Dedicated to
Professor Jakub Gutenbaum
on his 70th birthday

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On descriptor systems and related linear quadratic problem[1]

by

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Abstract: It is shown that a descriptor system under the condition of impulse controllability, Cobb (1984), may be converted, by means of linear transformations, to a system described in a state space and composed of state and output equations. The transformations determine one to one correspondence between the solutions of both the systems. It is noted that the control in a feedback form may not determine a unique solution of the descriptor system what is often overlooked in many previous papers. It is also shown that the LQ problem formulated in a descriptor space for the impulse observable system, Cobb (1984), may be converted by means of linear transformations to the usual LQ problem formulated in the state space. It is stressed that the second problem may be regular even then, when the weighting matrix of the control, in the cost functional of the first problem, is singular. The proposed approach simplifies the calculations related to the LQ problem solution significantly.
Keywords: descriptor system, linear quadratic problem.

1. Introduction

The LQ regular problem for descriptor systems is considered in many papers, e.g. Bender and Laub (1987), Cheng, Hong and Zhang (1988), Cobb (1983), Dai (1989), Lewis (1986), Wang, Shi, Zhang (1988). However, usually some stronger conditions and/or more complex calculations occur in them, than those appearing in the present paper. Also the lack of definition of the descriptor variable makes it difficult to formulate the LQ problem for descriptor system properly and this paper seems to be the first where this is done. It is worthwhile to

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stress that the approach proposed here gives significantly simpler calculations than those based on the geometric solution of Cobb (1983) or based on the minimization of Hamiltonian of Bender and Laub (1987). The technique used here is in some sense similar to those of Cheng, Hong and Zhang (1988) and Geerts (1994), but the authors of Cheng, Hong and Zhang (1988) make the more restrictive assumption that the system is strongly stabilizable and detectable, while in the present paper we assume that the system is impulse controllable and observable Cobb (1984). The reasonings performed here seem also to be simpler and more natural then those in Geerts (1994). Moreover, the final solution of Cheng, Hong and Zhang (1988) is given in the form of feedback control law dependent upon the descriptor variable. It is noted here that this kind of the feedback law does not guarantee the unique solution of the descriptor system equation. Additionally, in Cheng, Hong and Zhang (1988) the formula determining the optimal control contains a matrix, whose existence is only proved, but it is not shown how to calculate this matrix. Also, in many papers the so-called inconsistent initial conditions are allowed while in our paper we have limited ourselves only to the consistent initial conditions (see definition below). This seems to be more natural both from practical and mathematical points of view. The inconsistent initial conditions considered, e.g. in Geerts (1994), admit a discontinuity and related isolated point at the initial time. In this case, however, it is not clear how to understand the derivative of the descriptor variable calculated at this isolated point.

Consider the continuous-time descriptor system

$$E\,\dot{x}= Ax + Bu, \tag{1}$$

where $E \in R^{n\times n}$, $rankE = r < n$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x(t)$ is n-dimensional descriptor vector, and $u(t)$ is m-dimensional control vector. Let $C_0^\infty\left(R^k\right)$ be the space of continuous functions from $[0,\infty)$ into R^k which are smooth on $(0,\infty)$. Assume that the descriptor system (1) is regular, i.e. $\det(Es - A) \neq 0$. Under this assumption the system (1) has a unique solution for every consistent initial value x_0 and an appropriate control $u \in C_0^\infty(R^m)$. An initial value $x_0 \in R^n$ is called consistent if there exists a control $u \in C_0^\infty(R^m)$ such that the solution $x(x_0, u)$ of (1) belongs to $C_0^\infty(R^n)$. The space of these initial values (points) is denoted by I_c.

Consider the following cost functional

$$J_T(x_0,u) = \int_0^T \left(\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle\right) dt, \tag{2}$$

or

$$J(x_0,u) = \int_0^\infty \left(\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle\right) dt, \tag{3}$$

where

$$Q = \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{4}$$

$C_1 \in R^{l\times r}$, $C_2 \in R^{l\times(n-r)}$, Q is symmetric and positive semidefinite and $R \in R^{m\times m}$ is symmetric and positive definite. Now the following control problem may be formulated.

