Control and Cybernetics

vol. **28** (1999) No. 4

Multiobjective duality for the Markowitz portfolio optimization problem

by

Gert Wanka

Faculty of Mathematics, Technical University Chemnitz,
D-09107 Chemnitz, Germany
e-mail: wanka@mathematik.tu-chemnitz.de

Abstract: The classical Markowitz approach to portfolio selection leads to a biobjective optimization problem where the objectives are the expected return and the variance of a portfolio. In this paper a biobjective dual optimization problem to the Markowitz portfolio optimization problem is introduced and analyzed. For the Markowitz problem and its dual, weak and strong vector duality assertions are derived. The optimality conditions are also verified.

Keywords: portfolio optimization, duality, optimality conditions

1. Introduction

Consider n risky securities $S_1, \ldots, S_n$. Assume that r_i is the return on security S_i $(i = 1, \ldots, n)$, $\mu_i = E(r_i)$ is the expected value of return r_i, and the covariance between r_i and r_j is $\sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)]$. In particular, the variance of r_i is represented by σ_{ii}.

In order to reduce the risk, diversified portfolios are usually built for investments at the capital market. Let x_i be the share of the investor's capital that is allocated to security S_i. This defines a portfolio as a vector $x = (x_1, \ldots, x_n)^T$ with $x_i \geq 0$ for $i = 1, \ldots, n$ and $e^T x = 1$ where e is the n-dimensional vector of unit coordinates, $e = (1, \ldots, 1)^T$. The portfolio x is characterized by the expected return

$$E(x) = \sum_{i=1}^{n} x_i \mu_i$$

and the variance

$$V(x) = \sum^{n} \sigma_{ij} x_i x_j \ .$$

According to the Markowitz (1952) theory the investor intends to maximize the expected return and to minimize the risk measured with variance. Because these two objectives are typically in conflict, an adequate and reasonable solution notion is that of efficiency (Pareto-optimality). This leads to the classical and well-known portfolio optimization problem with two objectives (Markowitz, 1989, Linke, 1996, Elton,1991, and Sharpe, 1970):

$$\begin{array}{llll} (P) & F(x) & \to & v-\min \\ & & & \text{s.t. } e^T x = 1 \\ & & & x = (x_1, \ldots, x_n)^T \\ & & & x_i \geq 0, i = 1, \ldots, n \end{array}$$

where

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -E(x) \\ V(x) \end{pmatrix} = \begin{pmatrix} -\sum\limits_{i=1}^{n} \mu_i x_i \\ \sum\limits_{i,j=1}^{n} \sigma_{ij} x_i x_j \end{pmatrix}.$$

A point (portfolio) $x = (x_1, \ldots, x_n)^T$ that fulfils constraints $e^T x = 1$ and $x_i \geq 0, i = 1, \ldots, n$, is said to be an admissible point of (P). The aim of this paper is to present a dual multiobjective problem to the Markowitz portfolio optimization problem (P) and to prove the so-called weak and strong duality assertions. Moreover, we will use the strong duality assertions to derive the necessary and sufficient optimality conditions for the portfolio problem.

There are some comprehensive presentations of duality in multiobjective optimization, given by, for instance, Göpfert and Nehse (1990), Jahn (1986), Nakayama et al. (1985). Several authors applied general concepts of duality in vector optimization to specific problems or have established direct consideration for such problems independently from a general approach. A first dual pair in linear vector optimization was given by Gale et al. (1951). Later, Isermann (1978) introduced a dual problem in linear vector optimization which turns out to be a direct generalization of the scalar linear duality. Duality for geometric vector optimization was analyzed by Elster et al. (1989). Explicit formulations of dual problems also have been derived for multicriteria location and control-approximation problems by Tammer and Tammer (1991), Wanka (1991a, 1991b). However, to our best knowledge, multiobjective duality for the Markowitz portfolio optimization problem has not been investigated until now.

2. Solutions of the portfolio problem and the dual problem formulation

Recall the definitions of efficiency and proper efficiency (Göpfert, 1990; Jahn, DEFINITION 2.1 *An admissible point* $\overset{o}{x}= (\overset{o}{x}_1, \dots, \overset{o}{x}_n)^T$ *is said to be an efficient point (or solution) to (P) if there is no admissible point* $x = (x_1, \dots, x_n)^T$ *such that* $f_i(x) \le f_i(\overset{o}{x}), i = 1, 2,$ *and* $f_j(x) < f_j(\overset{o}{x})$ *for at least one index* $j \in \{1, 2\}$.

This is the usual definition of the efficiency (Pareto-optimality) in the case of two objectives.

