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Professor Jakub Gutenbaum
on his 70th birthday*

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Regularization of non-coercive quasi variational inequalities

by

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Abstract: This paper is devoted to the regularization of quasi-variational inequalities. The quasi-variational inequality is considered with multivalued operator. The operator involved is taken to be non-coercive and the data are assumed to be known approximately only. Under the assumption that the quasi-variational inequality be solvable, a weakly convergent approximation procedure is designed by means of the so-called Browder-Tikhonov regularization method.

Keywords: quasi-variational inequalities, regularization, monotone and pseudo-monotone operators, non-coercive, convergence.

1. Introduction

Throughout the paper, unless the contrary is stated, \mathcal{B} denotes a real reflexive Banach space and \mathcal{B}^* be its topological dual; $\langle \cdot, \cdot \rangle$ the associated pairing and $\| \cdot \|$ stands for the norm in \mathcal{B} as well as in \mathcal{B}^* . Let $\Omega \subset \mathcal{B}$ be nonempty, closed and convex. Consider the multivalued operators $\mathcal{F} : \mathcal{D}(\mathcal{F}) \subseteq \mathcal{B} \rightrightarrows \mathcal{P}(\mathcal{B}^*)$ ¹ and $\mathcal{K} : \Omega \rightrightarrows \mathcal{P}(\Omega)$, where for each $u \in \Omega$ the set $\mathcal{K}(u)$ is nonempty, closed and convex, the functional $\varphi : \mathcal{B} \rightarrow \bar{\mathbb{R}}$ and $f \in \mathcal{B}^*$ be arbitrary. The symbols " \rightrightarrows " and " \rightarrow " are used to specify the strong and the weak convergence, respectively.

¹By the notation $\mathcal{P}(A)$, we represent the so-called power set of A , i.e. the set of all of its subsets.

The present study is concerned with the following problem: find $y \in \mathcal{K}(y)$ and $F \in \mathcal{F}(y)$ such that

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in \mathcal{K}(y). \quad (1)$$

The above problem is referred to as a quasi-variational inequality (for short, QVI) and any element $y \in \Omega$ satisfying the above conditions is said to be a solution to $QVI(1)$. We shall denote by $\mathcal{S}(QVI)$, the set of all solutions to $QVI(1)$.

Special Cases:

(A): If the operator $\mathcal{F}(\cdot)$ is single-valued, $\varphi \equiv 0$ identically, then $QVI(1)$ recovers the following QVI : find $y \in \mathcal{K}(y)$ such that

$$\langle \mathcal{F}y - f, x - y \rangle \geq 0, \quad \forall x \in \mathcal{K}(y). \quad (2)$$

The above problem was introduced by Bensoussan and Lions (1973) in connection with a problem of impulse control. However, a more general treatment of the above problem was initiated by Mosco (1976).

(B): If the operator $\mathcal{F}(\cdot)$ is single-valued, $\varphi \equiv 0$ identically and $\mathcal{K}(u) \equiv \Omega$; $\forall u \in \Omega$, the $QVI(1)$ collapses to the problem: find $y \in \Omega$ such that

$$\langle \mathcal{F}y - f, x - y \rangle \geq 0, \quad \forall x \in \Omega. \quad (3)$$

The above problem is the celebrated Variational Inequality (for short, VI) introduced by Stampacchia (1964).

(C): If $\forall y \in \Omega$, $\mathcal{K}(y)$ is closed and convex cone with apex at the origin, $\varphi \equiv 0$ identically, then the $QVI(1)$ collapses to the generalized Quasi-complementarity system: find $y \in \Omega$ such that

$$y \in \mathcal{K}(y), \quad F \in \mathcal{F}(y) \cap \mathcal{K}^*(y), \quad \langle F, y \rangle = 0, \quad (4)$$

where $\mathcal{K}^*(y)$ denotes the (positive) polar of $\mathcal{K}(y)$. If $\mathcal{K}(u) \equiv \Omega$ identically, then (4) recovers the usual (nonlinear) Complementarity System (see Isac, 1993). The equivalence between $QVI(1)$ and (4) can be found in Giannesi (1997b), where a more general QVI has been studied.

