Control and Cybernetics

vol. 28 (1999) No. 4

On the construction of the common optimal market index in the Sharpe model[1]

by

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Abstract: The purpose of this paper is to determine one factor which represents the whole market behavior on the basis of the rates of return of all equities traded on this market. In the seminal Sharpe model the factor is an exogenous variable which is not determined by the model itself. This paper extends Sharpe's idea, as it assumes that the factor is a linear combination of all the rates of return of all traded equities. To determine the coefficients of this linear combination we minimize the loss function which expresses the weighted mean square deviation of all rates of return from their predictions, having given the linear combination form of the market index. It is found that the vector of linear coefficients has to be a nonzero eigenvector associated with the maximal eigenvalue of the appropriately transformed and estimated covariance matrix.

The optimal market index for the Warsaw Stock Exchange was compared with the standard index. It occurs that there is only a very small difference between the standard index of this market and the optimal index.

Keywords: portfolio analysis, Sharpe model, market index, linear regression, principal components

1. Introduction

The theory of portfolio selection was originally developed by Markowitz (1952, 1959). When analyzing financial market behavior Markowitz and Sharpe found

[1]This paper is a part of master thesis written by the author under supervision of Professor Karol Krzyżewski at the Faculty of Mathematics, Informatics and Mechanics, University of

that the majority of traded equities behave in the same fashion. That is, they are consistent with the whole of the market behavior. This finding resulted in a very simple and frequently used market behavior model called Sharpe model (Sharpe, 1963; Alexander and Francis, 1979; Elton and Gruber, 1991). It is assumed that the rate of return on the i-th equity behaves in the following way:

$$R^i = \alpha_i + \beta_i F + \varepsilon^i \quad \text{for } i = 1, \dots, k$$

$$\text{cov}(F, \varepsilon^i) = 0 \ \forall i, \quad \text{cov}(\varepsilon^i, \varepsilon^j) = 0 \ \forall i \neq j$$

$$E(\varepsilon^i) = 0, \quad \text{Var}(\varepsilon^i) = \sigma_i^2, \quad \text{Var}(F) \neq 0,$$

where:

R^i – the random variable which represents the rate of return of the i-th equity,
F – the random variable which represents common market risk,
ε^i – the random disturbance of the i-th rate of return.

In the Sharpe model it is assumed that there exists one common market risk for all rates of return. The market risk is, however, an exogenous variable which is not determined by the model itself. The common risk of a market F is often measured by market indices like S&P or Dow Jones. Parameters α_i and β_i are usually estimated from data using the method of least squares. Let me remind the procedure of estimation using this technique because these results will be used later on in the text. Assuming that we have the historical rates of return of k equities $\left(R_t^i\right)_{i=1,\dots,k}^{t=1,\dots,T}$ and the historical rates of return of the common market risk $(F_t)_{t=1,\dots,T}$, the estimators a_i and b_i have to minimize the following loss function:

$$L^i(a_i, b_i) = \sum_{t=1}^{T} (R_t^i - a_i - b_i F_t)^2,$$

which, after substitution $e_t^i = R_t^i - a_i - b_i F_t$, takes the form $L^i(a_i, b_i) = \sum_{t=1}^{T} (e_t^i)^2$.

The functions L^i for any i are convex, hence, the necessary conditions for the minimum are also sufficient. They can be expressed in the form of the well known system of normal equations:

$$\sum_{t=1}^{T} e_t^i F_t = 0, \quad \sum_{t=1}^{T} e_t^i = 0.$$

Solving the above system and assuming that $\sum_{t=1}^{T} (F_t - \overline{F})^2 \neq 0$, we get the classical linear regression estimators a_i and b_i:

$$b_i = \frac{\sum_{t=1}^{T} (R_t^i - \overline{R}^i)(F_t - \overline{F})}{\dots}, \quad a_i = \overline{R}^i - b_i \overline{F}, \qquad (1)$$

where $\overline{R}^i = \frac{1}{T}\sum_{t=1}^{T} R_t^i$ and $\overline{F} = \frac{1}{T}\sum_{t=1}^{T} F_t$.