DEFINITION 2.2 *An admissible point* $\overset{o}{x}= (\overset{o}{x}_1, \dots, \overset{o}{x}_n)^T$ *is said to be a properly efficient point (solution) to (P) if there exists a scalarizing vector* $\overset{o}{\lambda}= (\overset{o}{\lambda}_1, \overset{o}{\lambda}_2)^T$, $\overset{o}{\lambda}_i > 0$, $i = 1, 2$, *such that* $\overset{o}{\lambda}_1 \; f_1(\overset{o}{x}) + \overset{o}{\lambda}_2 \; f_2(\overset{o}{x}) \le \overset{o}{\lambda}_1 \; f_1(x) + \overset{o}{\lambda}_2 \; f_2(x)$ *for all admissible points* x.

Obviously, a properly efficient point is an efficient one. Note, however, that there exist different definitions of properly efficient solutions (Göpfert and Nehse, 1990), the classical one having been given by Geoffrion (1968). Relations between the different definitions have been explained by Göpfert and Nehse (1990). In particular, for convex objective functions the definition on the basis of linear scalarization (Definition 2.2) is an usual approach (Jahn, 1986; Göpfert and Nehse, 1990) and the properly efficient solutions defined in this way are also properly efficient in the sense of Geoffrion (Göpfert and Nehse, 1990).

Our aim is to formulate a multiobjective dual problem (P^*) to the portfolio optimization problem (P) and to verify the weak and strong duality assertions as well as the optimality conditions for properly efficient solutions to (P) and efficient solutions to the dual problem (P^*), respectively.

For the portfolio optimization problem (P) we introduce its dual (P^*) as the following bicriteria optimization problem:

$$(P^*) \quad G(y,z) = \begin{pmatrix} g_1(y,z) \\ g_2(y,z) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 - \sum\limits_{i,j=1}^{n} \sigma_{ij} y_i y_j \end{pmatrix} \quad \longrightarrow \quad \begin{array}{c} v - \max \\ (y,z) \in \tilde{B} \end{array}$$

with the dual variables $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$, $z = (z_1, z_2)^T \in \mathbb{R}^2$ and the set $\tilde{B}$ of constraints

$$\tilde{B} = \{ \; (y,z) \in \mathbb{R}^n \times \mathbb{R}^2 \quad : \quad \begin{array}{l} \exists \lambda_1 > 0, \lambda_2 > 0 \text{ such that} \\ \lambda_1 z_1 e + \lambda_2 (2\sigma y + z_2 e) \le -\lambda_1 \mu \; \} \end{array} \tag{1}$$

where vector $\mu = (\mu_1, \dots, \mu_n)^T$ is the vector of the expected returns and $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$ denotes the covariance matrix to returns $r_1, \dots, r_n$. The notation $v - \max$ means here the vector maximization in the sense of determination of efficient (Pareto-optimal) elements. An element (point) $(y,z) \in \tilde{B}$ is called admissible to (P^*). The definition of efficient solutions to (P^*) is analogous to

DEFINITION 2.3 *An admissible element* $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ *is said to be an efficient solution to* (P^*) *if there is no admissible element* $(y, z) \in \tilde{B}$ *such that* $g_i(y, z) \geq g_i(\overset{o}{y}, \overset{o}{z})$ *for* $i = 1, 2$ *and* $g_j(y, z) > g_j(\overset{o}{y}, \overset{o}{z})$ *for at least one index* $j \in \{1, 2\}$.

Hereafter, we call problem (P^*) the dual portfolio optimization problem. This is because (P^*) has properties that are characteristic for dual problems in multiobjective optimization.

Consider two multiobjective optimization problems:
a minimum problem

$$\begin{aligned} F(x) \rightarrow \ & v - \min \\ & x \in A \end{aligned} \tag{2}$$

and a maximum one

$$\begin{aligned} G(y) \rightarrow \ & v - \max \ . \\ & y \in B \end{aligned} \tag{3}$$

It is assumed that

$$F(x) = (f_1(x), \ldots, f_m(x))^T,\ G(y) = (g_1(y), \ldots, g_m(y))^T \in \mathbb{R}^m .$$

The usual partial ordering in $\mathbb{R}^m$ is defined by

$$u = (u_1, \ldots, u_m)^T \geq v = (v_1, \ldots, v_m)^T \text{ if } u_i \geq v_i \text{ for } i = 1, \ldots, m.$$

DEFINITION 2.4 *The property that there is no* $x \in A$ *and no* $y \in B$ *such that* $G(y) \geq F(x)$ *and* $G(y) \neq F(x)$ *is called the weak duality property for problems (2) and (3).*