In recent years the theory of QVI and VI has emerged as an important branch of applied and industrial mathematics. This theory provides us with a convenient mathematical apparatus for uniformly studying a wide range of problems arising in diverse fields as structural mechanics, elasticity, economics, optimization etc. (see for instance the books of Baiocchi and Capelo, 1984, and Kinderlehrer and Stampacchia, 1980).

A great number of results for $QVI(1)$ are available, when either the domain Ω is bounded or the operator $\mathcal{F}(\cdot)$ satisfies certain coerciveness conditions (see, for example a survey article by Harker and Pang, 1990, and references cited in Giannesi, 1996, 1997a, b). However, many engineering, economic and stochastic models lead to QVI (in particular to VI) with non-coercive operators defined on unbounded sets.

We briefly touch upon a problem of such nature: Let $\Xi \subseteq \mathbb{R}^N$ be a nonempty, bounded and open with smooth boundary Γ .

Consider the problem: find $u(x)$, with $x \in \Xi$, such that

$$\mathcal{A}(u) \equiv - \sum_{i=1}^N D^i \mathcal{A}_i(x, u, \nabla u) + \mathcal{A}_0(x, u, \nabla u) = f, \quad \text{in } \Xi. \quad (5)$$

The boundary conditions for (5) are as follows:

$$u \geq \Upsilon(u) \quad \text{on } \Gamma \quad (6)$$

$$\chi_a(u) \geq 0 \quad \text{on } \Gamma \quad (7)$$

$$\chi_a(u)(u - \Upsilon(u)) = 0 \quad \text{on } \Gamma \quad (8)$$

where $\Upsilon(\cdot)$ is the obstacle on the boundary Γ and is defined as :

$$\Upsilon(u(x)) = h(x) - \int_{\Gamma} \chi_a u(z) \mu(z) d\Gamma_z.$$

Here $h(\cdot)$ and $\mu(\cdot)$ are given on Γ and χ_a is the conormal derivative related to \mathcal{A} .

The above problem describes the temperature distribution (stationary) inside a material with thermally semipermeable boundary. This is for the case when the exterior temperature varies proportionally to certain average of the heat flux crossing the boundary. For a concrete description of the above problem the reader is referred to Garroni and Gossez (1983).

In the present situation it is of interest to consider the case when (5) has no lower order terms, that is, (5) is in the form:

$$\mathcal{A}(u) \equiv - \sum_{i=1}^N D^i \mathcal{A}_i(x, u, \nabla u) = f, \quad \text{in } \Xi.$$

Unfortunately, in this case, for certain source terms, a solution may fail to exist; this is due to the lack of coerciveness.

For \mathcal{VI} 's there have been many efforts to handle the lack of the coerciveness condition by a suitable regularization of the non-coercive problem. The central idea of these methods consists in regularizing the non-coercive problem by supplying a 'nice' operator which, along with the non-coercive operator, provides the desired properties. The present contribution is an extension of these ideas for the treatment of \mathcal{QVI} with non-coercive operators.

The rest of the paper is organized as follows: In the next section, we recall some results and concepts to be used throughout the paper. Section 3 presents certain auxiliary results and existence theorems. Some of the results of this section are applicable for the regularization of \mathcal{VI} with more general class of operators. Section 4 focuses on the regularization of \mathcal{QVI} .

2. Preliminaries

In order to make this paper self contained, we briefly set forth below some definitions and results which we use here. For more details, the reader is referred to Kluge (1979) and Zeidler (1990).

Let \mathcal{Z} denote a real reflexive Banach space, \mathcal{Z}^* be the topological dual of \mathcal{Z} , $\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ the associated pairing, and $\| \cdot \|_{\mathcal{Z}}$ be the norm in \mathcal{Z} as well as in \mathcal{Z}^* .

For a multivalued operator² \mathcal{A} from \mathcal{Z} to \mathcal{Z}^* , the set $\mathcal{D}(\mathcal{A}) := \{u \in \mathcal{Z} : Au \neq \emptyset\}$ denotes the (effective) domain of \mathcal{A} . We write it as $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightrightarrows \mathcal{P}(\mathcal{Z})$. We denote by $\mathcal{R}(\mathcal{A}) := \cup_{u \in \mathcal{D}(\mathcal{A})} Au$ and $\mathcal{G}(\mathcal{A}) := \{[u, v] \in \mathcal{Z} \times \mathcal{Z}^* ; u \in \mathcal{D}(\mathcal{A}), v \in Au\}$ the range and the graph of the operator \mathcal{A} , respectively.

