponentially decaying) at the origin, than the decay rates are algebraic (resp. logarithmic). This can be verified by solving the ODE problem (12) explicitly (see Lasiecka and Tataru, 1993). In the clamped and hinged case, the decay rates do not depend on γ.

REMARK 1.2 *Both clamped and hinged boundary conditions imposed on the plate model lead to decay rates which are uniform in γ. In the case of free boundary conditions, the analysis is more complex and it relies critically on the hyperbolicity property of the related Kirchoff operator. This fact prevents from having the decay rates independent of γ. However, the decay rates are still valid in the limit case $\gamma = 0$, but their derivation rests on the analyticity property of the thermal component of the plate.*

REMARK 1.3 *Other types of boundary conditions can be imposed on Γ_1. In fact, one can replace the Neumann boundary conditions by the Dirichlet boundary conditions, in which case the required geometric condition becomes $(x - x_0) \cdot \nu \leq 0$ on Γ_1. Note that we do not assume any geometric conditions imposed on dissipative portion of the boundary $\Gamma_0 \cup \Gamma_2$. This is in contrast with most of the literature on boundary stabilization of (uncoupled) wave or plate equations, Komornik (1994), Lagnese (1989).*

The remainder of this paper is devoted to the proof of Theorem 1.2. Here, we note that the main technical difficulties of the problem under study deal with the following features:

1. The plate equation has neither structural damping nor mechanical damping, which would have induced strong stability properties for the plate. This is in contrast with all other works available in the literature.
2. The PDE equation describing the plate model does not satisfy the Lopatinski condition in the case of *free* boundary conditions. As a consequence, the natural regularity of boundary traces (critical to the analysis) are much weaker and require more delicate arguments.
3. The value of γ in the plate model under consideration changes the character of dynamics (from analytic to hyperbolic). This forces a completely different treatment of the problem (mainly at the level of treating the

4. The coupling between the medium and the wall, represented by a "trace operator", induces a "strong" coupling (rather than "weak" coupling) in the structure. This effect produces uncontrolled "energy level" terms in the estimates – a notoriously difficult problem in the context of controllability and stabilization. Handling of this requires a special rescaling argument.

2. Preliminaries

We shall adopt the following notation:

$$|w|_{s,\Omega} \equiv |w|_{H^s(\Omega)}; \quad (u,v)_\Omega \equiv \int_\Omega uv d\Omega$$

The same notation will be used with Ω replaced by Γ, etc. For $s < 0$, the negative Sobolev's spaces are defined as duals (pivotals) to $H^{-s}(\Omega)$ with respect to $L_2(\Omega)$ topology.

Our goal is to show the uniform stability of the coupled PDE system (2), (3). We begin with a preliminary energy identity which illustrates the fact that the system is dissipative.

PROPOSITION 2.1 *With respect to the system of equations (2), (3), the following energy equality holds for all $T > 0$, $s < T$:*

$$E_\gamma(s) = E_\gamma(T) + 2\int_s^T (g(z_t), z_t)_{\Gamma_0} dt + 2\int_s^T |\nabla\theta|^2_{0,\Gamma_0} dt + 2\lambda \int_s^T |\theta|^2_{0,\partial\Gamma_0} dt \quad (16)$$

where, we recall, the 'energy' $E_\gamma(t)$ is defined by

$$\begin{aligned} E_\gamma(t) &\equiv |\nabla z|^2_{0,\Omega} + d|z|^2_{0,\Gamma_2} + |z_t|^2_{0,\Omega} + |w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0} \\ &+ a(w(t), w(t)) + |\theta|^2_{0,\Gamma_0} \end{aligned} \quad (17)$$

Proof. By applying the multipliers z_t on the wave equation, w_t on the elastic equation, θ on the thermal equation, and then integrating by parts, we obtain the above equality for smooth solutions. A density argument allows us to extend this inequality to all solutions of finite energy. ∎

Our basic strategy is to obtain first the estimates for the thermoelastic equations on Γ_0 and then, for the wave equation defined on Ω. A subtle point of the analysis is the treatment of the coupling and appropriate combination of the two estimates. Notice that the coupling introduces terms of the order of the energy.

We also note that our analysis differentiates the cases $\gamma > 0$ and $\gamma = 0$. When $\gamma = 0$ then the thermoelastic system represents an analytic semigroup, Liu and Renardy (1995), Lasiecka and Triggiani (1998). Instead, if $\gamma > 0$, the dynamics of the thermoelastic plate are hyperbolic-like with finite speed of the overall system are very different for the cases of $\gamma = 0$ and $\gamma > 0$. The difference between the two types of dynamics will play a critical role throughout the analysis, particularly at the level of treating trace regularity and the coupling between the two systems. Since several estimates presented in the course of the proof will require "sharp" information on the traces of solutions to plate equation, we find it convenient to collect these results below.

We consider the Kirchoff plate

$$\begin{aligned} & w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w = f \quad \text{on} \quad \Gamma_0 \times (0,T) \\ & \Delta w + (1-\mu)B_1 w = g_1 \\ & \frac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w - \gamma\frac{\partial}{\partial\nu}w_{tt} = g_2 \text{ on } \partial\Gamma_0 \times (0,T) \end{aligned} \tag{18}$$

LEMMA 2.1 Lasiecka and Triggiani (1999). *With reference to (18), $\gamma > 0$ and taking $f = 0, g_2 = 0$ the following regularity holds:*

$$|w(t)|^2_{2,\Gamma_0} + |w_t(t)|^2_{1,\Gamma_0} \le C_{\gamma,T}\left[|w(0)|^2_{2,\Gamma_0} + |w_t(0)|^2_{1,\Gamma_0} + \int_0^T |g_1|^2_{1/2,\partial\Gamma_0}\,dt\right] \tag{19}$$

We note that standard regularity results for Kirchoff plate require $g_1 \in H^{1/2}((0,T)\times\partial\Gamma_0)$, rather than the $1/2$ derivative in space only. The proof of Lemma 2.1, based on microlocal analysis arguments, is given in Lasiecka and Triggiani (1999).

LEMMA 2.2 *With reference to (18) with $\gamma > 0$ and taking $g_1 = g_2 = 0$, the following regularity holds: there exists a positive constant $\rho > 0$ such that*

$$\int_0^T |w|^2_{3/2+\rho,\partial\Gamma_0}dt \le C_{T,\gamma}\left[|w(0)|^2_{2,\Gamma_0} + |w_t(0)|^2_{1,\Gamma_0} + \int_0^T |f|^2_{-1+\rho,\Gamma_0}\,dt\right] \tag{20}$$

The value of ρ depends on the geometry Γ_0. However, it is always positive.

We note that the standard regularity result gives the above inequality with $\rho = 0$.