Find a subset L_c of I_c composed of such points $x_0 \in R^n$ that there exists a control $u_{op} \in C_0^\infty(R^m)$ for which $x(x_0, u_{op}) \in C_0^\infty(R^n)$ and for any other control $u \in C_0^\infty(R^m)$ such that $x(x_0, u) \in C_0^\infty(R^n)$ the following inequality holds: $J_T(x_0, u_{op}) \le J_T(x_0, u)$; for each $x_0 \in L_c$ find the optimal control u_{op}.

The earlier papers concerning this problem are those of Bender and Laub (1987), Cobb (1983) and Geerts (1994). The approach proposed here is quite different from that appearing in these papers. Here, some linear transformations are used to obtain a standard linear-quadratic regulator problem formulated in a state space.

The contribution of the paper is partially in showing that the regular, impulse controllable descriptor system may be transformed to the usual state and output equation and partially in showing that, if additionally the system is impulse observable, then the LQ problem for a descriptor system may be transformed to the usual LQ problem formulated in a state space.

2. Descriptor to state space transformation

Now the linear transformations will be defined which convert the system (1) to the usual state and output equation. It is well known that there exist nonsingular matrices $M, N \in R^{n\times n}$, such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where $I_r \in R^r$ is the identity matrix, and equation (1) can be transformed to the form

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u, \tag{6}$$

$$0 = A_3x_1 + A_4x_2 + B_2u, \tag{7}$$

where

$$N^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ x_1 \in R^r,\ x_2 \in R^{n-r}, \tag{8}$$

$$MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

$$A_1 \in R^{r\times r},\ A_2 \in R^{r\times(n-r)},\ A_3 \in R^{(n-r)\times r},\ A_4 \in R^{(n-r)\times(n-r)},$$

$$MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_1 \in R^{r\times m},\ B_2 \in R^{(n-r)\times m}.$$

Assume that

$$rank\,[A_4, B_2] = n - r. \tag{9}$$

In the literature this condition is known as impulse controllability, Cobb (1984). The same condition for the system (1) takes the form

$$\mathrm{Im}\,(E) + \mathrm{Im}\,(B) + A\,(Ker\,(E)) = R^n,$$

Geerts (1994), where $KerX$ and $\mathrm{Im}X$ denote the kernel and the image of a matrix X.

There exist non-singular matrices $U \in R^{(n-r)\times(n-r)}$, $V \in R^{(m+n-r)\times(m+n-r)}$ such that

$$U\,[A_4, B_2]\,V = [I_{n-r}, 0]\,. \tag{10}$$

Consider the new coordinates $\bar{x}\in R^r$, $\overline{y} \in R^{n-r}$, $\bar{u}\in R^m$ given by

$$\bar{x}= x_1, \begin{bmatrix} \overline{y} \\ \bar{u} \end{bmatrix} = V^{-1} \begin{bmatrix} x_2 \\ u \end{bmatrix}. \tag{11}$$

For the new coordinates the system (6)-(7) takes the form

$$\dot{\bar{x}}=\bar{A}_1\bar{x} + \bar{A}_2\,\overline{y}+ \bar{B}_1\bar{u}, \tag{12}$$

$$0 = -\overline{C}\,\bar{x} +\overline{y}, \tag{13}$$

where $\bar{A}_1= A_1$, $\left[\bar{A}_2, \bar{B}_1\right] = [A_2, B_1]\,V$, $\bar{A}_2\in R^{r\times(n-r)}$, $\bar{B}_1\in R^{r\times m}$, $\overline{C} = -UA_3$. Substituting $\overline{y}$ from (13) into (12) we obtain

$$\dot{\bar{x}}=\bar{A}\bar{x} + \bar{B}\bar{u}, \tag{14}$$

$$\overline{y} = \overline{C}\,\bar{x} \tag{15}$$

where $\bar{A}=\bar{A}_1 + \bar{A}_2\,\overline{C}$, $\bar{B}=\bar{B}_1$. Obviously, the variables $\bar{x}$, $\overline{y}$, $\bar{u}$ appearing in (12) and (13) are related to the variables x, u appearing in (1) by means of the transformation

$$\begin{bmatrix} \bar{x} \\ \overline{y} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} N^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{16}$$

Thus, the following theorem has been proved true:

THEOREM 2.1 *Let the condition (9) be fulfilled. Then:*

1. *The descriptor system (1) may be transformed to the form (14), (15), which may by interpreted as a state space model composed of state and output equations with state $\bar{x}$ and output $\overline{y}$, respectively.*