DEFINITION 2.1 *Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{Z} \rightrightarrows \mathcal{P}(\mathcal{Z}^*)$ and $[x, x^*], [z, z^*] \in \mathcal{G}(\mathcal{A})$ be arbitrary. The operator \mathcal{A} is said to be:*

(i) *monotone, iff*

$$\langle x^* - z^*, x - z \rangle_{\mathcal{Z}} \geq 0;$$

(ii) *strictly monotone, iff*

$$\langle x^* - z^*, x - z \rangle_{\mathcal{Z}} > 0, \quad \text{only if } x \neq z,$$

(iii) *strongly monotone, iff there exists a constant $c > 0$ such that*

$$\langle x^* - z^*, x - z \rangle_{\mathcal{Z}} \geq c \|x - z\|_{\mathcal{Z}}^2;$$

(iv) *maximal monotone, iff the graph of \mathcal{A} is not contained in the graph of any other monotone operator with the same domain.*

DEFINITION 2.2 *An operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{Z} \rightrightarrows \mathcal{P}(\mathcal{Z}^*)$ is said to be upper-semicontinuous (for short, u.s.c.) at $x \in \mathcal{D}(\mathcal{A})$, iff for any open neighbourhood \mathcal{V} of $\mathcal{A}(x)$ there is an open neighbourhood \mathcal{U} of x such that $Au \subset \mathcal{V}$ for each $u \in \mathcal{U}$. The operator \mathcal{A} is said to be u.s.c., iff it is u.s.c. at every point of its domain.*

The following definition of pseudo-monotone operator is due to Browder and Hess (1974). It generalizes the concept of a single-valued pseudo-monotone mapping, which was initially given by H. Brezis (see Zeidler, 1990).

DEFINITION 2.3 *An operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{Z} \rightrightarrows \mathcal{P}(\mathcal{Z}^*)$ is said to be pseudo-monotone, iff the following three conditions are fulfilled:*

(PM1) : *For each $x \in \mathcal{Z}$, the set $\mathcal{A}x$ be nonempty, bounded, closed and convex.*

(PM2) : *If $\{[x_n, x_n^*]\}_{n=1}^{\infty} \subset \mathcal{G}(\mathcal{A})$ be such that*

$$x_n \rightarrow x \text{ as } n \rightarrow \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle x_n^*, x_n - x \rangle_{\mathcal{Z}} \leq 0;$$

²Henceforth the term operator means a multivalued operator.

then for each $y \in \mathcal{Z}$ there exists $x^*(y) \in \mathcal{A}(x)$ with the property that

$$\liminf_{n \rightarrow \infty} \langle x_n^*, x_n - y \rangle_{\mathcal{Z}} \geq \langle x^*(y), x - y \rangle_{\mathcal{Z}}.$$

(PM3) : The restriction of \mathcal{A} to any finite dimensional subspace \mathcal{M} of \mathcal{Z} is weakly u.s.c. as an operator from \mathcal{M} to \mathcal{Z}^* .

REMARK 2.1 It is evident from the condition (PM1) above that the domain of the operator \mathcal{A} must be the whole space.³ It is well known that a maximal monotone operator defined on the whole space is pseudo-monotone.

Let us consider the following condition:

(PM4) : For each $x_0 \in \mathcal{Z}$ and each bounded subset \mathcal{M}_1 of \mathcal{Z} , there exists a constant $m(\mathcal{M}_1, x_0)$ such that

$$\langle x^*, x - x_0 \rangle_{\mathcal{Z}} \geq m(\mathcal{M}_1, x_0), \quad \forall [x^*, x] \in \mathcal{G}(\mathcal{A}), x \in \mathcal{M}_1.$$

It has been shown by Kenmochi (1974) that the conditions (PM1) (PM2) and (PM4) imply (PM3). It is not difficult to verify that the condition (PM4) is satisfied by all monotone operators defined on the whole space.