Proof. The result of Lemma 2.2 with $f = 0$ was proved in Avalos and Lasiecka (1997b) (see formula (2.70)). The estimate due to the additional forcing term f is obtained by a standard semigroup argument applied to the Kirchoff plate. ■

Our next result deals with the case of $\gamma = 0$, where the linear thermoelastic system represents an analytic semigroup. In this case, we have higher interior regularity of the solutions

LEMMA 2.3 *With reference to (18) with $f = \Delta\theta + f_1$, $\gamma = 0$ and taking $g_1 = -\theta, g_2 = -\frac{\partial}{\partial\nu}\theta$, where θ is the solution to the heat equation in (3), we obtain the following regularity:*

$$\int_0^T \left[|w(t)|^2_{3,\Gamma_0} + |w_t(t)|^2_{1,\Gamma_0}\right]dt \le C_T E_w(0) + C_T\int_0^T |f_1|^2_{-1,\Gamma_0}dt \tag{21}$$

Proof. It was shown in Lasiecka and Triggiani (1998) that the thermoelastic system with free boundary conditions generates an analytic semigroup e^{At} on the space

$$X \equiv H^2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$$

The domain of the generator A is given by

$$\begin{aligned} D(A) &= \{(w, w_t, \theta) \in H^4(\Omega) \times H^2(\Omega) \times H^2(\Omega); \\ &\quad \text{subject to the boundary conditions } in (3)\} \end{aligned}$$

By a standard interpolation result

$$D(A^{1/2}), D(A^{*1/2}) \subset H^3(\Omega) \times H^1(\Omega) \times H^1(\Omega) \tag{22}$$

Since it is also known, see Lasiecka and Triggiani (1998), that A is dissipative and invertible, analyticity of e^{At} implies, Bensoussan, Da Prato, Delfour and Mitter (1992) the following estimates

$$\int_0^T |A^{1/2} e^{At} x|_X^2 dt \leq C_T |x|_X^2$$

$$\int_0^T \left| A \int_0^t e^{A(t-s)} f(s) ds \right|_X^2 dt \leq C_T \int_0^T |f|_X^2 dt$$

Writing the solution to thermoelastic system as:

$$A^{1/2} \begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{At} A^{1/2} \begin{bmatrix} w(0) \\ w_t(0) \\ \theta(0) \end{bmatrix} + \int_0^t A e^{A(t-s)} A^{-1/2} \begin{bmatrix} 0 \\ f_1(s) \\ 0 \end{bmatrix} ds$$

we obtain

$$\begin{aligned} &\int_0^T \left| \begin{matrix} w \\ w_t \\ \theta \end{matrix} \right|_{D(A^{1/2})}^2 dt \leq \\ &\int_0^T \left| A^{1/2} e^{At} \begin{bmatrix} w(0) \\ w_t(0) \\ \theta(0) \end{bmatrix} \right|_X^2 + \int_0^T \left| A e^{A(t-s)} A^{-1/2} \begin{bmatrix} 0 \\ f_1(s) \\ 0 \end{bmatrix} \right|_X^2 ds \\ &\leq C_T E_w(0) + C_T \int_0^T \left| A^{-1/2} \begin{bmatrix} 0 \\ f_1 \\ 0 \end{bmatrix} \right|_X^2 dt \end{aligned} \tag{23}$$

Application of the inclusion in (22) completes the proof. ■

Finally, we recall another trace result valid for the Kirchoff plate with clamped boundary conditions which takes advantage of the fact that the clamped

LEMMA 2.4 *Let w be a solution of (3) with clamped boundary conditions and $\gamma \geq 0$. Then, the following trace regularity takes place:*

$$\begin{aligned}
&\int_0^T |\Delta w|^2_{0,\partial\Gamma_0} dt \leq \\
&C\int_0^T \left[|w|^2_{2,\Gamma_0} + |w_t|^2_{0,\Gamma_0} + \gamma|w_t|^2_{1,\Gamma_0} + |\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}\right] dt \\
&+C[E_{w,\gamma}(0) + E_{w,\gamma}(T)]
\end{aligned}$$

where the constants C do not depend on γ.

The proof of this Lemma is given in Avalos and Lasiecka (1997b, 1998a) for the equation without a z_t term. However, an identical argument can be applied to account for the z_t term.

3. Plate equation

Let the plate energy $E_{w,\gamma}(t)$ be defined as

$$E_{w,\gamma}(t) = |w_t(t)|^2_{L^2(\Gamma_0)} + \gamma|\nabla w_t(t)|^2_{L^2(\Gamma_0)} + a(w(t), w(t)) + |\theta(t)|^2_{0,\Gamma_0} \quad (24)$$

LEMMA 3.1 *With respect to the thermoelastic component of the model (3), with $\gamma \geq 0$, the following inequality holds: $\forall\, \epsilon > 0$, $\exists\, C_{\gamma,T,\epsilon}$ such that*

$$\begin{aligned}
&\int_0^T \left[a(w,w) + |w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt \leq \epsilon C_\gamma\left[E_{w,\gamma}(0) + E_{w,\gamma}(T)\right] \quad (25) \\
&+C_{T,\gamma,\epsilon}\int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}\right] dt + C_{T,\gamma,\epsilon} lot(w,\theta)
\end{aligned}$$

where $lot(w,\theta) = \sup_{t\in[0,T]}[|w(t)|^2_{1,\Gamma_0} dt + |\theta(t)|^2_{-1/4,\Gamma_0}]$. The constants C_γ do not depend on γ in the case of hinged and clamped boundary conditions. In the case of free boundary conditions, however, the constants C_γ may blow up with $\gamma \to 0$. However, if $\gamma = 0$, then the constant $C_0 \equiv C_{\gamma=0}$ is well defined.

Proof. We introduce some notation. Let $A_D u \equiv -\Delta u$; $D(A_D) = H^2(\Gamma_0) \times H^1_0(\Gamma_0)$. Let the Dirichlet map D be defined as follows:

$$D : L_2(\partial\Gamma_0) \to L_2(\Gamma_0);\ \Delta Dg = 0 \text{ in } \Gamma_0;\ Dg = g \text{ on } \partial\Gamma_0$$

and let γ_0 denote a restriction (trace) operator to $\partial\Gamma_0$. With the above notation one can write

$$\begin{aligned}
\Delta\theta &= \Delta\theta - \Delta D\gamma_0\theta = \Delta(\theta - D\gamma_0\theta) = -A_D(\theta - D\gamma_0\theta); \\
&\quad \text{in } H^1_0(\Omega)', \theta \in H^1(\Omega)
\end{aligned}$$

We apply the technique introduced in Avalos and Lasiecka (1997b, 1998a), that is, we multiply the first equation in (3) by $A_D^{-1}\theta$ and integrate from 0 to T to obtain

$$\int_0^T (w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta + z_t, A_D^{-1}\theta)_{L^2(\Gamma_0)}\, dt = 0. \tag{27}$$

We deal with each part separately:

(1) Using integration by parts, substitution of boundary conditions, the second equation of (3), formulas in (26) give (detailed computations are in Avalos and Lasiecka, 1998a, pp. 171-172).

