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Topological derivative for optimal control problems

by

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Abstract: The topological derivative is introduced for the extremal values of cost functionals for control problems. The optimal control problem considered in the paper is defined for the elliptic equation which models the deflection of an elastic membrane. The derivative measures the sensitivity of the optimal value of the cost with respect to changes in topology. A change in topology means removing a small ball from the interior of the domain of integration. The topological derivative can be used for obtaining the numerical solutions of the shape optimization problems.

Keywords: shape optimization, shape derivative, topological derivative, asymptotic expansion, control problems.

1. Introduction

In Sokołowski, Żochowski (1999a) the so-called topological derivative was introduced for a class of shape functionals. In the present paper the derivative is obtained for the optimal value of the cost functional for an optimal control problem. The optimal value of the cost defines a domain functional depending on the domain of integration of the elliptic state equation. The results on the topological derivative method have been obtained by the authors for the optimal shape design problems, Sokołowski, Żochowski (1999a, b). In particular, the topological derivative method justifies and generalizes the so called "bubble method" used, e.g., in Schumacher (1995) for numerical methods of topology optimization for the compliance functional in the context of 2D elasticity. It for the optimal value of the cost functional for some control problems. Such approach would allow consideration of the simultaneous structure design and control modifications.

2. Control problem

Let us consider the domain Ω in R^2 with piecewise smooth boundary, its subset D (also with the piecewise smooth boundary) and the control problem having the elliptic state equation,

$$\begin{aligned} -\Delta y &= \chi_D \cdot u \quad \text{in} \quad \Omega, \\ y &= 0 \quad \text{on} \quad \Gamma = \partial\Omega. \end{aligned}$$

For given $u \in L^2(D)$, $y = y(u)$ represents, e.g., the deflection of an elastic membrane, loaded by the vertical force u concentrated on D. The characteristic function of D is denoted χ_D, and Ω is the reference domain for the membrane. For such a system we define the cost functional

$$I(u) = \mathcal{I}(y(u), u) = \int_D F(y, u) d\Omega = \frac{1}{2} \int_D [(y - y_0)^2 + \alpha u^2] d\Omega,$$

which is minimized over the space of controls $u \in L^2(D)$, $\alpha > 0$, where y_0 is a given function. Minimization of $I(u)$ with respect to u means approximation of a given function y_0 in the region D by the deflection (shape) of an elastic membrane, using the smallest possible load u applied in D. The extremal value of the cost functional for this control problem defines the shape functional, depending on the geometrical domain Ω,

$$\mathcal{J}(\Omega) = \min_{u \in L^2(D)} \mathcal{I}(y(u), u).$$

Variation of the state $y'(v)$, corresponding to the variation v of the control

$$y(u + sv) = y(u) + sy'(v) \ ,$$

satisfies the equation

$$\begin{aligned} -\Delta y' &= \chi_D \cdot v \quad \text{in} \quad \Omega, \\ y' &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

and the variation $\delta I(u; v)$ of the cost I is given by

$$\delta I(u; v) = \int_D [F_y(y, u) \cdot y'(v) + F_u(y, u) \cdot v] \, d\Omega.$$

Introduction of the adjoint equation

$$-\Delta p = \chi_D \cdot F_y(y, u) \quad \text{in} \quad \Omega,$$

allows us to express the first term in the cost variation as follows

$$\begin{aligned}\int_D F_y \cdot y' \, d\Omega &= -\int_\Omega \Delta p \cdot y' \, d\Omega = \int_\Omega \nabla p \cdot \nabla y' \, d\Omega = \\ &= -\int_\Omega p \cdot \Delta y' \, d\Omega = \int_D pv \, d\Omega.\end{aligned}$$

Hence

$$\delta I(u; v) = \int_D [p + F_u(y, u)] \cdot v \, d\Omega$$

and the stationarity condition

$$\delta I(u; v) = 0 \ , \quad \forall v \in L^2(D)$$

takes on the form

$$F_u(y(u; x), u(x)) = -p(x) \quad \text{a.e. in} \quad D \ .$$

For the specific choice of the cost functional, this results in the equality $u = -\frac{1}{\alpha}p$ and gives the extremal value of the cost functional for the control problem in the following form

$$\mathcal{J}(\Omega) = \frac{1}{2} \int_D [(y - y_0)^2 + \frac{1}{\alpha} p^2] \, d\Omega,$$

where y, p are given as a solution of the coupled system of equations:

$$\begin{aligned}-\Delta y &= -\chi_D \cdot \frac{1}{\alpha} p \quad \text{in} \quad \Omega, \\ -\Delta p &= \chi_D \cdot (y - y_0) \quad \text{in} \quad \Omega, \\ y &= 0 \quad \text{on} \quad \Gamma, \\ p &= 0 \quad \text{on} \quad \Gamma.\end{aligned}$$