The estimator of the variance of the disturbance term σ_i^2 is given by the mean square error as

$$\hat{\sigma}^2(\varepsilon^i) = \frac{1}{T-1}\sum_{t=1}^{T}(R_t^i - a_i - b_i F_t)^2,$$

which after substitutions $R_t^i - \overline{R}^i = r_t^i$, $F_t - \overline{F} = f_t$ and (1), takes the following form:

$$\hat{\sigma}^2(\varepsilon^i) = \frac{1}{T-1}\left[\sum_{t=1}^{T}(r_t^i)^2 - \left(\sum_{t=1}^{T} r_t^i f_t\right)^2 \Big/ \sum_{t=1}^{T}(f_t)^2\right].$$

The purpose of this paper is to determine one factor which represents the whole market behavior on the basis of the rates of return of all equities traded on this market. It assumes that the factor is a linear combination of the rates of return of all traded equities. This extends the Sharpe model where the market risk is the exogenous variable which is not determined by the model itself (it comes into the model from nowhere). The paper also generalizes and clarifies the results obtained by Wierzbicki and Mnich (1995). To determine coefficients of this linear combination we minimize the loss function which expresses the weighted mean square deviation of all rates of return from their predictions, having given the linear combination form of the market index. It refers to the method of components (Morrison, 1976), however, the idea applied here is not the same. The result similar to that of the method of principal components is derived in the case of linear regression model and it is used to construct the stock market index.

2. The endogenous common risk model

In the Sharpe model, it is assumed that the factor F is given exogenously, which means that it is a random variable that comes into the model from the outside and is not determined by the model itself. Let us make the assumption that the factor F is a linear combination of the random variables R^i for $i = 1, \ldots, k$ with the nonzero vector of combination coefficients $y = (y_1, \ldots, y_k)^T$, i.e. $F(y) = \sum_{i=1}^{k} R^i y_i$. Let us denote by $F_t(y)$ the estimator of the factor $F(y)$ calculated on the basis of the sample R_t^i, for $i = 1, \ldots, k$, for the moments of time $t = 1, \ldots, T$: $F_t(y) = \sum_{i=1}^{k} R_t^i y_i$.

From now on, having given positive weights $w_1, w_2, \ldots, w_k > 0$, our task will be to find such a vector y and such vectors $a = (a_1, \ldots, a_k)^T$, $b = (b_1, \ldots, b_k)^T$ which minimize the following loss function:

$$L(a, b, y) = \sum^{k} \frac{w_i}{T-1} \sum^{T} [R_t^i - a_i - b_i F_t(y)]^2 \quad \text{where } w_i > 0 \ \forall i\,.$$

This is an extension to the idea of Wierzbicki and Mnich (1995, 1995a, 1995b), introduced, without any formal proof, for the cases when either all weights $w_i = 1$ or $w_i = 1/\hat{\sigma}_{ii}$ for $i = 1, \ldots, k$, where $\hat{\sigma}_{ii}$ is the estimator of variance of the i-th rate of return.

The minimization problem $\min\limits_{a,b,y} L(a, b, y)$ can be considered as the iterated minimization. First, for any y, we find such vector functions $a(y)$ and $b(y)$ that $L(a(y), b(y), y) = \min\limits_{a,b} L(a, b, y)$. In order to do it, we need to solve the sequence of classical regression problems. In the next step, we have to find such a vector y for which the loss function $\tilde{L}(y) = L(a(y), b(y), y)$ is minimized. Because the random disturbance term is now the function of y, denoted by $\varepsilon^i(y)$, see that

$$\tilde{L}(y) = \sum_{i=1}^{k} w_i \hat{\sigma}^2(\varepsilon^i(y)). \tag{2}$$

Hence, $\tilde{L}(y)$ can be thought of as an estimator of weighted sum of the variances of disturbance terms.

After the same substitutions like in the previous section, we get:

$$\tilde{L}(y) = \frac{1}{T-1} \sum_{i=1}^{k} \sum_{t=1}^{T} w_i (r_t^i)^2 - \frac{1}{T-1} \sum_{i=1}^{k} w_i \frac{\left(\sum_{t=1}^{T} r_t^i f_t(y)\right)^2}{\sum_{t=1}^{T} (f_t(y))^2}, \tag{3}$$

where: $f_t(y) = F_t(y) - \overline{F}(y) = \sum_{i=1}^{k} (R_t^i - \overline{R}^i) y_i$ and $\overline{F}(y) = \frac{1}{T} \sum_{t=1}^{T} F_t(y)$.