Definition 2.4 is a natural generalization of the so-called weak duality property within the usual scalar optimization with one objective function, i.e. $F(x) = f_1(x) \in \mathbb{R}$, $G(y) = g_1(y) \in \mathbb{R}$. The weak duality means that $G(y) \leq F(x)$ for all y and x admissible to the respective problems. This is briefly described by the formulation:

$$\sup G(y) \leq \inf F(x). \tag{4}$$

In general, a duality gap may occur which means that $\sup G(y) < \inf F(x)$. If inequality (4) is fulfilled as an equality, then, we say that the so-called strong duality occurs. Sometimes the strong duality property is understood in the stronger sense, that is, there exist solutions to $\inf F(x)$ and (or) $\sup G(y)$ such that $\max G(y) = G(\overset{o}{y}) = F(\overset{o}{x}) = \min F(x)$.

If multiobjective problems (2) and (3) with the weak duality properties have the problems have the strong vector duality property. One can distinguish between a weak and a strong form of the strong duality depending on the fact whether the equality $F(\overset{o}{x}) = G(\overset{o}{y})$ is fulfilled only for a certain (single) pair of points $\overset{o}{x}$ and $\overset{o}{y}$ or for all (properly) efficient elements to problems (2) and (3), respectively. Indeed, it follows from the weak duality property and the equality of the objective function values $G(\overset{o}{y}) = F(\overset{o}{x})$ that $\overset{o}{y}$ as well as $\overset{o}{x}$ are efficient to the corresponding problems (3) and (2).

The weak form of the strong duality property can be interpreted geometrically as touching of the image sets and also of the efficient frontiers of problems (2) and (3) in single points. Similarly, the strong form of the strong duality property means that the efficient frontiers coincide at least for all efficient points to (2) or to (3) or even for all efficient points to both problems (2) and (3). We have to distinguish between these different cases because it may happen that there are efficient points to problem (3) for which there is no corresponding efficient point to problem (2), and vice versa. In this situation we have only the coincidence of parts of the efficient frontiers of both problems. In other words, the efficient frontiers of problems (2) and (3) can have a common intersection or even coincide in the case of the strong duality property. Otherwise, under the assumption of the weak duality property there may be a duality gap as in scalar optimization.

From Definition 2.4 it follows that the property of weak duality provides an opportunity to construct lower bounds for efficient solutions of the primal problem (2) and upper bounds for efficient points of the dual problem (3), as in scalar optimization. For example, if we are given an admissible point $\overset{o}{y}$ to the dual problem (3), then $G(\overset{o}{y})$ represents a lower bound in the sense that there are no admissible points x to the primal problem (2) such that $F(x) \leq G(\overset{o}{y})$ and $F(x) \neq G(\overset{o}{y})$ with respect to the partial ordering considered for problems (2) and (3), respectively. If we find an admissible point $\overset{o}{x}$ fulfilling $F(\overset{o}{x}) = G(\overset{o}{y})$, then its efficiency is guaranteed. In the case of the strong duality properties, one can solve the dual problem (3) getting an efficient solution $\overset{o}{y}$ and the corresponding $G(\overset{o}{y})$. This yields the objective function value $F(\overset{o}{x}) = G(\overset{o}{y})$ of a primal efficient solution $\overset{o}{x}$, i.e. the remaining problem is to solve the equation $F(x) = g$ with the known right hand side $g = G(\overset{o}{y})$.

An additional opportunity is the formulation of optimality conditions to the primal and dual problem by means of strong duality. This gives also conditions, equations or inequalities etc., for the determination of efficient solutions. Thus the assertions of duality play a useful role both in scalar optimization and in multiobjective optimization.

3. Weak duality

In the remainder of the paper we will point out weak duality as well as strong duality for the portfolio optimization problem (P) and its dual (P^*). We start with the weak duality theorem (cf. Definition 2.4).

THEOREM 3.1 *There is no admissible point x to (P) and no admissible point (y, z) to (P^*) such that $G(y, z) \geq F(x)$ and $G(y, z) \neq F(x)$.*

Proof. Let us assume that the assertion of Theorem 3.1 is not true. Then there exist x admissible to (P) and (y, z) admissible to (P^*) with corresponding numbers $\lambda_1 > 0$ and $\lambda_2 > 0$ and a vector $k = (k_1, k_2)^T \neq (0, 0)^T$, $k_1 \geq 0$, $k_2 \geq 0$ satisfying the equation $G(y, z) = F(x) + k$. This implies that

$$\begin{array}{rcl} \lambda_1 f_1(x) + \lambda_2 f_2(x) & = & \lambda_1 g_1(y, z) + \lambda_2 g_2(y, z) - \lambda_1 k_1 - \lambda_2 k_2 \\ & < & \lambda_1 g_1(y, z) + \lambda_2 g_2(y, z). \end{array}$$

On the other hand, we will show that

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) \geq \lambda_1 g_1(y, z) + \lambda_2 g_2(y, z)$$

which leads to contradiction.