3. Topological derivative

The variation of the geometrical domain Ω resulting in the change of the topological characteristic consists in removing a small ball centered at the point $x_0 \in \text{int}(\Omega \setminus D)$, such that $B(x_0, \rho) \subset \Omega \setminus D$ for sufficiently small $0 < \rho < \rho_0$. Denoting $\Omega_\rho = \Omega \setminus B(x_0, \rho)$, we define the optimal value of the cost functional for the control problem defined in the domain of integration Ω_ρ as a function depending on small parameter $\rho > 0$ and the point $x_0 \in \Omega$,

$$\mathcal{J}(\rho) = \mathcal{J}(x_0, \rho) = \frac{1}{2} \int [(\ldots)^2 + \frac{1}{\alpha} \ldots^2] \, d\Omega$$

The state and adjoint variables y_ρ, p_ρ are given by the unique solutions to the following optimality system,

$$\begin{aligned}
-\Delta y_\rho &= -\chi_D \cdot \frac{1}{\alpha} p_\rho \quad \text{in} \quad \Omega_\rho\,, \\
-\Delta p_\rho &= \chi_D \cdot (y_\rho - y_0) \quad \text{in} \quad \Omega_\rho\,, \\
y_\rho &= 0 \quad \text{on} \quad \Gamma, \\
p_\rho &= 0 \quad \text{on} \quad \Gamma, \\
\partial y_\rho / \partial n &= 0 \quad \text{on} \quad \Gamma_\rho\,, \\
\partial p_\rho / \partial n &= 0 \quad \text{on} \quad \Gamma_\rho\,,
\end{aligned}$$

where $\Gamma_\rho = \partial B(x_0, \rho)$. Observe that we have imposed the free edge condition on the boundaries of holes.

Our objective is to analyse the behaviour of $J(\rho) = \mathcal{J}(\Omega_\rho)$ as $\rho \to 0+$. To this end we evaluate the limits of derivatives $J'(\rho), J''(\rho)$ for $\rho \downarrow 0$. Using the formulae given in Sokołowski, Zolesio (1992) and Sokołowski, Żochowski (1999a), the following form of the shape derivative of $\mathcal{J}(\Omega_\rho)$ is obtained,

$$J'(\rho) = \lim_{s \to 0} \frac{1}{s}(J(\rho + s) - J(\rho)) = \int_D [(y_\rho - y_0) y'_\rho + \frac{1}{\alpha} p_\rho p'_\rho]\, d\Omega,$$

where y'_ρ, p'_ρ are strong shape derivatives, Sokołowski, Zolesio (1992), of solutions y_ρ, p_ρ to the state equation and the adjoint state equation, respectively. The shape derivatives y'_ρ, p'_ρ satisfy the following equations in the weak forms:

$$\int_{\Omega_\rho} [\nabla y'_\rho \cdot \nabla \phi_1 + \frac{1}{\alpha} \chi_D \cdot p'_\rho \phi_1] d\Omega = \int_{\Gamma_\rho} \frac{\partial y_\rho}{\partial \tau} \frac{\partial \phi_1}{\partial \tau} d\Gamma$$

$$\int_{\Omega_\rho} [\nabla p'_\rho \cdot \nabla \phi_2 - \chi_D \cdot y'_\rho \phi_2] d\Omega = \int_{\Gamma_\rho} \frac{\partial p_\rho}{\partial \tau} \frac{\partial \phi_2}{\partial \tau} d\Gamma$$

for all test functions $\phi_1, \phi_2 \in H_0^1(\Omega) \cap H^2(\Omega)$, where $\partial/\partial\tau$ denotes the derivative in the tangential direction to the boundary of the hole. In order to simplify the form of the derivative $J'(\rho)$, we must introduce the second level adjoint variables ξ_ρ, η_ρ, defined by the following system of equations:

$$\begin{aligned}
-\Delta \xi_\rho - \chi_D \cdot \eta_\rho &= \chi_D \cdot (y_\rho - y_0) \quad \text{in} \quad \Omega_\rho\,, \\
-\Delta \eta_\rho + \frac{1}{\alpha} \chi_D \cdot \xi_\rho &= \frac{1}{\alpha} \chi_D \cdot p_\rho \quad \text{in} \quad \Omega_\rho\,, \\
\xi_\rho &= 0 \quad \text{on} \quad \Gamma, \\
\eta_\rho &= 0 \quad \text{on} \quad \Gamma, \\
\partial \xi_\rho / \partial n &= 0 \quad \text{on} \quad \Gamma_\rho\,, \\
\partial \eta_\rho / \partial n &= 0 \quad \text{on} \quad \Gamma_\rho\,,
\end{aligned}$$

or, in the weak form, with the test functions $\psi_1, \psi_2 \in H_0^1(\Omega)$:

$$\int_{\Omega_\rho} [\nabla\eta_\rho \cdot \nabla\psi_2 + \frac{1}{\alpha}\chi_D \cdot \xi_\rho\psi_2] d\Omega = \int_D \frac{1}{\alpha} p_\rho \psi_2 \, d\Omega.$$

After substituting $\phi_1 := \xi_\rho$, $\phi_2 := \eta_\rho$, $\psi_1 := y'_\rho$, $\psi_2 := p'_\rho$ these integral identities take on the forms

$$\int_{\Omega_\rho} [\nabla y'_\rho \cdot \nabla\xi_\rho + \frac{1}{\alpha}\chi_D \cdot p'_\rho\xi_\rho] d\Omega = \int_{\Gamma_\rho} \frac{\partial y_\rho}{\partial\tau}\frac{\partial\xi_\rho}{\partial\tau} d\Gamma,$$

$$\int_{\Omega_\rho} [\nabla p'_\rho \cdot \nabla\eta_\rho - \chi_D \cdot y'_\rho\eta_\rho] d\Omega = \int_{\Gamma_\rho} \frac{\partial p_\rho}{\partial\tau}\frac{\partial\eta_\rho}{\partial\tau} d\Gamma,$$

$$\int_{\Omega_\rho} [\nabla\xi_\rho \cdot \nabla y'_\rho - \chi_D \cdot \eta_\rho y'_\rho] d\Omega = \int_D (y_\rho - y_0) y'_\rho \, d\Omega,$$

$$\int_{\Omega_\rho} [\nabla\eta_\rho \cdot \nabla p'_\rho + \frac{1}{\alpha}\chi_D \cdot \xi_\rho p'_\rho] d\Omega = \int_D \frac{1}{\alpha} p_\rho p'_\rho \, d\Omega,$$

and as a result we get the following expression for the first order shape derivative of the functional $\rho \to \mathcal{J}(\Omega_\rho) = J(\rho)$:

$$J'(\rho) = \int_{\Gamma_\rho} [\frac{\partial y_\rho}{\partial\tau}\frac{\partial\xi_\rho}{\partial\tau} + \frac{\partial p_\rho}{\partial\tau}\frac{\partial\eta_\rho}{\partial\tau}] \, d\Gamma = \int_{\Gamma_\rho} G \, d\Gamma.$$

In order to obtain the limit of $J'(\rho)$ for $\rho \downarrow 0$, we use the asymptotic expansions for the solutions of elliptic equtions in the neighbourhood of the small circular hole with respect to the radius of the hole, see Göhde (1985), Herwig (1989), Sokołowski, Żochowski (1999a), and we refer to Section 4 for the details on asymptotic expansions. Observe that in the case of our control problem the functions y_ρ, p_ρ are not solutions to single equations, but to a coupled system of equations. However, y_ρ, p_ρ are harmonic outside of D, and it can be shown that the same type of expansions as obtained for the single equation can be derived for y_ρ, p_ρ. This can be expressed in the following way. Let

$$\nabla y(x_0) = [a, b]^T \, , \quad x_0 \in \Omega \setminus D \, .$$

The solution y_ρ as a function of polar coordinates r, θ in a neighbourhood of the ball $B(x_0, \rho)$ can be expressed for $r \geq \rho$ as follows:

$$y_\rho = y + a\frac{\rho^2}{r}\cos\theta + b\frac{\rho^2}{r}\sin\theta + \mathcal{R}$$