$\forall\, \epsilon, \epsilon_1, \epsilon_2 > 0, \quad \exists C_\epsilon,\, C_{\epsilon_1,\epsilon_2}$ such that

$$\left\{\begin{array}{l} \left|\int_0^T \left(w_{tt} - \gamma\Delta w_{tt}, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt - \int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt\right| \\ \leq |w_t|_{0,\Gamma_0}|A_D^{-1}\theta|_{0,\Gamma_0}\big|_0^T + \gamma|\nabla w_t|_{0,\Gamma_0}|\nabla A_D^{-1}\theta|_{0,\Gamma_0}\big|_0^T \\ \quad + \int_0^T [|w_t|_{0,\Gamma_0}|\theta|_{0,\Gamma_0} + \gamma|\nabla w_t|_{0,\Gamma_0}|\nabla\theta|_{0,\Gamma_0}]\, dt + \epsilon\int_0^T |w|^2_{2,\Omega} \\ \quad + \int_0^T [C_\epsilon|\theta|^2_{1,\Omega} + Ca(D\gamma_0 w, w)]dta + \int_0^T \int \Gamma_0 z_t D\gamma_0 w\, d\Gamma_0 dt \\ \leq \epsilon_1 C\left[E_{w,\gamma}(0) + E_{w,\gamma}(T)\right] a + C\int_0^T |z_t|^2_{-1,\Gamma_0}dt \\ \quad +\varepsilon\int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt + C_\varepsilon\int_0^T |\theta|^2_{1,\Gamma_0} \\ \quad +\epsilon\int_0^T |w|^2_{2,\Omega}dt + C\int_0^T a(D\gamma_0 w, w)dt + C_{\epsilon_1,\epsilon_2}\sup_{t\in[0,T]}|\theta(t)|^2_{-1/2+\epsilon_2,\Gamma_0} \end{array}\right. \tag{28}$$

In the last step we used

$$\begin{array}{l} |w_t|_{0,\Gamma_0}|A_D^{-1}\theta|_{0,\Gamma_0} + \gamma|\nabla w_t|_{0,\Gamma_0}|\nabla A_D^{-1}\theta|_{0,\Gamma_0} \leq \\ \epsilon_1[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}] + C_{\epsilon_1}|\nabla A_D^{-1}\theta|^2_{0,\Gamma_0} \\ \leq \epsilon_1[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}] + C_{\epsilon_1,\epsilon_2}|\theta|^2_{-1/2+\epsilon_2,\Gamma_0} \end{array}$$

Note that neither of the constants C and C_ε depend on T or γ.

(2) Another integration by parts and application of boundary conditions gives that in the case of *clamped or hinged* boundary conditions

$$\begin{array}{l} \displaystyle\int_0^T \left(\Delta^2 w, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt = \\ \displaystyle -\int_0^T \left(\Delta w, \frac{\partial}{\partial\nu}A_D^{-1}\theta\right)_{\partial\Gamma_0} dt + \int_0^T a(w, A_D^{-1}\theta)dt \end{array} \tag{29}$$

and in the case of *free* boundary conditions

$$\int_0^T \left(\Delta^2 w, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt =$$

(30)

By applying the estimate in Lemma 2.4 in order to eliminate the trace of Δw for the clamped case, we obtain the following inequality valid in the *clamped and hinged* case

$$\int_0^T \left(\Delta^2 w, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt \leq$$
$$\epsilon \int_0^T \left[|w|_{2,\Gamma_0} + |w_t|^2_{0,\Gamma_0} + \gamma|\nabla w|^2_{0,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}\right] dt$$
$$+\epsilon[E_{w,\gamma}(0) + E_{w,\gamma}(T)] + C_\epsilon \int_0^T |\theta|^2_{1,\Gamma_0} dt \tag{31}$$

Instead, in the *free* case we simply have

$$\int_0^T \left(\Delta^2 w, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt \leq \epsilon \int_0^T |w|_{2,\Gamma_0} dt + C_\epsilon \int_0^T |\theta|^2_{1,\Gamma_0} dt \tag{32}$$

(3) Application of the standard Green's formula gives:

$$\int_0^T \left(\Delta\theta, A_D^{-1}\theta\right)_{L^2(\Gamma_0)} dt \leq C \int_0^T |\theta|^2_{1,\Omega} dt \tag{33}$$

(4) Finally, for the last term in (27):

$$\left|\int_0^T \left(z_t, A_D^{-1}\theta\right)_{L^2(\Gamma_0)}\right| dt \leq \int_0^T |z_t|_{-1,\Gamma_0} |A_D^{-1}\theta|_{1,\Gamma_0} dt \leq$$
$$C \int_0^T |z_t|^2_{-1,\Gamma_0} dt + C \int_0^T |\theta|^2_{1,\Gamma_0} dt \tag{34}$$

Combining equations (28) - (34) results in the following:

Proposition 3.1 *For any $T > 0$ and ε, ϵ_i, $i = 1, 2$ small enough there exist positive constants $C > 0, C_\epsilon, C_{\epsilon_i}$, independent of γ such that*

$$(1 - 2\varepsilon) \int_0^T \left[|w_t|^2{}_{0,\Gamma_0} + \gamma|\nabla w_t|^2{}_{0,\Gamma_0}\right] dt \leq$$
$$\epsilon_1 C \left[E_{w,\gamma}(0) + E_{w,\gamma}(T)\right] + C_\epsilon \int_0^T |\theta|^2_{1,\Gamma_0} dt$$
$$+C \int_0^T |z_t|^2_{-1,\Gamma_0} dt + \varepsilon \int_0^T |w|^2_{2,\Gamma_0} dt$$
$$+C \int^T a(D\gamma_0 w, w) dt + C_{\epsilon_1,\epsilon_2} \sup |\theta(t)|^2_{-1/2+\epsilon_2,\Gamma_0} \tag{35}$$

The main technical issue to be tackled now is the estimate for the term $\int_0^T a(D\gamma_0 w, w)dt$ in (35) in the case of *free boundary conditions*. Notice that this term disappears in the *hinged or clamped* case. In the case of free boundary conditions, this estimate will introduce dependence of the constants on the parameter γ. Moreover, the cases $\gamma > 0$ and $\gamma = 0$ need to be treated separately.

PROPOSITION 3.2

- *Let $\gamma > 0$. $\forall \epsilon, \epsilon_0 > 0$, there exist constants $C_{\epsilon,T,\gamma}, C_{\epsilon_0,\epsilon,T,\gamma}$ such that the following estimates are valid in the case of free boundary conditions*
$$\int_0^T a(D\gamma_0 w, w)dt \le \epsilon \int_0^T |w|^2_{2,\Gamma_0} dt + C_{T,\gamma,\epsilon} \int_0^T [|z_t|^2_{-1,\Gamma_0} + |\theta|^2_{1,\Gamma_0}]dt$$
$$+\epsilon_0 C_{T,\gamma} E_{\gamma,w}(0) + C_{T,\gamma,\epsilon,\epsilon_0} lot(w,\theta)$$
- *Let $\gamma = 0$. $\forall \epsilon > 0\ \exists C_{T,\epsilon}$ such that*
$$\int_0^T a(D\gamma_0 w, w)dt \le \epsilon C_T E_w(0) + C_T \int_0^T |z_t|^2_{-1,\Gamma_0} dt + C_{T,\epsilon,} lot(w,\theta) \quad (36)$$

Proof. Follows through several steps. We write $w \equiv w_1 + w_2$, where w_1 corresponds to the plate equation (3) with the zero initial data and no forcing team. The basic semigroup estimates and "sharp" regularity result in Lemma 2.1 applied to the Kirchoff plate give:

$$\begin{aligned} |w_1(t)|^2_{2,\Gamma_0} + |w_{1t}|^2_{1,\Gamma_0} &\le C_{T,\gamma} \int_0^T \left[|\Delta\theta|^2_{-1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0} + |\theta|^2_{1/2,\partial\Gamma_0} \right] dt \\ &\le C_{T,\gamma} \int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0} \right] dt \end{aligned} \quad (37)$$

where in the last step we use the boundary conditions satisfied by the θ variable as well as the trace theorem.