From now on we will assume that the estimator of covariance matrix $\hat{C}$ is nonsingular. The first term of the right hand side of (3) does not depend on y. Hence, minimization of function (2) can be reduced to maximization of the second term of (3) which, after substitution of the appropriate formulas in place of f_t and r_t^i can be expressed as follows:

$$\frac{1}{T-1} \sum_{i=1}^{k} w_i \frac{\left(\sum\limits_{t=1}^{T} r_t^i f_t(y)\right)^2}{\sum\limits_{t=1}^{T} (f_t(y))^2} = \frac{\sum\limits_{i=1}^{k} w_i \left(\sum\limits_{j=1}^{k} \hat{\sigma}_{ij} y_j\right)^2}{\sum\limits_{i,j=1}^{k} \hat{\sigma}_{ij} y_i y_j} = \frac{(\hat{C}y)^T W (\hat{C}y)}{y^T \hat{C} y},$$

where $\hat{C} = (\hat{\sigma}_{ij})_{i,j=1,\ldots,k}$ – estimator of covariance matrix, $W = \mathrm{diag}(w_1, \ldots, w_k)$ and

$$\hat{\sigma}_{ij} = \frac{1}{T-1} \sum_{t=1}^{T} (R_t^i - \overline{R}^i)(R_t^j - \overline{R}^j).$$

Thus, the minimization problem has been finally transformed into the following maximization problem:

$$\max \frac{(\hat{C}y)^T W (\hat{C}y)}{\ldots} \tag{4}$$

THEOREM 2.1 *The solution of the maximization problem (4) is a vector y^* such that $y^* = W^{\frac{1}{2}} z^*$, where z^* is a nonzero eigenvector corresponding to the maximal eigenvalue of the matrix $W^{\frac{1}{2}} \hat{C} W^{\frac{1}{2}}$.*

Proof. After substitution $z = W^{\frac{1}{2}} \hat{C} y$, problem (4) takes the form:

$$\max_{y \neq 0} \frac{(\hat{C}y)^T W (\hat{C}y)}{y^T \hat{C} y} = 1 \Big/ \min_{z \neq 0} \frac{z^T W^{-\frac{1}{2}} \hat{C}^{-1} W^{-\frac{1}{2}} z}{z^T z}$$

For any symmetric matrix A, $\min_{z \neq 0} \frac{z^T A z}{z^T z} = \min_{\eta_i \in \mathrm{spec}(A)} \eta_i$ and the minimum is exactly attained at an eigenvector which corresponds to the minimal eigenvalue of the matrix A. Moreover, if matrix A is nonsingular and positively defined, then

$$\min_{\eta_i \in \mathrm{spec}(A)} \eta_i = \min_{\lambda_i \in \mathrm{spec}(A^{-1})} 1/\lambda_i = 1 \Big/ \max_{\lambda_i \in \mathrm{spec}(A^{-1})} \lambda_i \ .$$

Any eigenvector associated with minimal eigenvalue of matrix A is an eigenvector which corresponds to the maximal eigenvalue of the inverse A^{-1}. Hence, the optimum is attained at the nonzero eigenvector z^* associated with the maximal eigenvalue of the matrix $W^{\frac{1}{2}} \hat{C} W^{\frac{1}{2}} = (W^{-\frac{1}{2}} \hat{C}^{-1} W^{-\frac{1}{2}})^{-1}$. After simple transformation we get $y^* = W^{\frac{1}{2}} z^*$, which completes the proof. ■

COROLLARY 2.1 *It is possible that there exist many optimal factors - as many as the number of nonzero eigenvectors which correspond to the highest eigenvalue of the matrix $W^{\frac{1}{2}} \hat{C} W^{\frac{1}{2}}$.*

It follows from Corollary 2.1 that the length of the vector y is not important and we can normalize it to equalize the sum of coordinates to one, provided that they do not sum up to zero. The i-th coordinate of the vector can be interpreted as the percentage contribution of the i-th asset in the portfolio associated with the optimal factor, assuming the possibility of short sale (y_i can be negative).