where

$$\mathcal{R} = \rho^2 \, [O(\frac{\rho}{r}) + l(\rho, r)],$$

and $l(\rho, r)$ may contain finite powers of $\ln\rho, \ln r$. Hence $\mathcal{R} = O(\rho^{2-\epsilon})$ for any

y in the neighbourhood of x_0 and using the Taylor expansion for y, we have the following expansion for y_ρ,

$$y_\rho = y(x_0) + a(\frac{\rho^2}{r} + r)\cos\theta + b(\frac{\rho^2}{r} + r)\sin\theta + O(\rho^{2-\epsilon}),$$

where $y(x_0)$ denotes the value at x_0 of the solution to the elliptic equation in the domain Ω, i.e., in the full domain without hole.

The above formulae are given in the polar coordinate system with the center at x_0, which coincides with the center of the ball. In particular, from the expansion it follows that, see Sokołowski, Żochowski (1999a) for a proof,

$$\frac{\partial y_\rho}{\partial\tau} = \frac{1}{\rho}\frac{\partial y_\rho}{\partial\theta}|_{r=\rho} = 2(-a\sin\theta + b\cos\theta) + O(\rho^{1-\epsilon}).$$

Now, using these expansions for $y_\rho, p_\rho, \xi_\rho, \eta_\rho$, it can be shown that

$$\lim_{\rho\to 0+} J'(\rho) = 0,$$

and therefore we evaluate the second derivative $J''(\rho)$. It should be noted, that the existence of asymptotic expansions for functions ξ_ρ, η_ρ requires a separate proof which is given in Sokołowski, Żochowski (1999a), since in the system of equations for ξ_ρ, η_ρ not only the geometrical domains of integration, but also right–hand sides depend on the parameter ρ. The use of appropriate formulae given, e.g., in Sokołowski, Zolesio (1992) and Sokołowski, Żochowski (1999a), leads to the following form of the second order derivative

$$J''(\rho) = \int_{\Gamma_\rho} [\frac{d}{d\rho}G - \frac{\partial}{\partial n}G]\, d\Gamma + \frac{1}{\rho}\int_{\Gamma_\rho} G\, d\Gamma.$$

Again, it follows that the first integral vanishes as $\rho \to 0+$ and we obtain

$$\begin{aligned} J''(0) &= \lim_{\rho\to 0+} J''(\rho) = \\ &= \lim_{\rho\to 0+} \frac{1}{\rho}\int_{\Gamma_\rho} [\frac{\partial y_\rho}{\partial\tau}\frac{\partial\xi_\rho}{\partial\tau} + \frac{\partial p_\rho}{\partial\tau}\frac{\partial\eta_\rho}{\partial\tau}]\, d\Gamma. \end{aligned}$$

Using once more the asymptotic expansions and the explicit form of the above integral allows us to perform the passage to the limit which results in the final formula for the second order derivative at $\rho = 0^+$.

THEOREM 3.1 *The topological derivative*

$$\mathcal{T}(x_0) = \lim_{\rho\to 0^+} \frac{\mathcal{J}(\Omega\setminus B(x_0,\rho)) - \mathcal{J}(\Omega)}{|B(x_0,\rho)|}$$

is given by the following formula

$$\mathcal{T}(x_0) = 2/\pi J''(0; x_0) = 8\cdot[\nabla y(x_0)\cdot\nabla\xi(x_0) + \nabla p(x_0)\cdot\nabla\eta(x_0)]\ ,$$

The formula gives the map of the second derivative of the functional $J(\rho)$ at $\rho = 0^+$ as a function of the point $x_0 \in \Omega$. The negative value of $J''(0; x_0)$ indicates that removing from Ω a sufficiently small ball around x_0 would result in decreasing of the optimal value $\mathcal{J}(\Omega)$ of the cost functional for control problem, giving rise to the new, improved design, but with different topological characteristics.

4. The asymptotic expansion

For the convenience of the reader we provide a proof of the asymptotic expansions which are used in the present paper as well as in Sokołowski, Żochowski (1999a). We consider a scalar equation.