Therefore, by regularity of the Dirichlet map $D \in \mathcal{L}(H^{3/2}(\partial\Gamma_0) \to H^2(\Gamma_0))$ and trace theory

$$\int_0^T a(D\gamma_0 w_1, w_1)dt \le C \int_0^T |w_1|^2_{2,\Gamma_0} dt \le C_{T,\gamma} \int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0} \right] dt \quad (38)$$

which gives the right estimate for the first component w_1. As for the second component w_2, we notice that w_2 satisfies the homogeneous Kirchoff plate equation with nonzero initial data and no forcing term. Thus, the "sharp" regularity result of Lemma 2.2 applies and gives

$$\int_0^T |w_2|^2_{3/2+\rho,\partial\Gamma_0} dt \le C_{T,\gamma} \left[|w(0)|^2_{2,\Gamma_0} + |w_t(0)|^2_{1,\Gamma_0} \right] \quad (39)$$

Hence, by using trace theory and regularity of the Dirichlet map, moment inequality and the estimate in (39) we obtain:

$\int^T \qquad \int^T$

$$\leq \epsilon \int_0^T |w_2|^2_{2,\Gamma_0} dt + C_\epsilon \int_0^T |w_2|^2_{3/2,\partial\Gamma_0} dt$$
$$\leq \epsilon \int_0^T |w_2|^2_{2,\Gamma_0} dt + \epsilon_0 \int_0^T |w_2|^2_{3/2+\rho,\partial\Gamma_0} dt + C_{\epsilon,\epsilon_0} \int_0^T |w_2|^2_{1/2,\partial\Gamma_0} dt \quad (40)$$

But, by (38),

$$\begin{aligned}\int_0^T |w_2|^2_{1/2,\partial\Gamma_0} dt &\leq C \int_0^T |w_2|^2_{1,\Gamma_0} \leq C \int_0^T |w_1|^2_{1,\Gamma_0} + \int_0^T |w|^2_{1,\Gamma_0} \\ &\leq C_T, \gamma \int_0^T [|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}] dt + C_T lot(w,\theta) \quad (41)\end{aligned}$$

Hence

$$\int_0^T a(D\gamma_0 w_2, w_2) dt \leq \epsilon \int_0^T |w_2|^2_{2,\Gamma_0} dt + \epsilon_0 C_{T,\gamma} E_{\gamma,w}(0)$$
$$+ C_{T,\epsilon,\epsilon_0} lot(w,\theta) + C_{T,\gamma,\epsilon,\epsilon_0} \int_0^T [|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}] dt$$

Combining (38) and (42) and taking $\epsilon_2 < 1/4$ gives the desired result in Proposition 3.2 for $\gamma > 0$.

The argument for $\gamma = 0$ is as follows. We apply the result of Lemma 2.3 and the interpolation inequality

$$\begin{aligned}\int_0^T a(D\gamma_0 w, w) dt &\leq C \int_0^T |w|^2_{2,\Gamma_0} dt \leq \epsilon \int_0^T |w|^2_{3,\Gamma_0} dt + C_\epsilon \int_0^T |w|^2_{1,\Gamma_0} dt \\ &\leq \epsilon C_T E_w(0) + \epsilon C_T \int_0^T |z_t|^2_{-1,\Gamma_0} + C_\epsilon lot(w,\theta) \quad (42)\end{aligned}$$

This gives the desired inequality for the case of $\gamma = 0$. ∎

From Propositions 3.1 and 3.2 after taking ϵ_0, ϵ small enough so that $\epsilon_0 C_{T,\gamma}$, $\epsilon C_T \leq \epsilon_1 C_\gamma$ we obtain

PROPOSITION 3.3 *Let $\gamma \geq 0$. $\forall \epsilon$, ϵ_1 $\exists$ $C_{\gamma,T,\epsilon,\epsilon_1}$ such that*

$$\int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma |\nabla w_t|^2_{0,\Gamma_0}\right] dt \leq \epsilon_1 C_\gamma [E_\gamma(0) + E_\gamma(T)] + \epsilon \int_0^T |w|^2_{2,\Gamma_0}$$
$$+ C_{T,\gamma,\epsilon,\epsilon_1} \int_0^T [|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}] dt + C_{T,\gamma,\epsilon,\epsilon_1} lot(w,\theta)$$

Moreover, the constants C_γ do not depend on γ in the hinged and clamped cases.

Continuation of the proof of Lemma 3.1. Next, we multiply the same equation we obtain

$$(w_t, w)_{\Gamma_0}\big|_0^T + \gamma\,(\nabla w_t, \nabla w)_{\Gamma_0}\big|_0^T - \left[\int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt\right]$$
$$= -\int_0^T a(w,w)\,dt + \int_0^T (\nabla\theta, \nabla w)_{\Gamma_0}\,dt - \int_0^T (\theta, \frac{\partial}{\partial\nu}w)_{\partial\Gamma_0} dt - \int_0^T (z_t, w)_{\Gamma_0}\,dt$$

In the case of *clamped* or *hinged* boundary conditions, the only difference in the expression is that the boundary integral $\int_0^T(\theta, \frac{\partial}{\partial\nu}w)_{\partial\Gamma_0}$ does not appear. Taking norms and using the trace theorem gives

$$\int_0^T a(w,w)dt \leq$$
$$\left|(w_t, w)_{\Gamma_0}\big|_0^T + \gamma\,(\nabla w_t, \nabla w)_{\Gamma_0}\big|_0^T - \int_0^T \left[|w_t|^2_{\Gamma_0} + \gamma|\nabla w_t|^2_{\Gamma_0}\right] dt\right|$$
$$+\left|\int_0^T (\nabla\theta, \nabla w)_{\Gamma_0}\,dt\right| + \left|\int_0^T (\theta, \frac{\partial}{\partial\nu}w)_{\partial\Gamma_0} dt\right| - \left|\int_0^T (z_t, w)_{\Gamma_0}\,dt\right|$$