DEFINITION 2.4 Let $\phi : \mathcal{Z} \rightarrow \bar{\mathbb{R}}$. The functional ϕ is said to be:

- (i) proper, iff it takes nowhere the value $-\infty$ and is not identically equal to $+\infty$;
- (ii) sequentially lower-semicontinuous (for short, l.s.c.), iff

$$\lim_{n \rightarrow \infty} y_n \rightarrow y \implies \liminf_{n \rightarrow \infty} \phi(y_n) \geq \phi(y);$$

- (iii) convex, iff

$$\phi(tx + (1-t)z) \leq t\phi(x) + (1-t)\phi(z); \quad \forall x, z \in \mathcal{D}(\phi), \quad t \in [0, 1],$$

- (iv) strictly convex, iff ϕ is convex and

$$2\phi[(x+z)/2] < \phi(x) + \phi(z).$$

We conclude this section by recalling a fixed point theorem. For the proof the reader is referred to Kluge (1979).

THEOREM 2.1 Let $\mathcal{C} \subset \mathcal{Z}$ be convex and weakly closed, $\mathcal{S} : \mathcal{C} \rightrightarrows \mathcal{P}(\mathcal{C})$ be such that $\forall u \in \mathcal{C}$, $\mathcal{S}(u) \neq \emptyset$ closed and convex, and the graph $\mathcal{G}(\mathcal{S})$ be weakly closed. Assume that either the set \mathcal{C} or $\mathcal{S}(\mathcal{C})$ be bounded. Then \mathcal{S} has at least one fixed point in \mathcal{C} .

³It is possible to define the concept of pseudo-monotone operator on proper subsets; see for example Browder and Hess (1974) where pseudo-monotone operators are also defined on convex sets.

3. Auxiliary results and existence theorems

Consider the following \mathcal{VI} : find $y \in C$ and $F \in \mathcal{F}(y)$ such that

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in C. \quad (9)$$

Let $\mathcal{S}(\mathcal{VI})$ be the set of all solutions to $\mathcal{VI}(9)$.

We begin with the following:

PROPOSITION 3.1 *Let $C \subseteq B$ be nonempty, closed and convex, $\mathcal{F} : \mathcal{D}(\mathcal{F}) \subseteq B \rightrightarrows \mathcal{P}(B^*)$ be maximal monotone and satisfy ⁴ $\text{int } \mathcal{D}(\mathcal{F}) \supseteq C$, $\varphi : B \rightarrow \bar{\mathbb{R}}$ be proper, convex and l.s.c.. Then, $y \in C$ is a solution to $\mathcal{VI}(9)$, iff y is a solution to the following system: find $y \in C$ such that*

$$\langle F^* - f, x - y \rangle \geq \varphi(y) - \varphi(x); \quad \forall x \in C, \quad \forall F^* \in \mathcal{F}(x). \quad (10)$$

Proof. "Only if ". Let $y \in C$ be a solution to $\mathcal{VI}(9)$. By the definition of monotonicity of the operator \mathcal{F} , $\forall x, \forall y$ and $\forall F^* \in \mathcal{F}(x), \forall F \in \mathcal{F}(y)$, we have:

$$\langle F^* - F, x - y \rangle \geq 0;$$

and then:

$$\begin{aligned} \langle F^*, y - x \rangle &\leq \langle F, y - x \rangle \\ &\leq \langle f, y - x \rangle + \varphi(x) - \varphi(y). \end{aligned}$$

The above inequality can be written as

$$\langle F^* - f, x - y \rangle \geq \varphi(y) - \varphi(x).$$

That is, $y \in C$ is a solution to $\mathcal{VI}(10)$.

"If ". Let $y \in C$ be a solution to $\mathcal{VI}(10)$. Consider an arbitrary $z \in C$ and a sequence $\{t_n\}_{n=1}^{\infty} \downarrow 0$ with $t_n \in]0, 1], n \in \mathcal{N}$. The convexity of C implies that $z_n := (1 - t_n)y + t_n z \in C$. Let $F_n \in \mathcal{F}(z_n)$. At $x := z_n$, the inequality (10) becomes:

$$t_n \langle F_n - f, z - y \rangle \geq \varphi(y) - \varphi(z_n), \quad \forall n \in \mathcal{N}, \quad \forall F_n \in \mathcal{F}(z_n).$$

By exploiting the convexity of φ and dividing both sides by t_n , the above inequality implies:

$$\langle F_n - f, z - y \rangle \geq \varphi(y) - \varphi(z), \quad \forall n \in \mathcal{N}, \quad \forall F_n \in \mathcal{F}(z_n).$$

By the local boundedness of a monotone operator at every interior point of its domain, we infer that the sequence $\{F_n\}_{n=1}^{\infty}$ is bounded and hence by the reflexivity of the space B (and hence also of the dual B^*) it is weakly compact (as $n \rightarrow \infty$). Therefore as $t_n \downarrow 0$, $z_n \rightarrow y$ and $F_n \rightharpoonup F(z)$. By the maximal monotonicity, we deduce $F(z) \in \mathcal{F}(y)$.