We may note that from the inequality defining the set of constraints $\tilde{B}$ of (P^*) with $x \geq 0$ (i.e. $x_i \geq 0\,,\ i = 1, \ldots, n$) it follows that

$$0 \geq x^T [\lambda_1(\mu + z_1 e) + \lambda_2(2\sigma y + z_2 e)]\,.$$

The above inequality allows to obtain

$$\begin{array}{rcl} \lambda_1 f_1(x) + \lambda_2 f_2(x) = & & \lambda_1(-E(x)) + \lambda_2 V(x) \\ & = & \lambda_1(-\sum\limits_{i=1}^{n} \mu_i x_i) + \lambda_2 \sum\limits_{i,j=1}^{n} x_i x_j \sigma_{ij} \\ & \geq & \lambda_1(-x^T \mu + x^T(\mu + z_1 e) \\ & + & \lambda_2[\sum\limits_{i,j=1}^{n} x_i x_j \sigma_{ij} + x^T(2\sigma y + z_2 e)] \\ & = & \lambda_1 z_1 x^T e + \lambda_2 x^T \sigma x + 2\lambda_2 x^T \sigma y + \lambda_2 z_2 x^T e \\ & = & \lambda_1 z_1 + \lambda_2 z_2 + \lambda_2 (x + 2y)^T \sigma x \end{array} \tag{5}$$

because of $x^T e = e^T x = 1$ and $\sigma^T = \sigma$, i.e., $x^T \sigma y = y^T \sigma^T x = y^T \sigma x$.

Using the Schwarz inequality for positive semidefinite symmetric matrices (see the following Lemma 3.1), i.e.

$$\begin{array}{rl} -2y^T \sigma x \leq & 2(y^T \sigma y)^{\frac{1}{2}} (x^T \sigma x)^{\frac{1}{2}} \leq y^T \sigma y + x^T \sigma x \\ (\text{with } 2ab \leq & a^2 + b^2\,,\ a = (y^T \sigma y)^{\frac{1}{2}}\,,\ b = (x^T \sigma x)^{\frac{1}{2}})\,, \end{array} \tag{6}$$

we obtain

By substituting (7) into (5) we get

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) \geq \lambda_1 z_1 + \lambda_2 z_2 - \lambda_2 y^T \sigma y = \lambda_1 g_1(y,z) + \lambda_2 g_2(y,z)$$

which completes the proof. ∎

LEMMA 3.1 *Let $\sigma = \sigma^T$ be a positive semidefinite (n,n)-matrix. Then.*

$$|y^T \sigma x| \leq (y^T \sigma y)^{\frac{1}{2}} (x^T \sigma x)^{\frac{1}{2}} \,. \tag{8}$$

Proof. It is well known that for a positive definite matrix $\sigma = \sigma^T$, $\langle y, x\rangle_\sigma := y^T \sigma x$ defines a scalar product. Thus, inequality (8) represents the Schwarz inequality for that scalar product. If σ is only positive semidefinite, several cases have to be considered. For $y^T \sigma x = 0$, inequality (8) is trivially fulfilled. If $y^T \sigma x \neq 0$, inequality (8) can be proven in a similar manner as for positive definite matrix σ. Namely, starting with

$$0 \leq (y - \lambda x)^T \sigma (y - \lambda x) = y^T \sigma y - \lambda x^T \sigma y - \lambda y^T \sigma x + \lambda^2 x^T \sigma x$$

and substituting

$$\lambda = (y^T \sigma y)(y^T \sigma x)^{-1} \,,$$

we obtain

$$0 \leq -y^T \sigma y + (y^T \sigma y)^2 (y^T \sigma x)^{-2} (x^T \sigma x). \tag{9}$$

Therefore, for y such that $y^T \sigma y \neq 0$ inequality (8) follows from (9).