Noticing that

$$\left|(w_t, w)_{L^2(\Gamma_0)}\big|_0^T + \gamma\,(\nabla w_t, \nabla w)_{L^2(\Gamma_0)}\Big|_0^T\right| \leq \epsilon\left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right]\big|_0^T$$
$$+C_\epsilon\left[|w|^2_{0,\Gamma_0} + \gamma|\nabla w|^2_{0,\Gamma_0}\right]\big|_0^T \leq \epsilon[E_{w,\gamma}(0) + E_{w,\gamma}(T)] + C_\epsilon lot(w,\theta)$$

we obtain

$$\int_0^T a(w,w)dt \leq \epsilon_0\left[E_{w,\gamma}(0) + E_{w,\gamma}(T)\right] + C_1\int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt$$
$$+\varepsilon\int_0^T |w|^2_{2,\Gamma_0}\,dt + C_\varepsilon\int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}\right] dt + C_{\epsilon_0} lot(w,\theta)$$

Thus, we have that $\forall\,\epsilon, \epsilon_1 > 0$ there exist suitable constants $C_{\epsilon,\epsilon_1} > 0$ such that

$$(1-\varepsilon)\int_0^T a(w,w)\,dt \leq \epsilon_1 C\left[E_{w,\gamma}(0) + E_{w,\gamma}(T)\right] +$$
$$C_1\int_0^T \left[|w_t|^2_{0,\Gamma_0} + \gamma|\nabla w_t|^2_{0,\Gamma_0}\right] dt$$
$$+C_\epsilon\int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0}\right] dt + C_{\epsilon_1} lot(w,\theta) \tag{43}$$

If the ϵ of equations (43) and the one of Proposition 3.3 is small enough, these equations can be combined to produce the inequality

$$+C_{T,\gamma,\epsilon}\int_0^T \left(|\theta|^2_{1,\Gamma_0}+|z_t|^2_{-1,\Gamma_0}\right)\,dt] + C_{T,\gamma,\epsilon_1}lot(w,\theta) \tag{44}$$

which is the desired result in Lemma 3.1 for $\gamma \geq 0$. ■

4. Wave equation

Let $E_z(t)$ be the energy defined by

$$E_z(t) = |z_t(t)|^2_{0,\Omega} + |\nabla z(t)|^2_{0,\Omega} + d|z(t)|^2_{0,\Gamma_2} \tag{45}$$

LEMMA 4.1 *Consider the wave equation with boundary conditions*

$$\begin{cases} z_{tt} = \Delta z \quad \text{in} \quad \Omega \\ \frac{\partial}{\partial \nu} z = 0 \quad \text{on} \quad \Gamma_1, \;\; \frac{\partial}{\partial \nu} z + dz = -g(z_t) \quad \text{on} \quad \Gamma_2 \\ \frac{\partial}{\partial \nu} z = -g(z_t) + w_t \quad \text{on} \quad \Gamma_0; \end{cases} \tag{46}$$

Then, for any $\alpha < T/2$ there exist suitable constants C, possibly depending on α, such that

$$\int_\alpha^{T-\alpha} E_z(t)\,dt \leq C\,[E_z(\alpha) + E_z(T-\alpha)]$$
$$+C_T\int_0^T [|z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} + |w_t|^2_{0,\Gamma_0}]\,dt + C_T lot(z) \tag{47}$$

where $lot(z) \leq C\int_0^T [|z|^2_{1-\delta,\Omega} + |z_t|^2_{-\delta,\Omega}]dt;\;\; \delta > 0.$

Proof. The following trace regularity valid for the wave equation is necessary for the proof.

Consider the wave equation

$$\left.\begin{array}{lll} z_{tt} = \Delta z & \text{in} & \Omega \times (0,T) \\ \frac{\partial}{\partial \nu} z = 0 & \text{on} & \Gamma_1 \times (0,T) \end{array}\right\} \tag{48}$$

LEMMA 4.2 *Let z be a solution to (48) with the interior regularity*

$$z \in C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega));$$

and the following boundary regularity

$$\frac{\partial}{\partial \nu} z, z_t \in L_2((0,T)\times \Gamma_0 \cup \Gamma_2)$$

Let $T > 0$ be arbitrary and let α be an arbitrary small constant such that $\alpha < \frac{T}{2}$. Then we have that:

$$\int^{T-\alpha}\int \left|\frac{\partial}{\partial \nu} z\right|^2 \leq C_{T,\alpha}\left[\int^T\left[\left|\frac{\partial}{\partial \nu} z\right|^2 + |z_t|^2_{0,\Gamma_0\cup\Gamma_2}\right] dt + lot(z)\right]$$

We note that the above inequality does not follow from the standard trace theory and the assumed interior regularity. This is an independent trace regularity result. The proof of this Lemma follows from Lemma 7.1 in Lasiecka and Triggiani (1992).

As the first step in proving Lemma 4.1, we use the multipliers method with the multipliers $h \cdot \nabla z$ and $z\, div\, h$, where $h(x)$ is an appropriately constructed vector field. In the special case when Γ_1 is flat, one takes a classical radial vector field $h(x) \equiv (x - x_0)$ with $x_0 \in \Gamma_1$. However, in a more general convex case, a more complicated construction is necessary. This construction is based on Lemma 4.13 in Tataru (1999). Indeed, following Tataru (1999), we define a C^2 convex function $l(x)$, which is supported in the neighborhood of Γ_1 and enjoys the following properties:

$$\Gamma_1 = \{x \in R^2; l(x) = 0\}; \quad \frac{\partial}{\partial \nu} l = 1 \text{ on } \Gamma_1$$

Due to the convexity and regularity of Γ_1, such a function $l(x)$ can be always constructed. Let x_0 be a point in R^3 such that $(x - x_0) \cdot \nu \leq 0$ on Γ_1. Since Γ_1 is convex, such a point (outside Ω) can always be found. Next, define

$$h(x) \equiv (x - x_0) - \nabla[l(x)(x - x_0) \cdot \nu_e(x)] + \lambda l(x) \nabla l(x); \lambda > 0$$

where ν_e denotes an extension of normal derivative ν into a collar neighborhood of the boundary Γ_1. One easily verifies that due to the relations $\frac{\partial}{\partial \nu} l = 1; l = 0$; on Γ_1, we have that $h \cdot \nu = 0$ on Γ_1. Moreover, selecting an appropriately large constant λ, an appropriately small neighborhood of the boundary Γ_1, and recalling the convexity of l, we obtain that $J(h) > 0$, where $J(h)$ denotes the Jacobian of the vector field h. Indeed, to establish this it is enough to show that $J(h) > 0$ on Γ_1. Straightforward computations yield

$$J(h) = I - J(\nabla l)(x - x_0) \cdot \nu_e + lJ(\nabla((x - x_0) \cdot \nu_e)) - \nabla l \nabla^T ((x - x_0) \cdot \nu_e)$$
$$[\nabla l \nabla^{\underline{T}} ((x - x_0) \cdot \nu_e)]^T + \lambda \nabla l \nabla^T l + \lambda l J(\nabla l) \tag{49}$$