⁴Here the notation $\text{int } E$, represents the interior of the set E .

Therefore, for each $z \in \mathcal{C}$, there exists $F(z) \in \mathcal{F}(y)$ such that

$$\langle F(z) - f, z - y \rangle \geq \varphi(y) - \varphi(z).$$

This, in view of Proposition 3.3 below, implies that there exists $F \in \mathcal{F}(y)$ such that

$$\langle F - f, z - y \rangle \geq \varphi(y) - \varphi(z), \quad \forall z \in \mathcal{C},$$

which shows that y is a solution to (9). This completes the proof. \blacksquare

REMARK 3.1 *The above result is an extension of the classic Minty Lemma (see Kinderlehrer and Stampacchia, 1980) to the multivalued VI. The formulation (10) is known as Minty Variational Inequality (for short, MVI) for VI (9) and plays a very prominent role in issues such as regularization and penalization.*

REMARK 3.2 *The ‘Only if’ part requires neither the convexity of the functional nor the condition that \mathcal{F} be maximal and satisfy $\text{int } \mathcal{D}(\mathcal{F}) \supseteq \Omega$; the ‘If’ part does not require the monotonicity of \mathcal{F} . The assumption that φ is l.s.c. is to make Proposition 3.3 applicable. For the same purpose, we have also imposed the maximal monotonicity of \mathcal{F} to deduce that $\mathcal{F}(y)$ is nonempty, closed, convex and from the condition that $\text{int } \mathcal{D}(\mathcal{F}) \supseteq \Omega$ we have derived the (local) boundedness of $\mathcal{F}(\cdot)$.*

PROPOSITION 3.2 *Assume that the hypotheses of Proposition 3.1 hold. Then, the set of all solutions to VI(9) is closed and convex.*

Proof. Let us assume that $\mathcal{S}(\text{VI}) \neq \emptyset$, otherwise the statement is trivially true. For $z \in \mathcal{C}$, define a functional

$$H_z(\cdot) = \varphi(\cdot) - \varphi(z) - \langle F^* - f, z - \cdot \rangle$$

where $F^* \in \mathcal{F}(z)$.

From the fact that φ is convex and l.s.c., we infer that the functional H_z is also convex and lower-semicontinuous.

Hence, the set $\{x : H_z(x) \leq 0\}$ is closed and convex. However, in view of the preceding result

$$\mathcal{S}(\text{VI}) = \bigcap_{z \in \mathcal{C}} \{x : H_z(x) \leq 0\}.$$

Clearly $\mathcal{S}(\text{VI})$ is closed and convex. This completes the proof. \blacksquare

PROPOSITION 3.3 *Let $\mathcal{C} \subseteq \mathcal{B}$ be nonempty, closed and convex, $\mathcal{C}^* \subseteq \mathcal{B}^*$ be nonempty, closed, convex and bounded, $\varphi : \mathcal{B} \rightarrow \bar{\mathbb{R}}$ be proper, convex and l.s.c., and $y \in \mathcal{C}$ be arbitrary. Assume that for each $x \in \mathcal{C}$ there exists $x^*(x) \in \mathcal{C}^*$ such that*

$$\langle x^*(x), x - y \rangle \geq \varphi(y) - \varphi(x). \quad (11)$$

Then, there exists $y^ \in \mathcal{C}^*$ such that*

$$\langle y^*, x - y \rangle \geq \varphi(y) - \varphi(x); \quad \forall x \in \mathcal{C}. \quad (12)$$

Proof. Let the conclusion of Proposition 3.3 be false. Then, for each $x^* \in \mathcal{C}^*$, there exists at least one $x \in \mathcal{C}$ such that

$$\langle x^*, x - y \rangle < \varphi(y) - \varphi(x). \quad (13)$$

Let $x \in \mathcal{C}$ be arbitrary. We define:

$$\mathcal{S}_x := \{x^* \in \mathcal{C}^* : \langle x^*, x - y \rangle < \varphi(y) - \varphi(x)\}.$$