For y such that $y^T \sigma y = 0$, inequality (9) cannot be divided by $y^T \sigma y$. Moreover, (8) is violated because $y^T \sigma x \neq 0$. This situation, however, is impossible since $y^T \sigma x \neq 0$ implies also $y^T \sigma y \neq 0$. To verify this, let us assume that $y^T \sigma y = 0$ for some $y \neq 0$. Let $\lambda_1, \ldots, \lambda_k > 0$, $\lambda_{k+1} = \ldots = \lambda_n = 0$, $k \leq n-1$, be the eigenvalues of σ with the corresponding system of orthonormal eigenvectors y^i to λ_i, $i = 1, \ldots, n$. Let $y = \sum\limits_{i=1}^{n} \alpha_i y^i$. Since $\sigma y^i = 0$ for $i = k+1, \ldots, n$, we obtain

$$\begin{aligned} 0 = y^T \sigma y &= \left(\sum_{i=1}^{n} \alpha_i y^i\right)^T \sigma \left(\sum_{j=1}^{n} \alpha_j y^j\right) \\ &= \left(\sum_{i=1}^{n} \alpha_i y^i\right)^T \left(\sum_{j=1}^{k} \alpha_j \lambda_j y^j\right) = \sum_{i=1}^{k} \lambda_i \alpha_i^2 \,. \end{aligned}$$

This implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. Thus,

$$y = \sum_{i=k+1}^{n} \alpha_i y^i \quad \text{and} \quad \sigma y = \sum_{i=k+1}^{n} \alpha_i \sigma y^i = 0 \,,$$

that is, from $y^T \sigma y = 0$ it necessarily follows that $\sigma y = 0$. Hence, there is $y^T \sigma x = x^T \sigma y = 0$ which contradicts $y^T \sigma x \neq 0$.

4. The scalarized problem

In order to establish the strong vector duality we need the following proposition concerning optimality conditions for the scalarized portfolio optimization problem (P_λ) $(\lambda = (\lambda_1, \lambda_2),\ \lambda_i > 0, i = 1, 2)$ corresponding to (P):

$$(P_\lambda) \qquad \inf_{\substack{\text{s.t. } e^T x = 1 \\ x_i \geq 0,\ i = 1, \dots, n}} \{-\lambda_1 \mu^T x + \lambda_2 x^T \sigma x\}.$$

This is a quadratic programming problem. For such problems there exists a well elaborated duality theory (Elster et al., 1977; Dorn, 1960). For our special problem (P_λ) a suitable dual problem is:

$$(P_\lambda^*) \qquad \sup_{\substack{y \in \mathbb{R}^n, w \in \mathbb{R} \\ 2\lambda_2 \sigma y + we \leq -\lambda_1 \mu}} \{-\lambda_2 y^T \sigma y + w\}.$$

Due to the classical (scalar) duality theory, problems (P_λ) and (P_λ^*) satisfy the strong duality property, i.e. $\inf(P_\lambda) = \sup(P_\lambda^*)$. Moreover, the dual problem (P_λ^*) is always solvable. By means of the duality, the optimality conditions can be derived.

Proposition 4.1 *Let $\bar{x}$ be a solution to (P_λ). There exists a solution $(\bar{y}, \bar{w})$ to (P_λ^*) fulfilling the following optimality conditions:*
i) $\bar{y}^T \sigma \bar{y} + \bar{x}^T \sigma \bar{x} + 2\bar{y}^T \sigma \bar{x} = 0$,
ii) $\bar{x}^T (2\lambda_2 \sigma \bar{y} + \bar{w} e + \lambda_1 \mu) = 0$.

Proof. Because of the strong duality and the solvability of (P_λ^*) the optimal objective function values of (P_λ) and (P_λ^*) for the solutions $\bar{x}$ and $(\bar{y}, \bar{w})$, respectively, are equal. That is

$$0 = -\lambda_1 \mu^T \bar{x} + \lambda_2 \bar{x}^T \sigma \bar{x} + \lambda_2 \bar{y}^T \sigma \bar{y} - \bar{w}. \tag{10}$$

It is straightforward to verify the following equality:

$$0 = 2\lambda_2 \bar{y}^T \sigma \bar{x} - \bar{x}^T (2\lambda_2 \sigma \bar{y} + \bar{w} e) + \bar{w}. \tag{11}$$

Adding (11) to (10), we obtain

$$\begin{aligned} 0 &= -\lambda_1 \mu^T \bar{x} + \lambda_2 \bar{x}^T \sigma \bar{x} + \lambda_2 \bar{y}^T \sigma \bar{y} - \bar{w} \\ &\quad + 2\lambda_2 \bar{y}^T \sigma \bar{x} - \bar{x}^T (2\lambda_2 \sigma \bar{y} + \bar{w} e) + \bar{w} \\ &= \lambda_2 [\bar{y}^T \sigma \bar{y} + \bar{x}^T \sigma \bar{x} + 2\bar{y}^T \sigma \bar{x}] + [-\bar{x}^T (2\lambda_2 \sigma \bar{y} + \bar{w} e + \lambda_1 \mu)]. \end{aligned} \tag{12}$$