The above formula, when restricted to Γ_1 gives

$$J(h) = I - J(\nabla l)(x - x_0) \cdot \nu - \nabla l \nabla^T ((x - x_0) \cdot \nu_e) - [\nabla l \nabla^T ((x - x_0) \cdot \nu_e)]^T$$
$$+ \lambda \nabla l \nabla^T l \tag{50}$$

Hence, for all points on Γ_1 and vector $u \in R^3$ we have

$$(J(h)u, u)_{R^3} = |u|^2 - (J(\nabla l)u, u)_{R^3}(x - x_0) \cdot \nu - 2\nabla l \cdot u \nabla((x - x_0) \cdot \nu_e) \cdot u$$
$$+ \lambda(\nabla l \cdot u)^2 \tag{51}$$

Due to convexity of l and the condition $(x - x_0) \cdot \nu \leq 0$ on Γ_1, we obtain

$$(J(h)u, u)_{R^3} \geq |u|^2 + \lambda(\nabla l \cdot u)^2 - \epsilon|u|^2 - \frac{1}{4\epsilon}(\nabla l \cdot u)^2 |\nabla((x - x_0) \cdot \nu_e)|^2$$

where in the last step we have selected $\lambda > \frac{1}{4\epsilon}|\nabla((x-x_0)\cdot\nu_e)|^2$. This proves the strict positivity of the Jacobian $J(h)$ and leads to the construction of a field h such that $h\cdot\nu = 0$ on Γ_1 and $J(h) > 0$ in Ω.

With the above construction, we apply standard, by now, multiplier calculations which lead to:

$$\begin{aligned}&\int_s^T E_z(t)dt \le C[E_z(s)+E_z(T)]+\\ &C\int_s^T\int_{\Gamma_0\cup\Gamma_2}[z_t^2+g^2(z_t)+z^2]\,d(\Gamma_0\cup\Gamma_2)\\ &+C\int_s^T|w_t|^2_{0,\Gamma_0}dt + C\int_s^T\left|\frac{\partial}{\partial\tau}z\right|^2_{0,\Gamma_0\cup\Gamma_2}dt + C_T\,lot(z)\end{aligned} \tag{53}$$

The term which needs to be further estimated is the last boundary term in (53), which involves tangential derivatives not controlled by the energy norms. To accomplish this we shall use the result of Lemma 4.2. This gives

$$\begin{aligned}&\int_\alpha^{T-\alpha}\left|\frac{\partial}{\partial\tau}z\right|^2_{0,\Gamma_0\cup\Gamma_2}dt \le\\ &C_T\int_0^T[|z_t|^2_{0,\Gamma_2\cup\Gamma_0}+|\frac{\partial}{\partial\nu}z|^2_{0,\Gamma_2\cup\Gamma_0}]dt + C_T lot(z)\\ &\le C_T\int_0^T\left[|z_t|^2_{0,\Gamma_2\cup\Gamma_0}+|g(z_t)|^2_{0,\Gamma_2\cup\Gamma_0}+|w_t|^2_{0,\Gamma_0}\right]dt + C\,lot(z)\end{aligned} \tag{54}$$

Next we apply (53) with s replaced by α and T replaced by $T-\alpha$, and use classical trace theory to absorb $\int_{\Gamma_0\cup\Gamma_2}z^2dx$

$$\begin{aligned}&\int_\alpha^{T-\alpha}E_z(t)\,dt \le C\,[E_z(\alpha)+E_z(T-\alpha)]\\ &\le C_T\int_0^T\left[|z_t|^2_{0,\Gamma_2\cup\Gamma_0}+|g(z_t)|^2_{0,\Gamma_2\cup\Gamma_0}+|w_t|^2_{0,\Gamma_0}\right]dt + C_T\,lot(z)\end{aligned} \tag{55}$$

which provides the desired conclusion in Lemma 4.1. ■

5. Coupled system

For the final analysis, we will combine the energy estimates obtained for the plate and wave equations, and then absorb the lower terms by means of a standard compactness/uniqueness argument.

PROPOSITION 5.1 *Let $T > 0$ be sufficiently large. The following estimate holds for the solution to (2) and (3).*

$$\int_0^T E_\gamma(t)dt + E_\gamma(T) \le C_{\gamma,T}\int_0^T\left[|\theta|^2_{1,\Gamma_0}+|z_t|^2_{0,\Gamma_0\cup\Gamma_2}+|g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2}\right]dt$$

where the constants $C_{\gamma,T}$ do not depend on γ for the hinged and clamped boundary conditions.

Proof. We shall combine equations (25) and (47). We multiply (25) by a suitably large constant A_T (possibly depending on T) and add the result to (47). This gives:

$$\begin{aligned} & A_T \int_0^T E_{w,\gamma}(t)dt + \int_\alpha^{T-\alpha} E_z(t)dt \leq \\ & \epsilon C_\gamma A_T [E_{w,\gamma}(0) + E_{w,\gamma}(T)] + C[E_z \alpha + E_z(T-\alpha)] \\ & + A_T\, C_{T,\gamma,\epsilon} \int_0^T \left[|\theta|^2_{1,\Gamma_0} + |z_t|^2_{-1,\Gamma_0} \right] dt \\ & + C_T \int_0^T \left[|z_t|^2_{0,\Gamma_0 \cup \Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} + |w_t|^2_{0,\Gamma_0} \right] dt \\ & + A_T\, C_{T,\gamma,\epsilon}\, lot(w,\theta) + C_T\, lot(z) \end{aligned} \tag{57}$$

We take $A_T > 2C_T$, which allows us to eliminate the term with $|w_t|^2_{0,\Gamma_0}$ from the right hand side of inequality in (57) . We also select small $\epsilon = \epsilon(T)$, so that $\epsilon C_\gamma A_T \leq C$. This gives:

$$\begin{aligned} & \int_0^T E_{w,\gamma}(t)dt + \int_\alpha^{T-\alpha} E_z(t)\, dt \leq \\ & C_\gamma \left[E_{w,\gamma}(0) + E_{w,\gamma}(T) + E_z(\alpha) + E_z(T-\alpha) \right] \\ & + C_{\gamma,T} \int_0^T \left(|\theta|^2_{1,\Gamma_0} + |z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |z_t|^2_{-1,\Gamma_0} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} \right) dt \\ & + C_{T,\gamma}[lot(w,\theta) + lot(z)] \end{aligned} \tag{58}$$

Hence

$$\begin{aligned} & \int_\alpha^{T-\alpha} E_\gamma(t)\, dt \leq C_\gamma \left[E_\gamma(0) + E_\gamma(T) + E_\gamma(\alpha) + E_\gamma(T-\alpha) \right] \\ & + C_{\gamma,T} \int_0^T \left(|\theta|^2_{1,\Gamma_0} + |z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} \right) dt \\ & + C_{T,\gamma}[lot(w,\theta) + lot(z)] \end{aligned} \tag{59}$$