2. *There is one-to-one correspondence between the solutions* $x(x_0, u)$ *of the descriptor system (1) and the solutions* $\bar{x}\ (\bar{x}\ (0), \bar{u})$ *of the state space system (14), (15); the correspondence between* x, u, *and* $\bar{x}$, $\overline{y} = \overline{C}\,\bar{x}$, $\bar{u}$ *is described by the transformation (16).*
3. *The descriptor system (12), (13) (equivalent to ((14), (15)), for any control* $\bar{u}\ (t)$ *or* $\bar{u}\ (x)$ *has the unique solution* $\begin{bmatrix} \bar{x}\ (t) \\ \overline{y}(t) \end{bmatrix}$ *dependent on* $\overline{x}(0), \overline{u}$ *and lying in* $r-$ *dimensional subspace of* R^n *determined by* $S = \left\{ \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} : \overline{y} = \overline{C}\,\bar{x} \right\}$.

EXAMPLE 2.1 *Consider the system described by*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{17}$$

so taking the form (6). It is easy to show that the condition (9) is fulfilled. Applying the transformation (11) in the form $\overline{x} = x_1$, $\overline{y} = u$, $\overline{u} = x_2$ *we obtain*

$$\dot{\overline{x}} = \overline{u}, \ \overline{y} = -\overline{x}. \tag{18}$$

It is easy to note that the initial state $\overline{x}(0)$ *and the control* $\overline{u} = \overline{u}(t)$ *- as a function of time, or* $\overline{u} = \overline{u}(\overline{x})$ *- in the feedback form determines the unique solutions* $\overline{x}(t)$ *and* $\overline{u}(t)$ *of (18), and* $x(t)$ *and* $u(t)$ *of (17). On the other hand the consistent initial state* $x_1(0)$ *and control given in the feedback form* $u = -x_1$ *does not determine the unique solution of (17). In fact, by substituting* $u = -x_1$ *to (17) we obtain*

$$\dot{x}_1 = -x_1 + x_2 \tag{19}$$

and for different x_2 *we get infinitely many solutions of the equation (19) and (17). Note that for the control given in the form of a function of time* $u = u(t)$ *we obtain the unique solution of the equation(17) since it is regular, Geerts (1994).*

REMARK 2.1 *If* A_4 *is singular then the transformation (11) may change the roles of the components of the vectors* x_2 *and* u. *Some components of* x_2 *may take the role of control and some components of* u *- the role of descriptor variable. Therefore, for the control determined in a feedback form* $u = u(x_1)$ *the descriptor system (1) may have a non-unique solution. This is an essential difference between the descriptor system and the state space system, which is frequently overlooked in the literature. Note that the transformed 'control'* $\bar{u} = \bar{u}\ (\bar{x})$ *in a feedback form determines a unique solution of the system (12), (13). On the other hand, if* A_4 *is nonsingular then the control* $u = u(x_1)$ *in the feedback form determines the unique solution of the system (1).*

3. LQ problem solution

Now the quadratic form appearing in the cost functionals (2) and (3) will be transformed. From (13) and (16) it results that

$$\begin{bmatrix} x \\ u \end{bmatrix} = T \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix},$$

where

$$T = \begin{bmatrix} N & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I_r & 0 \\ \overline{C} & 0 \\ 0 & I_m \end{bmatrix},$$

and so

$$\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle =$$

$$\left\langle T' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} T \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}, \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}, \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right\rangle, \quad (20)$$

where $S_1 \in R^{r\times r}, S_2 \in R^{r\times m}, S_3 \in R^{m\times r}, S_4 \in R^{m\times m}$. The transformed LQ problem (14), (20) is regular iff $S_4 > 0$.

LEMMA 3.1 *Matrix S_4 is positive definite iff*

$$rank \begin{bmatrix} A_4 \\ C \end{bmatrix} = n - r, \quad (21)$$

where $C = C_1 N_2 + C_2 N_4$, $N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}$, $N_1 \in R^{r\times r}$, $N_2 \in R^{r\times (n-r)}$, $N_3 \in R^{(n-r)\times r}$, $N_4 \in R^{(n-r)\times(n-r)}$ *and* C_1, C_2 *are given by (4).*

Proof. We first observe that $S_4 = \begin{bmatrix} V_2'C' & V_4'\sqrt{R} \end{bmatrix} \begin{bmatrix} CV_2 \\ \sqrt{R}V_4 \end{bmatrix}$, and so it is sufficient to prove that $rank \begin{bmatrix} CV_2 \\ \sqrt{R}V_4 \end{bmatrix} = m$ iff $rank \begin{bmatrix} A_4 \\ C \end{bmatrix} = n - r$, where V is given by (10) and $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$, $V_1 \in R^{(n-r)\times(n-r)}, V_2 \in R^{(n-r)\times m}, V_3 \in R^{m\times(n-r)}, V_4 \in R^{m\times m}$.