Due to (6) and since $\bar{x}$ and $(\bar{y}, \bar{z})$ are admissible to (P_λ) and (P_λ^*), respectively, the two expressions in the last line of (12) are nonnegative. Hence, equation (12) implies that the terms inside the square brackets are equal to zero, i.e. *i)*

REMARK 4.1 *The second condition of Proposition 4.1 can be interpreted as the complementary slackness condition.*

Note that if conditions $i)$ and $ii)$ of Proposition 4.1 are satisfied for $\bar{x}$ and $(\bar{y}, \bar{w})$ admissible to (P_λ) and (P^*_λ), respectively, then $\bar{x}$ and $(\bar{y}, \bar{w})$ are (optimal) solutions to the corresponding problems. This results from the proof of Proposition 4.1: if we begin with (12), we obtain (11) and (10). But (10) is the equality of the objective function values of (P_λ) and (P^*_λ) and, therefore, $\bar{x}$ is a solution to (P_λ) and $(\bar{y}, \bar{w})$ is a solution to (P^*_λ), because they are admissible. Thus, we get the following assertion:

PROPOSITION 4.2 *Let $\bar{x}$ be feasible to (P_λ) and $(\bar{y}, \bar{w})$ feasible to (P^*_λ). Moreover, let conditions i) and ii) from Proposition 4.1 be satisfied. Then $\bar{x}$ and $(\bar{y}, \bar{w})$ are optimal solutions to (P_λ) and (P^*_λ), respectively.*

REMARK 4.2 *Condition i) of Proposition 4.1 is equivalent to*

$$\bar{x}^T \sigma(\bar{x} + \bar{y}) = 0 \quad \textit{and} \quad \bar{y}^T \sigma(\bar{x} + \bar{y}) = 0 \,. \tag{13}$$

Proof. Adding up the two equations of (13) implies immediately $i)$.

To verify (13) starting from condition $i)$ let us analyze inequality (6). From $i)$ we see, replacing x with $\bar{x}$ and y with $\bar{y}$, that inequality (6) is fulfilled as equation. Hence,

$$-2\bar{x}^T \sigma \bar{y} = 2(\bar{y}^T \sigma \bar{y})^{\frac{1}{2}} (\bar{x}^T \sigma \bar{x})^{\frac{1}{2}} = \bar{y}^T \sigma \bar{y} + \bar{x}^T \sigma \bar{x} \,.$$

Substituting $a = (\bar{y}^T \sigma \bar{y})^{\frac{1}{2}}$, $b = (\bar{x}^T \sigma \bar{x})^{\frac{1}{2}}$, we obtain $2ab = a^2 + b^2$ which is equivalent to $a = b$, i.e. $\bar{y}^T \sigma \bar{y} = \bar{x}^T \sigma \bar{x}$. With condition $i)$ this yields

$$\begin{aligned} 0 = {} & \tfrac{1}{2}[\bar{y}^T \sigma \bar{y} + \bar{x}^T \sigma \bar{x} + 2\bar{y}^T \sigma \bar{x}] = \tfrac{1}{2}[2\bar{x}^T \sigma \bar{x} + 2\bar{y}^T \sigma \bar{x}] \\ = {} & (\bar{x} + \bar{y})^T \sigma \bar{x} = \bar{x}^T \sigma (\bar{x} + \bar{y}) \end{aligned}$$

and, analogously, $0 = \bar{y}^T \sigma(\bar{x} + \bar{y})$. Therefore, (13) is true. ■

5. Strong duality

Now, we are able to prove the strong duality theorem for the multiobjective portfolio optimization problem (P) and its dual (P^*).

THEOREM 5.1 *If $\overset{o}{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ to (P^*) which holds the equality of the objective function values*

$$F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z}) \,.$$

Moreover, with the corresponding scalarizing numbers $\overset{o}{\lambda}_i > 0$, $i = 1, 2$ (see

i) $\overset{o}{y}^T \sigma \overset{o}{y} + \overset{o}{x}^T \sigma \overset{o}{x} + 2 \overset{o}{y}^T \sigma \overset{o}{x} = 0$
implying
$\overset{o}{x}^T \sigma(\overset{o}{x} + \overset{o}{y}) = 0\,,\ \overset{o}{y}^T \sigma(\overset{o}{x} + \overset{o}{y}) = 0\,,$
ii) $\overset{o}{x}^T\ [\overset{o}{\lambda}_1\ (\mu + \overset{o}{z}_1\ e) + \overset{o}{\lambda}_2\ (2\sigma\ \overset{o}{y} + \overset{o}{z}_2\ e)] = 0\ .$