Here, we recall, the energy $E_\gamma(t)$ is defined as in (17). Our next step is to use dissipativity of the energy to eliminate terms involving α. By using the energy identity in Proposition 2.1 and the simple inequality

$$\left(\int_0^\alpha + \int_{T-\alpha}^T \right) E_\gamma(t)dt \leq 2\alpha E_\gamma(0)$$

we obtain

$$\int^T \ldots$$

$$C_{T,\gamma}\int_0^T \left(|\theta|^2_{1,\Gamma_0} + |z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2}\right)\, dt$$
$$+C_{T,\gamma}[lot(w,\theta) + lot(z)] \tag{60}$$

Once more using the energy relation gives that for $t \le T$

$$E_\gamma(t) \ge E_\gamma(T) - C_{T,\gamma}\int_0^T \left(|\theta|^2_{1,\Gamma_0} + |z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{\Gamma_0\cup\Gamma_2}\right)\, dt$$

Hence

$$TE_\gamma(T) \le C_\gamma E_\gamma(T) + C_{T,\gamma}\int_0^T \left(|\theta|^2_{1,\Gamma_0} + |z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{\Gamma_0\cup\Gamma_2}\right)\, dt$$
$$+C_{T,\gamma}[lot(w,\theta) + lot(z)] \tag{61}$$

Combining (61), (60), and taking $T > C_\gamma$ leads to the desired conclusion in Proposition 5.1. ■

Our next step is to eliminate the lower order terms from the inequality in Proposition 5.1. This is done via the usual compactness and uniqueness argument.

PROPOSITION 5.2 *Let T be sufficiently large. With respect to the coupled PDE system (2), (3) there exists a constant $C_T > 0$ such that*

$$lot(w,\theta) + lot(z) \le C_T\int_0^T \left(|z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} + |\theta|^2_{1,\Gamma_0}\right)\, dt \tag{62}$$

c_T is independent of γ in the hinged and clamped cases.

Proof. The conclusion follows by a contradiction from the usual compactness and uniqueness argument. Since this argument is standard, we shall not report all the details. We shall only point out the main steps. The compactness of $lot(w,\theta) + lot(z)$, with respect to topology induced by the energy E_γ, $\gamma > 0$, follows from the compact embeddings

$$H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)$$
$$\subset H^2(\Gamma_0)\times H^1(\Gamma_0)\times H^1(\Omega)\times L_2(\Omega);\quad \epsilon > 0$$

As for the uniqueness part, we deal with the following overdetermined system:

$$\begin{array}{ll}
\tilde z_{tt} = \Delta \tilde z & \text{on } [0,T]\times\Omega\\
\tilde z_t = 0 & \text{on } [0,T]\times\Gamma_0\cup\Gamma_2\\
\frac{\partial}{\partial\nu}\tilde z = 0 & \text{on } [0,T]\times\Gamma_1\\
\frac{\partial \tilde z}{\partial\nu} = \tilde w_t & \text{on } [0,T]\times\Gamma_0
\end{array} \tag{63}$$

$$\begin{array}{ll}
\tilde{w}_{tt} - \gamma \Delta \tilde{w}_{tt} + \Delta^2 \tilde{w} = 0 & \text{on } [0,T] \times \Gamma_0 \\
\tilde{\theta} \equiv 0; \ \Delta \tilde{w}_t = 0 & \text{on } [0,T] \times \Gamma_0 \\
\Delta \tilde{w} + (1-\mu) B_1 \tilde{w} = 0 & \text{on } [0,T] \times \partial\Gamma_0 \\
\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1-\mu) B_2 \tilde{w} - \gamma \frac{\partial}{\partial \nu} \tilde{w}_{tt} = 0 & \text{on } [0,T] \times \partial\Gamma_0
\end{array}$$

$$\{\tilde{z}(0), \tilde{z}_t(0), \tilde{w}(0), \tilde{w}_t(0), \tilde{\theta}(0)\} = \{\tilde{z}_0, \tilde{z}_1, \tilde{w}_0, \tilde{w}_1, \tilde{\theta_0}\} \in Y$$

Our aim is to show that the above overdetermined system admits only the zero solution. Since we also have

$$\tilde{w}_{ttt} = \gamma \Delta \tilde{w}_{ttt} - \Delta^2 \tilde{w}_t \equiv 0, \text{on } [0,T] \times \Gamma_0$$

we obtain

$$\begin{array}{ll}
\Delta^2 \tilde{w}_t = 0 & \text{on } [0,T] \times \Gamma_0 \\
\tilde{\theta} \equiv 0 & \text{on } [0,T] \times \Gamma_0 \\
\Delta \tilde{w}_t + (1-\mu) B_1 \tilde{w}_t = 0 & \text{on } [0,T] \times \partial\Gamma_0 \\
\frac{\partial}{\partial \nu} \Delta \tilde{w}_t + (1-\mu) B_2 \tilde{w}_t = 0 & \text{on } [0,T] \times \partial\Gamma_0
\end{array}$$

Therefore, by the uniqueness of solutions to elliptic equations, $\tilde{w}_t \equiv 0$, and by going back to the wave equation we obtain

$$\begin{array}{ll}
\tilde{z}_{tt} = \Delta \tilde{z} & \text{on } [0,T] \times \Omega \\
\tilde{z}_t = 0 & \text{on } [0,T] \times \Gamma_0 \cup \Gamma_2 \\
\frac{\partial}{\partial \nu} \tilde{z} = 0 & \text{on } [0,T] \times \Gamma_1 \\
\frac{\partial \tilde{z}}{\partial \nu} = 0 & \text{on } [0,T] \times \Gamma_0 \\
\frac{\partial}{\partial \nu} \tilde{z} + d\tilde{z} = 0 & \text{on } [0,T] \times \Gamma_2
\end{array} \tag{64}$$

Holmgren's Theorem implies $\tilde{z}_t \equiv 0$. This reduces the entire problem to the following static equations

$$\begin{array}{ll}
\Delta^2 \tilde{w} = 0 & \text{on } [0,T] \times \Gamma_0 \\
\tilde{\theta} \equiv 0 & \text{on } [0,T] \times \Gamma_0 \\
\Delta \tilde{w} + (1-\mu) B_1 \tilde{w} = 0 & \text{on } [0,T] \times \partial\Gamma_0 \\
\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1-\mu) B_2 \tilde{w} = 0 & \text{on } [0,T] \times \partial\Gamma_0
\end{array} \tag{65}$$

$$\begin{array}{ll}
\Delta \tilde{z} = 0 & \text{on } [0,T] \times \Omega \\
\frac{\partial}{\partial \nu} \tilde{z} = 0 & \text{on } [0,T] \times \Gamma_1 \cup \Gamma_0 \\
\frac{\partial}{\partial \nu} \tilde{z} + d\tilde{z} = 0 & \text{on } [0,T] \times \Gamma_2
\end{array} \tag{66}$$

By the uniqueness of elliptic equations (note that $l, d > 0$) we conclude that $\tilde{z} = \tilde{w} = \tilde{\theta} = 0$ as desired. ■

Completion of the proof of Theorem 1.2

Combining the results of Propositions 5.1 and Proposition 5.2, we obtain

$\int^T$

where the constants $C_{T,\gamma}$ do not depend on γ in the *clamped and hinged* case.