Suppose that

$$rank \begin{bmatrix} A_4 \\ C \end{bmatrix} = n - r \quad (22)$$

and, on the contrary, that there exists a non-zero vector $w \in R^m$ such that

$$CV_2 w = 0 \quad (23)$$

$$\sqrt{R}V_4 w = 0. \tag{24}$$

From (10) it results that $A_4V_2 + B_2V_4 = 0$, and so by (24) we conclude that

$$A_4V_2w = 0. \tag{25}$$

It must be that $V_2w \neq 0$, otherwise from (24) it results that $V\begin{bmatrix} 0 \\ w \end{bmatrix} = 0$, which contradicts the invertibility of V. From (24) and (25) we have $V_2w \in Ker\begin{bmatrix} A_4 \\ C \end{bmatrix}$, contrary to (22).

Suppose now that

$$rank\begin{bmatrix} CV_2 \\ \sqrt{R}V_4 \end{bmatrix} = m, \tag{26}$$

and there exists $v \in R^{n-r}$, $v \neq 0$ such that $\begin{bmatrix} A_4 \\ C \end{bmatrix} v = 0$. From the definition of the matrices U and V we get

$$\begin{bmatrix} U & 0 \\ 0 & I_{l+m} \end{bmatrix}\begin{bmatrix} A_4 & B_2 \\ C & 0 \\ 0 & \sqrt{R} \end{bmatrix} V = \begin{bmatrix} I_{n-r} & 0 \\ CV_1 & CV_2 \\ \sqrt{R}V_3 & \sqrt{R}V_4 \end{bmatrix}. \tag{27}$$

Let

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^{-1}\begin{bmatrix} v \\ 0 \end{bmatrix}, \tag{28}$$

$z_1 \in R^{n-r}$, $z_2 \in R^m$, $\begin{bmatrix} v \\ 0 \end{bmatrix} \in R^{n-r+m}$. Multiplying (27) from the right hand side by $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ yields

$$0 = \begin{bmatrix} z_1 \\ CV_1z_1 + CV_2z_2 \\ \sqrt{R}V_3z_1 + \sqrt{R}V_4z_2 \end{bmatrix}. \tag{29}$$

It results from (29) that $z_1 = 0, CV_2z_2 = 0, \sqrt{R}V_4z_2 = 0$, hence $z_2 \in Ker\begin{bmatrix} CV_2 \\ \sqrt{R}V_4 \end{bmatrix}$ and finally that $z_2 = 0$. This contradicts (28) because V is invertible. ■

In the literature, the condition (21) is known as impulse observability, Cobb (1984). If conditions (9) and (21) are fulfilled then the following way of solving the LQ problem may by used: Convert the descriptor equation (1) to state space equation (14) and the cost functional (2) or (3) taking into account (20). Solve the standard problem (14)-(2) or (14)-(3) formulated in the state space. Let

$$\bar{u}_{op} = -\bar{K}\bar{x} \tag{30}$$

be the feedback control law of the standard LQ problem. Applying the transformation (11) to the equation (13) and (30) and accounting that $\overline{C} = -UA_3$ we obtain

$$u_{op} = -Kx_1, \tag{31}$$

$$x_2 = -Lx_1, \tag{32}$$

where

$$K = -V_3\overline{C} + V_4\overline{K} \tag{33}$$

$$L = -V_1\overline{C} + V_2\overline{K}. \tag{34}$$

Finally, the subspace of the consistent initial conditions is determined by

$$L_c = \left\{ \begin{bmatrix} x_{10} \\ -Lx_{10} \end{bmatrix} \in R^n : x_{10} \in R^r \right\}. \tag{35}$$

The formulas (31) and (32) determine the feedback law and the subspace in which the solutions of the LQ problem lie, respectively.

The properties of the LQ problem solutions for the descriptor system (1) are summarized below.

REMARK 3.1 *1. The control u in the feedback form (31) may not determine uniquely neither the solution x_1 of (14) nor the solution x of (1). The unique solution is obtained when* $\det(A_4) \neq 0$.