Proof. Let $\overset{o}{x}$ be properly efficient to (P). From Definition 2.2 of proper efficiency it follows that there exists a corresponding scalarizing vector $\overset{o}{\lambda} = (\overset{o}{\lambda}_1, \overset{o}{\lambda}_2)^T, \overset{o}{\lambda}_i > 0, i = 1, 2$, such that $\overset{o}{\lambda}^T\ F(x) \geq \overset{o}{\lambda}^T\ F(\overset{o}{x})$ for all admissible x to (P). In other words, $\overset{o}{x}$ is a solution to the scalarized problem $(P_{\overset{o}{\lambda}})$. With the assigned dual problem $(P^*_{\overset{o}{\lambda}})$ we have the strong duality property and also the existence of a dual solution $(\overset{o}{y}, \overset{o}{w})$, i.e. $\min(P_{\overset{o}{\lambda}}) = \max(P^*_{\overset{o}{\lambda}})$. With conditions *i)* and *ii)* of Proposition 4.1, condition *i)* of Theorem 5.1 is satisfied (it has exactly the same form as condition *i)* of Proposition 4.1). Moreover, there is

$$\overset{o}{x}^T\ (2\ \overset{o}{\lambda}_2\ \sigma\ \overset{o}{y} + \overset{o}{w}\ e + \overset{o}{\lambda}_1\ \mu) = 0\,. \tag{14}$$

We define $\overset{o}{z} = (\overset{o}{z}_1, \overset{o}{z}_2)^T$ by

$$\overset{o}{z}_1 := -\mu^T\ \overset{o}{x}, \quad \overset{o}{z}_2 := -2\ \overset{o}{y}^T \sigma\ \overset{o}{x}\ . \tag{15}$$

Let us first establish the strong duality relationship $F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z})$. Following this, we check the admissibility of $(\overset{o}{y}, \overset{o}{z})$ with respect to (P^*). Consider the components of $G(\overset{o}{y}, \overset{o}{z}) = (g_1(\overset{o}{y}, \overset{o}{z}), g_2(\overset{o}{y}, \overset{o}{z}))^T$. Due to (15) and *i)*, there is

$$\begin{aligned} g_1(\overset{o}{y}, \overset{o}{z}) &= \overset{o}{z}_1 = -\mu^T\ \overset{o}{x} = -\sum_{i=1}^{n} \mu_i\ \overset{o}{x}_i = -E(\overset{o}{x}) = f_1(\overset{o}{x})\,, \\ g_2(\overset{o}{y}, \overset{o}{z}) &= \overset{o}{z}_2 - \overset{o}{y}^T \sigma \overset{o}{y} = -2\ \overset{o}{y}^T\ \sigma\ \overset{o}{x} - \overset{o}{y}^T\ \sigma\ \overset{o}{y} \\ &= \overset{o}{x}^T\ \sigma\ \overset{o}{x} = \sum_{i,j=1}^{n} \sigma_{ij}\ \overset{o}{x}_i \overset{o}{x}_j = V(\overset{o}{x}) = f_2(\overset{o}{x}) \end{aligned}$$

which means $F(\overset{o}{x}) = G(\overset{o}{y}, \overset{o}{z})$. To point out $(\overset{o}{y}, \overset{o}{z}) \in \tilde{B}$ (i.e. admissibility), we calculate, taking into consideration (14) and (15),

$$\begin{aligned} \overset{o}{\lambda}_1 \overset{o}{z}_1 + \overset{o}{\lambda}_2 \overset{o}{z}_2 &= -\ \overset{o}{\lambda}_1\ \mu^T\ \overset{o}{x} - 2\ \overset{o}{\lambda}_2 \overset{o}{y}^T\ \sigma\ \overset{o}{x} \\ &= \overset{o}{x}^T\ (-\ \overset{o}{\lambda}_1\ \mu - 2\ \overset{o}{\lambda}_2\ \sigma\ \overset{o}{y}) = \overset{o}{x}^T\ \overset{o}{w}\ e = \overset{o}{w}\ e^T\ \overset{o}{x} = \overset{o}{w}\ . \end{aligned} \tag{16}$$

Notice that $(\overset{o}{y}, \overset{o}{w})$ is admissible to $(P^*_{\overset{o}{\lambda}})$ (it is even a solution of $(P^*_{\overset{o}{\lambda}})$). Therefore, it satisfies the inequality