If the weights have the form $w_i = 1/\hat{\sigma}_{ii}$, for $i = 1, \dots, k$, then $W^{\frac{1}{2}} \hat{C} W^{\frac{1}{2}} = \hat{K}$, where $\hat{K}$ is the estimated correlation matrix. Therefore, z^* is a nonzero eigenvector of the estimated correlation matrix $\hat{K}$, and $y^* = W^{\frac{1}{2}} z^*$. Note that this result is different from that suggested by Wierzbicki and Mnich (1995, 1995a, 1995b).

Assuming that the weights $w_i = 1$ for $i = 1, \dots, k$, we get $W^{\frac{1}{2}} \hat{C} W^{\frac{1}{2}} = \hat{C}$. Hence, $y^* = z^*$ and y^* is a nonzero eigenvector of the estimated covariance matrix $\hat{C}$. Then, the estimators b_i and $\hat{\sigma}^2(F)$ take the simple forms:

$$b_i = \frac{y_i^*}{\|y^*\|_2^2} \quad \text{and} \quad \hat{\sigma}^2(F(y^*)) = (y^*)^T \hat{C} y^* = \max_{\lambda_i \in \mathrm{spec}(\hat{C})} \lambda_i \|y^*\|_2^2 \, .$$

3. Construction of the optimal index and the dynamic market portfolio

Let $\hat{C}_t$ be the estimated nonsingular covariance matrix of returns of equities until time t (inclusive). It is possible to find the sequence of optimal factors associated with the sequence of vectors $\{y^t\}_{t=1,\dots,n}$, which are eigenvectors associated with the maximal eigenvalues of the sequence of estimated covariance matrices $\{\hat{C}_t\}_{t=1,\dots,n}$. Vectors y^t for $t = 1,\dots,n$ can be normalized $\sum_{i=1}^{k} y_i^t = 1$ for any t, to represent the investment portfolios, provided that $\sum_{i=1}^{k} y_i^t \neq 0$ for any t. Such a normalized sequence of vectors will be called hereafter the dynamic market portfolio. We can compute, for any t, the value of the above portfolio $V_t(y) = \sum_{i=1}^{k} P_i^t y_i^t$ and the rate of return, assuming that $V_t(y) \neq 0$:

$$R_t(y) = \frac{V_{t+1}(y) - V_t(y)}{V_t(y)},$$

where:

P_i^t – the price of the i-th asset at time t,

y_i^t – the percentage contribution of the i-th asset in the dynamic market portfolio at time t,

$V_t(y)$ – the value of portfolio at time t,

$R_t(y)$ – the rate of return of the portfolio y between time moments t and $t+1$.

This approach has one serious disadvantage. We must assert that the matrices $\{\hat{C}_t\}_{t=1,\dots,n}$ have to be nonsingular. The necessary condition for the nonsingularity of the matrix is that for any $i = 1,\dots,k$ the number of observed returns R_t^i must be strictly greater than the degree of the matrix. Hence, it is impossible to construct such a factor in the case of emerging markets, where the historical time series are not long enough.

4. Illustrative example

In order to illustrate the optimal index we analyze the data from the Warsaw Stock Exchange from the period January 1996 – April 1997. Due to the fact that during this period new equities appeared on the market, the number of examined equities is restricted to these which were traded on the market for a long enough period of time. Otherwise, the time series would have empty places or would not have an appropriate length of time to assert nonsingularity of the covariance matrix. Moreover, the analysis takes into account a one day investment horizon period, which means that only daily rates of return are considered. The dynamic market portfolio is built on 65 equities. The total length of the time series of daily rates of return is 324 and the number of moments of time for which the optimal factor is computed is 175. We want to compare the optimal index with WIG (Warsaw Stock Exchange Index). This is done in Table 1 and shown in

	Optimal index	WIG
Expected rate of return	0.003396	0.003456
Standard deviation	0.015997	0.016080

Table 1. The comparison of WIG and the optimal index.