2. The control $\bar{u}$ in the feedback form (30) determines the unique solution $\bar{x}$ and x of the equations (14) and (1) respectively.

3. The open-loop optimal control $u_{op}(t)$ described by (14), (31) and (32) determines the unique solution x of the system (1).

EXAMPLE 3.1 *Consider the example of Bender and Laub (1987) and Cobb (1983) in which the system is described by equation (17) and the cost functional takes the form*

$$J(x_0, u) = \int_0^\infty \left(\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t), x(t) \right\rangle + u^2(t) \right) dt. \tag{36}$$

Using the terminology of Cobb (1984) the system is impulse controllable and impulse observable. Applying the same transformation as in Example 1 we obtain equation (14) in the form

$$\dot{\bar{x}} = \bar{u}, \tag{37}$$

and the new cost functional is

$$J(\bar{x}_{10}, \bar{u}) = \int_0^\infty \left(2\,\bar{x}^2\,(t) + \bar{u}^2\,(t) \right) dt. \tag{38}$$

The solution of the standard LQP is $\bar{u}_{op} = -\sqrt{2}\,\bar{x}$. *Finally we have*

$$L_c = \left\{ \begin{bmatrix} x_{10} \\ -\sqrt{2}x_{10} \end{bmatrix} : x_{10} \in R \right\}$$

and the optimal control takes the form

$$u_{op} = -x_1, \tag{39}$$

$$x_2 = -\sqrt{2}x_1. \tag{40}$$

The solution to problem (17)-(36) given by (40) and (39) is the same as in Bender and Laub (1987) and Cobb (1983) but the calculations are significantly simpler.

From the proof of Lemma 2 it results that the condition (21) of impulse observability is a necessary condition of regularity of the converted LQ problem. If additionally the matrix R is positive defined, then this condition becomes also a sufficient one. Then the LQ problem for a descriptor system may be regular even when the matrix R is singular. This will be illustrated in the next example.

EXAMPLE 3.2 *Consider the following system*

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_2 + u_2, \\ 0 &= x_1 + u_1 + u_2, \end{aligned} \tag{41}$$

with the cost functional

$$J(x_0, u) = \int_0^\infty \left(x_1^2 + x_2^2 + u_1^2\right) dt. \tag{42}$$

The converted equation (14) and (15) takes the form

$$\begin{aligned} \dot{\overline{x}} &= \overline{x} - \overline{u}_1 + \overline{u}_2, \\ \overline{y} &= -\overline{x} - \overline{u}_1, \end{aligned} \tag{43}$$

with cost functional

$$\overline{J}(x_0, u) = \int_0^\infty \left(x^2 + u_1^2 + u_2^2\right) dt. \tag{44}$$

The converted LQ problem (43)-(44) is regular though the matrix R in (42) is singular.

4. Conclusion

It is shown that the descriptor system under the condition of impulse controllability may by converted, by means of linear transformations, to the state space model composed of the state and output equation (Theorem 2.1). The observation resulting from Example 2.1 is that the control in a feedback form may not determine a unique solution of the descriptor system, contrary to the system described in a state space. This observation is often overlooked in the literature (e.g. Bender and Laub, 1987, Cheng, Hong and Zhang, 1988 and many others), but it has been noticed in Geerts (1994). It may be essential in some problem formulations for descriptor systems.

It is also shown that the LQ problem for descriptor system may be solved by converting it to the standard LQ problem formulated in a state space (Lemma 3.1). Example 3.1 illustrates the advantage of this approach, especially if we compare the calculations with those of Bender and Laub (1987) or Cobb (1983). From the proof of Lemma 3.1 it results that the condition (21) of impulse observability is a necessary condition of regularity of the converted LQ problem. If, additionally, the matrix R is positive definite then this condition becomes also a sufficient one. Example 3.2 shows that even for a singular matrix R, the converted LQ problem may be regular. This last point has been already noticed in Geerts (1994).

One should realize that while developing a model of a real system it is usually more reasonable to choose the state space model completed by some algebraic equations than a descriptor one. Since there is no formal definition of the descriptor variable there is some freedom in choosing one. In connection with this the consideration of the descriptor system with rectangular coefficient matrices like that in Geerts (1994) seems to be artificial from the point of view of application. In reality, the latter system may be easily converted to the system with quadratic coefficient matrices using the approach proposed in the present paper, i.e., by changing the role of some of the descriptor variables.

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