By using the assumptions imposed on the nonlinear function g and splitting the region of integration into two: $z_t \le 1$ and $z_t > 1$ we also obtain:

$$\int_0^T \left[|z_t|^2_{0,\Gamma_0\cup\Gamma_2} + |g(z_t)|^2_{0,\Gamma_0\cup\Gamma_2} + |\theta|^2_{1,\Gamma_0}\right] dt \le$$
$$C_{T,m,M}[I+h_0]\int_0^T \int_{\Gamma_0\cup\Gamma_2} [g(z_t)z_t\, dx + |\theta^2_{1,\Gamma_0}|dt] \tag{68}$$

where we have used Jensen's inequality. Combining (67) and (68) and recalling monotonicity of h_0 we obtain:

$$E_\gamma(T) \le C_{T,\gamma,m,M}[I+h_0]\int_0^T \int_{\Gamma_0\cup\Gamma_2} [g(z_t)z_t\, dx + |\theta|^2_{1,\Gamma_0}]\, dt$$
$$= C_{T,\gamma,m,M}[I+h_0][E_\gamma(0) - E_\gamma(T)] \tag{69}$$

where in the last step we have used the energy relation. Since $[I+h_0]$ is invertible, this gives

$$[I+h_0]^{-1}(C^{-1}_{\gamma,T,m,M}E_\gamma(T)) \le E_\gamma(0) - E_\gamma(T) \tag{70}$$

and

$$p(E_\gamma(T)) + E_\gamma(T) \le E_\gamma(0)$$

with p defined by the Theorem 1.2. The final conclusion of Theorem 1.2 follows now from application of Lemma 3.1 in Lasiecka and Tataru (1993).

References

AVALOS, G. (1996) The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustic. *Abstract and Applied Analysis*, **1**, 203-218.

AVALOS, G. and LASIECKA, I. (1996) Differential Riccati equation for the active control of a problem in structural acoustics. *JOTA*, **91**, 695-728.

AVALOS, G. and LASIECKA, I. (1997A) Uniform decay rates of solutions to a structural acoustic model with nonlinear dissipation. *Applied Mathematics and Computations, S.C.*, **8**, 2, 287-312.

AVALOS, G. and LASIECKA, I. (1997B) Exponential stability of a thermoelastic system without mechanical dissipation. An invited paper in a special issue honoring Pierre Grisvard; *Rendiconti Di Instituto Di Matematica Dell'Università di Trieste*, E. Mitidieri and Philippe Clement, eds., Suppl. Vol. **XXVIII**, 1-27.

AVALOS, G. and LASIECKA, I. (1998A) Exponential stability of a thermoelastic system with free boundary conditions without mechanical dissipation.

AVALOS, G. and LASIECKA, I. (1998B) The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system. *Semigroup Forum.* **57**, 278-292.

BANKS, H.T., SILCOX, R.J. and SMITH, R.C. (1993) The modeling and control of acoustic/structure interaction problems via piezoceramic actuators: 2-d numerical examples. *ASME Journal of Vibration and Acoustics*, **2**, 343-390.

BANKS, H.T., SMITH, R.C. and WANG, Y. (1995) The modeling of piezoceramic patch interactions with shells, plates, and beams. *Quarterly of Applied Mathematics*, **53**, 2, 353-381.

BARBU, V. (1976) *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, Leyden.

BEALE, J. (1976) Spectral properties of an acoustic boundary condition. *Indiana Univ. Math. J.*, **9**, 895-917.

BENSOUSSAN, A., DA PRATO, G., DELFOUR, M. and MITTER, S. (1992) *Representation and Control of Infinite Dimensional Systems.* Birkhäuser, Boston.

CAMURDAN, M. (1999) Uniform stability of a coupled structural acoustic system by boundary dissipation. *Abstract and Applied Analysis*, to appear.

CAMURDAN, M. and TRIGGIANI, R. (1997) Control problems in noise reduction: The case of two coupled hyperbolic equations. In: *Proceedings of SPIE's 4th Annual Symposium on Smart Structures and Materials*, volume Mathematics and Control in Smart Structures.

CRAWLEY, E.F. and DE LUIS, J. (1987) Use of piezoceramic actuators as elements of intelligent structures. *AIAA Journal*, **25**, 1373-1385.

FAHROO, F. and WANG, C. (1999) A new model for acoustic interaction and its exponential stability. *Quaterly in Applied Mathematics*, **LVII**, 157-179.

FULLER, C.R., GIBBS, G.P. and SILCOX, R.J. (1990) Simultaneous active control of flexural and extensional power flow in beams. *Journal of Intelligent Materials, Systems and Structures*, **1**.

GRISVARD, P. (1967) Characterization de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, **25**, 40-63.

LAGNESE, J. (1989) *Boundary Stabilization of Thin Plates.* SIAM, Philadelphia.

LASIECKA, I. (1994) Existence and uniqueness of the solutions to second order abstract equations with nonlinear and non-monotone boundary conditions. *J. Nonlinear Analysis, Methods and Applications*, **23**, 797-823.

LASIECKA, I. (1999) Boundary stabilization of a 3-dimensional structural acoustic model. *Journal de Mathematiques Pure et Appliquee*, **78**, 203-232.

LASIECKA, I. and TATARU, D. (1993) Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential and Integral Equations*, **6**, 507-533.

LASIECKA, I. and TRIGGIANI, R. (1990) Sharp regularity theory for second data. *Ann. Mat. Pura Applicata*, **IV**, CLVII, 285-367.

LASIECKA, I. and TRIGGIANI, R. (1991) Sharp regularity results for mixed second order hyperbolic equations of Neumann type. Part II: General boundary data. *J. Diff. Eq.*, **94**, 112-164.

LASIECKA, I. and TRIGGIANI, R. (1992) Uniform stabilization of the wave equation with Dirichlet or Neumann Feedback control without geometric conditions. *Applied Mathematics and Optimization.* **25**, 189-224.

LASIECKA, I. and TRIGGIANI, R. (1993) Sharp trace estimates of solutions to Kirchhoff an Euler-Bernoulli equations. *Applied Mathematics and Optimization*, **28**, 277-306.

LASIECKA, I. and TRIGGIANI, R. (1998) Analyticity of thermoelastic semigroups with free boundary conditions. *Annali di Scuola Normale Superiore*, **XXVII**.

LASIECKA, I. and TRIGGIANI, R. (1999) Structural decomposition of thermoelastic semigroups with rotational forces. *Semigroup Forum*, to appear.

LIU, Z. and RENARDY, M. (1995) A note on the equations of a thermoelastic plate. *Appl. Math. Lett.*, **8**, 3, 1-6.

MORSE, P.M. and INGARD, K.U. (1968) *Theoretical Acoustics.* McGraw-Hill.

PAZY, A. (1983) *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences; **44**, Springer-Verlag.

SIMON, J. (1987) Compact sets in the space $L^p(0,T;B)$. *Ann. Math. Pura Appl.*, **4 CXLVI**, 65-96.

TATARU, D. (1999) Carleman estimates, unique continuation and applications. Preprint, to appear.