Replacing $\mathring{w}$ with (16), we obtain

$$2\,\mathring{\lambda}_2\,\sigma\,\mathring{y} + \mathring{\lambda}_1\mathring{z}_1\,e + \mathring{\lambda}_2\mathring{z}_2\,e \leq -\,\mathring{\lambda}_1\,\mu$$

and, further

$$\mathring{\lambda}_1\,(\mu + \mathring{z}_1\,e) + \mathring{\lambda}_2\,(2\sigma\,\mathring{y} + \mathring{z}_2\,e) \leq 0\,.$$

The last inequality is required to guarantee $(\mathring{y}, \mathring{z}) \in \tilde{B}$. Finally, the equation $F(\mathring{x}) = G(\mathring{y}, \mathring{z})$ and the weak duality assertion of Theorem 3.1 shows that $(\mathring{y}, \mathring{z})$ is efficient to (P^*).

Equations (14) and (16) imply condition *ii*) of Theorem 5.1. This completes the proof. ■

6. Conclusions

In this paper, we have considered a dual multiobjective optimization problem (P^*) to the classical Markowitz portfolio optimization problem (P). For the Markowitz problem we have considered the so-called properly efficient solutions while for the dual problem – the efficient solutions. The weak and strong duality assertions have been derived (Theorems 3.1 and 5.1). In particular, a linear scalarization of the Markowitz problem was used to gain these results. Moreover, the strong duality assertion has been used to obtain the necessary optimality conditions. We note, without proof, that these conditions turn out to be sufficient, too. For the scalarized problems (P_λ) and (P_λ^*) this has been expressed in Proposition 4.2.

In scalar linear programming, it is possible and also useful to give an economic interpretation of the dual problem and of the dual variables (Padberg, 1995). Such economic or capital market interpretation could also be an interesting completion of our mathematical investigations but, until now, we were unable to give one. We want to address this problem and involve researchers in the area of capital market theory.

Finally, let us point out that our approach can also be applied to construct dual multiobjective problems to some generalizations of the classical Markowitz portfolio problem, e.g. to portfolio problems allowing short sales or to problems with other types of constraints. This will be presented in a forthcoming paper.

References

DORN, W.S. (1960) Duality in quadratic programming. *Quart. Appl. Math.*, **18**, 2, 155-162.

ELSTER, R., GERTH, CH. and GÖPFERT, A. (1989) Duality in geometric vector optimization. *Optimization*, **20**, 4, 457-476.

ELSTER, K.-H., REINHARDT, R., SCHÄUBLE, M. and DONAT, G. (1977)

ELTON, E.J. and GRUBER, M.J. (1991) *Modern Portfolio Theory and Investment Analysis.* J. Wiley, New York.

GALE, D., KUHN, H.W. and TUCKER, A.W. (1951) Linear programming and the theory of games. In T.C. Koopmans, ed., *Activity Analysis of Production and Allocation.* J. Wiley/Chapman and Hall, New York, 317-329.

GEOFFRION, A.M.(1968) Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Application*, **22**, 3, 618-630.

GÖPFERT, A. and NEHSE, R. (1990) *Vektoroptimierung.* B.G. Teubner, Leipzig.

ISERMANN, H. (1978) On some relations between a dual pair of multiple objective linear programs. *ZOR-Methods and Models of Operations Research*, **22**, 1, 33-41.

JAHN, J. (1986) *Mathematical Vector Optimization in Partially Ordered Linear Spaces.* P. Lang, Frankfurt a. M.

LINKE, M. (1996) *Relative Portfolio-Optimierung unter Berücksichtigung von Benchmarks und Liabilities.* Josef Eul-Verlag, Bergisch Gladbach.

MARKOWITZ, H.M. (1952) Portfolio selection. *Journal of Finance*, **7**, 1, 77-91.

MARKOWITZ, H.M. (1989) *Mean-Variance Analysis in Portfolio Choice and Capital Markets.* Basil Backwell, Oxford.

NAKAYAMA, H., SAWARAGI, Y. and TANINO, T. (1985) *Theory of Multiobjective Optimization.* Academic Press, London.

PADBERG, M. (1995) *Linear Optimization and Extensions.* Springer-Verlag, Berlin.

SHARPE, W.F. (1970) *Portfolio Theory and Capital Markets.* Mc Graw-Hill, New York.

TAMMER, CH. and TAMMER, K. (1991) Generalization and sharpening of some duality relations for a class of vector optimization problems. *ZOR-Methods and Models of Operations Research*, **35**, 249-265.

WANKA, G. (1991a) On duality in the vectorial control-approximation problem. *ZOR-Methods and Models of Operations Research*, **35**, 309-320.

WANKA, G. (1991b) Duality in vectorial control approximation problems with inequality restrictions. *Optimization*, **22**, 755-764.