Let us recall that in the polar coordinate system

$$\Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\phi\phi}, \tag{1}$$

and, if w is radially symmetric,

$$\Delta w = w_{rr} + \frac{1}{r} w_r. \tag{2}$$

Let $R_0 < R_1$ and $P(R_0, R_1)$ be a ring

$$P(R_0, R_1) = \{x \in \mathbb{R}^2 \mid R_0 < |x| < R_1\}.$$

Define also the circle $C(R)$

$$C(R) = \{x \in \mathbb{R}^2 \mid |x| = R\}.$$

Assume that $0 < \rho < \frac{1}{2} R_0$ and $R_0 < R$. We define the boundary value problems, parametrized by ρ:

$$\begin{aligned} \Delta w_\rho &= 0 \quad \text{in} \quad P(\rho, R), \\ w_\rho &= 0 \quad \text{on} \quad C(R), \\ \frac{\partial w_\rho}{\partial n} &= -\rho \cdot h_\rho \quad \text{on} \quad C(\rho). \end{aligned} \tag{3}$$

Furthermore, we assume that the function h_ρ is continuous on $C(\rho)$ and bounded,

$$\|h_\rho\|_{L^2(C(\rho))} \le \Lambda_1,$$

uniformly with respect to ρ.

LEMMA 4.1 *The function w_ρ satisfies the following conditions:*
i) on any curve $\Gamma \subset P(R_0, R)$ it may be expressed as

$$w_\rho = \rho^2 \cdot g_\rho,$$

where $|g_\rho| \le \Lambda_2(\Lambda_1)$ on Γ;
ii) on $C(\rho)$ it satisfies inequality

Proof. The function h_ρ has the Fourier series expansion

$$h_\rho = c_\rho + \sum_{k=1}^{\infty}(a_{\rho,k}\sin k\phi + b_{\rho,k}\cos k\phi),$$

and in addition

$$c_\rho^2 + \sum_{k=1}^{\infty}(a_{\rho,k}^2 + b_{\rho,k}^2) \le \Lambda_1^2. \tag{4}$$

We determine the solution w_ρ of (3) also in the form of a series.

The first term, corresponding to c_ρ, is radially symmetric, and taking into account (2), has the following representation

$$w_\rho^0 = A + B\ln r.$$

From the boundary condition

$$\frac{\partial w_\rho}{\partial n} = -\rho \cdot c_\rho \quad \text{on} \quad C(\rho),$$

it follows that

$$\begin{aligned} A + B\ln R &= 0, \\ B\frac{1}{\rho} &= \rho c_\rho, \end{aligned}$$

hence

$$B = \rho^2 c_\rho, \qquad A = -\rho^2 c_\rho \ln R,$$

where $|c_\rho| \le \Lambda_1$.

Finally,

$$w_\rho^0 = \rho^2 c_\rho \ln r - \rho^2 c_\rho \ln R. \tag{5}$$

Consider now the term corresponding to the boundary condition

$$\frac{\partial w_\rho^k}{\partial n} = -\rho \cdot a_{\rho,k} \cdot \sin k\phi \quad \text{on} \quad C(\rho),$$

for $k \ge 1$ (the cosine term can be treated in the same way). We seek the solution in the form

$$w_\rho^k = v(r) \cdot \sin k\phi,$$

and, considering (1), get the representation

Again, from the boundary conditions it follows that

$$\begin{aligned} AR^k + B\frac{1}{R^k} &= 0, \\ kA\rho^{k-1} - kB\frac{1}{\rho^{k+1}} &= \rho a_{\rho,k}. \end{aligned}$$

Hence

$$w_\rho^k = \frac{\rho^{k+2} \cdot a_{\rho,k}}{k(R^{2k} + \rho^{k-1})}(r^k - \frac{R^{2k}}{r^k}) \sin k\phi, \tag{6}$$

where $|a_{\rho,k}| \leq \Lambda_1$.

Now, substituting $r := \rho$ in (5),(6) we get

$$\begin{aligned} |w_\rho^0(\rho)| &\leq \Lambda_2(\Lambda_1)\rho^2 |\ln \rho|, \\ |w_\rho^k(\rho)| &\leq \Lambda_3(\Lambda_1)\rho^2. \end{aligned}$$

The convergence of the series for the solution w_ρ for small ρ follows immediately from (4). ■

4.1. Derivation of the asymptotic expansions

The solution u_ρ to the equation

$$\begin{aligned} \Delta u_\rho &= f \quad \text{in} \quad \Omega_\rho, \\ u_\rho &= g \quad \text{on} \quad \Gamma, \\ \frac{\partial u_\rho}{\partial n} &= 0 \quad \text{on} \quad \Gamma_\rho = C(\rho). \end{aligned} \tag{7}$$

can be represented in the form

$$u_\rho = u_0 + s_\rho + v_\rho \tag{8}$$

where

$$s_\rho = \frac{\rho^2}{r^2}(a \cos\phi + b \sin\phi)$$

and $[a, b] = \nabla u_0(0)$.