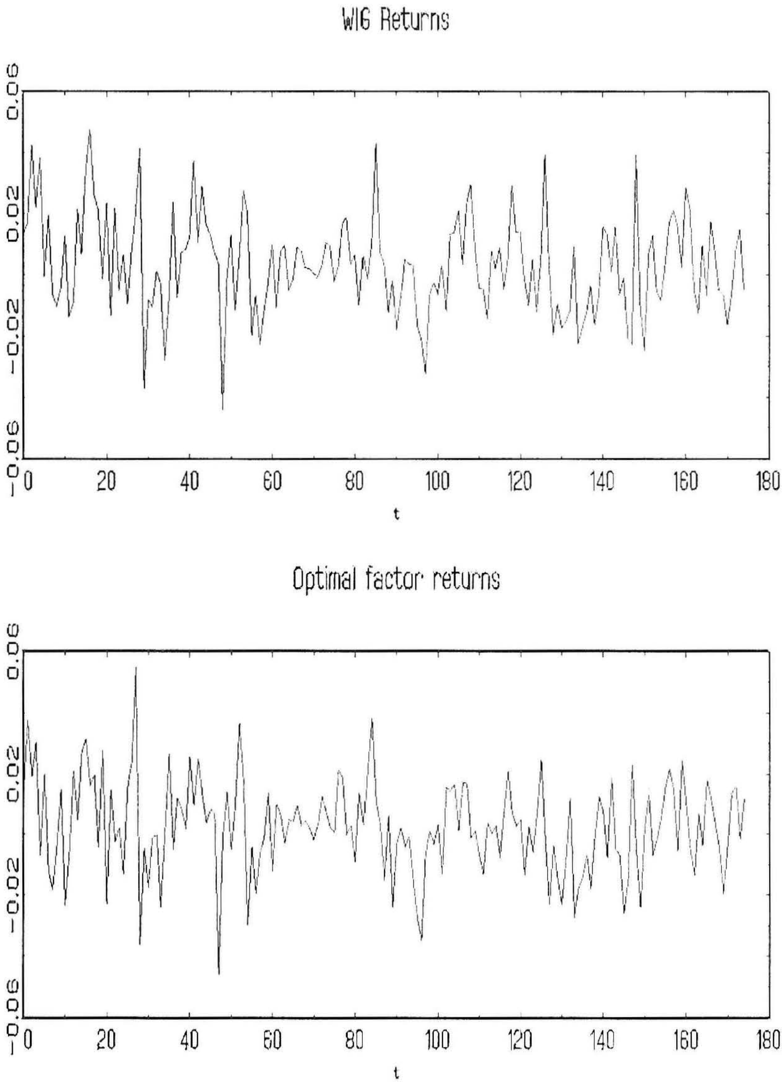

Figure 1. The rate of return of the optimal index associated with the dynamic market portfolio and the returns of the WIG index.

One can easily notice that the behavior of the optimal index which corresponds to the dynamic market portfolio $\{y^t\}_{t=1,\dots,n}$ is almost the same as the behavior of WIG. There is very little difference between the mean rate of return and the risk measured by the standard deviation. Thus, we can conclude that WIG is very close to the optimal index for the market.

5. Concluding remarks

We assume that there exists one common risk for the whole market, which can be represented by the linear combination of the rates of return of all listed equities. The loss function (2) which represents the weighted sum of the estimators of the variances of disturbance terms is minimized. Other types of loss functions (i.e. absolute error loss function) are left for further research. It occurs that the vector which minimizes the loss function is a transformed eigenvector of the transformed estimated covariance matrix, corresponding to the maximal eigenvalue, assuming that this estimated covariance matrix is not singular. It would be interesting, particularly from the practical point of view, to relax the assumption about nonsingularity of the estimated covariance matrix. This is proposed for further research.

Next, the dynamic market portfolio whose value represents the optimal index of the stock market is constructed as the sequence of normalized eigenvectors associated with the sequence of estimated covariance matrices. These matrices are the nonsingular estimated covariance matrices of the rates of return on equities until the particular points of time for which the dynamic portfolio is created. The optimal market index computed for the Warsaw Stock Exchange behaves very similarly to the existing market index called WIG (Warsaw Stock Exchange Index). Further research is suggested to create the optimal indices for other markets and to compare them with the existing indices on these markets.

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