In Ω_ρ the function v_ρ satisfies the boundary value problem:

$$\begin{aligned} \Delta v_\rho &= 0 \quad \text{in} \quad \Omega_\rho, \\ v_\rho &= -s_\rho \quad \text{on} \quad \Gamma, \\ \frac{\partial v_\rho}{\partial n} &= -\rho \cdot h_\rho \quad \text{on} \quad \Gamma_\rho = C(\rho). \end{aligned} \tag{9}$$

We consider only Dirichlet conditions on Γ, but mixed conditions are treated in

The function $h_\rho = -\frac{1}{\rho}\frac{\partial u_0}{\partial n}\frac{\partial s_\rho}{\partial n}$ satisfies the inequality

$$|h_\rho| \le \Lambda_1 \tag{10}$$

for any $x \in C(\rho)$. This follows from the fact that there exist $0 < R_0 < R$ such that

$$B(R_0) \subset \Omega \subset B(R),$$

and we consider only $\rho < \frac{1}{2}R_0$.

Next, we express v_ρ in the form of the series

$$v_\rho = p_\rho^1 + q_\rho^1 + \rho \cdot (p_\rho^2 + q_\rho^2) + \dots \tag{11}$$

where the functions p_ρ^1, q_ρ^1 satisfy the BVP's

$$\begin{aligned} \Delta p_\rho^1 &= 0 \quad \text{in} \quad P(\rho, R), \\ p_\rho^1 &= \quad \text{on} \quad C(R), \\ \frac{\partial p_\rho^1}{\partial n} &= -\rho \cdot h_\rho \quad \text{on} \quad C(\rho), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \Delta q_\rho^1 &= 0 \quad \text{in} \quad \Omega, \\ q_\rho^1 &= -s_\rho - p_\rho^1 \quad \text{on} \quad \Gamma. \end{aligned} \tag{13}$$

Observe that $|s_\rho| \le \Lambda_0 \cdot \rho^2$ on Γ.

From the Lemma,

$$\begin{aligned} |p_\rho^1| &\le \Lambda_3(\Lambda_1)\rho^2 |\ln \rho| \quad \text{on} \quad C(\rho), \\ |p_\rho^1| &\le \Lambda_2(\Lambda_1)\rho^2 \quad \text{on} \quad \Gamma. \end{aligned}$$

Similarly, from the form of s_ρ it follows that

$$|\frac{\partial q_\rho^1}{\partial n}| \le \Lambda_4(\Lambda_1)\rho^2 \tag{14}$$

on $C(\rho)$, since both s_ρ and p_ρ^1 are uniformly bounded by ρ^2 on Γ.

The next term in the series satisfies the BVPs:

$$\begin{aligned} \Delta p_\rho^2 &= 0 \quad \text{in} \quad P(\rho, R), \\ p_\rho^2 &= 0 \quad \text{on} \quad C(R), \\ \frac{\partial p_\rho^2}{\partial n} &= -\frac{\partial q_\rho^1}{\partial n} \quad \text{on} \quad C(\rho), \end{aligned} \tag{15}$$

and

$$\Delta q_\rho^2 = 0 \quad \text{in} \quad \Omega, \tag{16}$$

It is obvious from (14) and the Lemma that it constitutes a higher order correction.

In general, the idea is as follows: p_ρ^k satisfies the condition on $C(\rho)$ and simultaneously disturbs the condition on Γ. Then, q_ρ^k corrects the condition on Γ, but changes again the condition on $C(\rho)$. However, this new disturbance is by an order of magnitude smaller.

The fact that all the constants Λ_i are uniform with respect to ρ guarantees the convergence of the series (11).

The final conclusion results from (8) and the internal regularity of u_0. In $B(R_0)$ its value in $P(\rho, 2\rho)$ may be expressed as

$$u_0 = u_0(0) + ar\cos\phi + br\sin\phi + O(\rho^2), \tag{17}$$

where $O(\rho^2)$ is uniform with respect to ρ. Substitution into (8) gives

$$u_0 = u_0(0) + (r + \frac{\rho^2}{r})(a\cos\phi + b\sin\phi) + O(\rho^2) + v_\rho, \tag{18}$$

where

$$|v_\rho| \leq \Lambda\rho^2|\ln\rho|.$$

5. Numerical example

We consider the domain $\Omega = [-3,3]\times[-3,3]$ and its subset $D = [-1,1]\times[-1,1]$. The boundary conditions are the same as in the previous sections, and the reference solution is

$$y_0 = 9 - r^2.$$

The control penalty parameter is $\alpha = 0.2$. In Fig.1 we see the solution, which should on the central square approximate y_0. Indeed, the graph of the difference $y - y_0$ in Fig.2 confirms this. Fig.4 shows the contour map of $J''(0;x)$. The darker the shade, the smaller the value of the function. Noting the dark patches, we locate the places where the material should be weakened in order to decrease the extremal value of the goal functional (better accuracy at smaller cost). Observe that they coincide with the regions of high control value, see Fig.3, as should be expected.

6. Conclusions

The methodology described in this paper works well in these examples, where we can get the asymptotic expansions of the solution to the state equation.

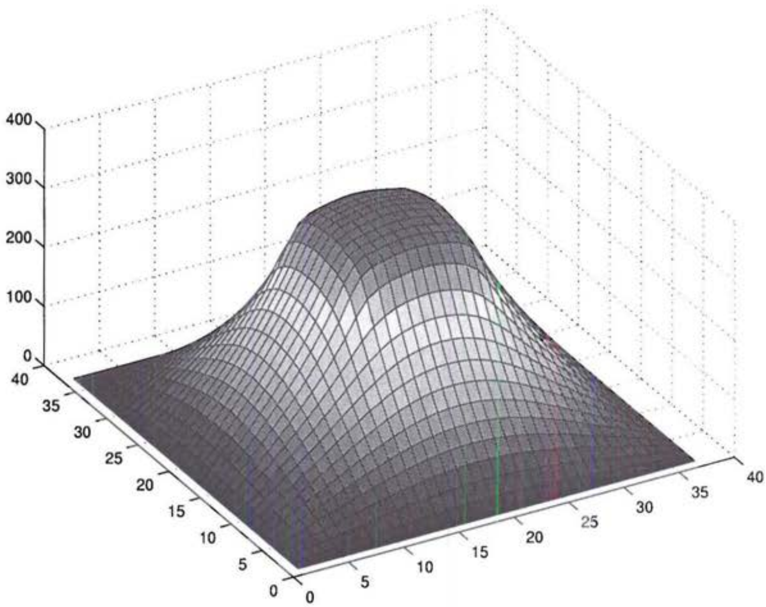

Figure 1. The solution y.

functions y_ρ, p_ρ cease to be harmonic around holes, since they satisfy the system of equations

$$\begin{aligned} -\Delta y &= -\frac{1}{\alpha} p \quad \text{in} \quad \Omega, \\ -\Delta p &= (y - y_0) \quad \text{in} \quad \Omega, \\ y &= 0 \quad \text{on} \quad \Gamma, \\ p &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

which may be transformed for the single variable to

$$\Delta^2 p + k^2 p = 0.$$

The asymptotic expansions are in this case much more complicated. We have presented the simpler case in order not to obscure the idea of application of the topological derivative in control problems.

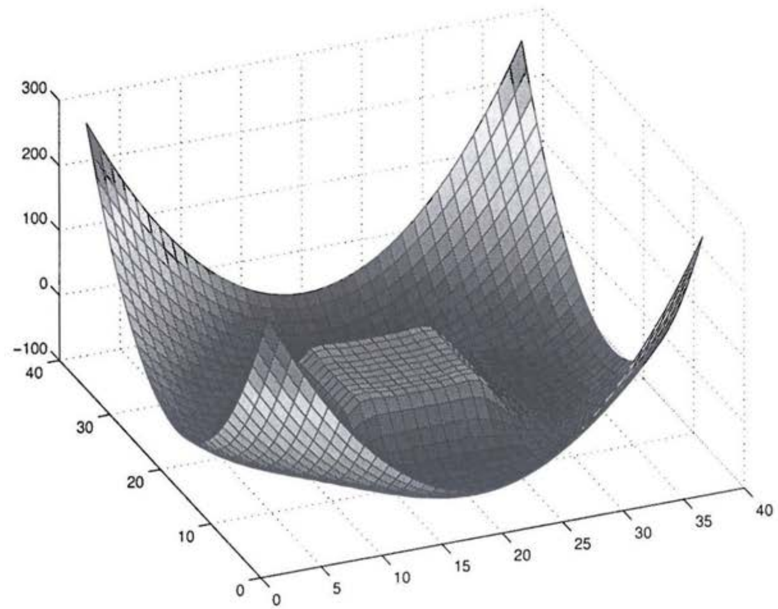

Figure 2. The difference between solution y and y_0.

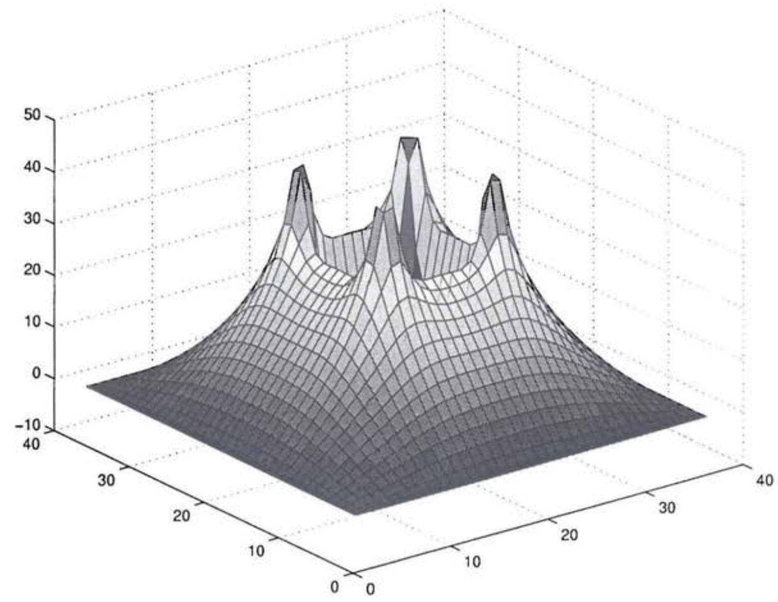

Figure 3. The value of the control

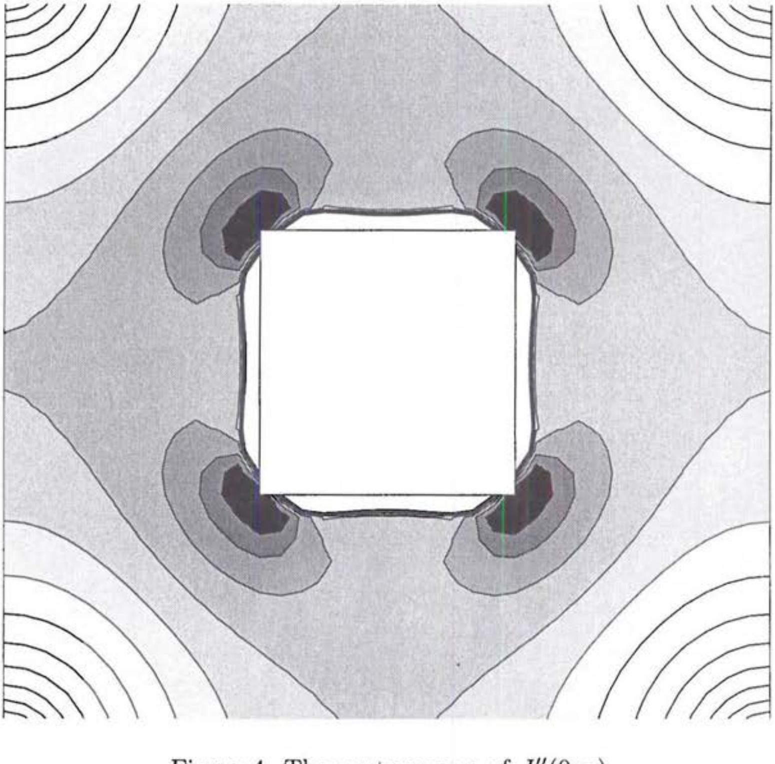

Figure 4. The contour map of $J''(0;x)$.

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