We infer that for each $x \in \mathcal{C}$, the corresponding set \mathcal{S}_x is open (in the weak topology of the space \mathcal{B}^*).⁵ Since the space \mathcal{B}^* is reflexive, the set \mathcal{C}^* is weakly compact and hence we can always extract a finite set $\{x_1, x_2, \dots, x_n\} \subseteq \mathcal{C}$, so that the corresponding sets $\{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n\}$ constitutes a finite covering of \mathcal{C}^* .

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be a partition of unity such that each λ_i , $i = 1, 2, \dots, n$, is a continuous function on \mathcal{C}^* (again in the weak topology) and satisfies $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^n \lambda_i(x^*) = 1$, $\forall x^* \in \mathcal{C}^*$.

We define a mapping $\mathcal{J} : \mathcal{C}^* \rightarrow \mathcal{C}$ such that

$$\mathcal{J}(x^*) = \sum_{i=1}^n \lambda_i(x^*)(x_i).$$

From the fact that $\mathcal{J}(x^*)$ is a convex combination of the elements x_i with (weakly) continuous coefficients, we deduce that $\mathcal{J}(\cdot)$ is also (weakly) continuous.

We have

$$\begin{aligned} \langle x^*, \mathcal{J}(x^*) - y \rangle &= \left\langle x^*, \sum_{i=1}^n \lambda_i(x^*) x_i - y \right\rangle \\ &= \sum_{i=1}^n \lambda_i(x^*) \langle x^*, x_i - y \rangle \\ &< \varphi(y) - \sum_{i=1}^n \lambda_i(x^*) \varphi(x_i). \end{aligned} \quad (14)$$

By exploiting the convexity of the functional $\varphi(\cdot)$, we deduce that

$$\varphi(\mathcal{J}(x^*)) \leq \sum_{i=1}^n \lambda_i(x^*) \varphi(x_i).$$

This, when combined with (14), yields

$$\langle x^*, \mathcal{J}(x^*) - y \rangle < \varphi(y) - \varphi(\mathcal{J}(x^*)). \quad (15)$$

To finish the proof, we now establish the impossibility of (15).

⁵If $\varphi(x) = \infty$, we take the corresponding set to be empty.

We define two mappings, namely: $\Phi : \mathcal{C} \rightrightarrows \mathcal{P}(\mathcal{C}^*)$ and $\Psi : \mathcal{C}^* \rightrightarrows \mathcal{P}(\mathcal{C}^*)$, which are related by the following relation:

$$\Psi(x^*) := \Phi(\mathcal{J}(x^*)), \tag{16}$$

where the mapping Φ assigns to each $x \in \mathcal{C}$, the set

$$\Phi(x) := \{x^* \in \mathcal{C}^* : \langle x^*, x - y \rangle \geq \varphi(y) - \varphi(x)\}.$$

Clearly, $\forall x \in \mathcal{C}$, the set $\Phi(x)$ is nonempty, closed and weakly compact (due to the boundedness of the set \mathcal{C}^*). The mapping Φ is u.s.c., when \mathcal{C}^* is supplied by the weak topology, see Browder (1968).

In view of (16), $\forall x^* \in \mathcal{C}^*$, the set $\Psi(x^*)$ is nonempty, closed, convex and weakly compact and the mapping Ψ is u.s.c. (in the weak topology of the set \mathcal{C}^*). Consequently, by the Tychonov Theorem, see Theorem 4 in Browder (1968), Ψ has a fixed point, that is, there exists $x^* \in \Psi(x^*)$.

This confirms the existence of $x^* \in \mathcal{C}^*$, such that

$$\langle x^*, \mathcal{J}(x^*) - y \rangle \geq \varphi(y) - \varphi(\mathcal{J}(x^*)),$$

which contradicts (15) (since the relation (15) is, indeed, valid $\forall x^* \in \mathcal{C}^*$). This completes the proof. ■

We give an existence theorem for (9).

THEOREM 3.1 *Let $\mathcal{F} : \mathcal{B} \rightrightarrows \mathcal{B}^*$ satisfies (PM1), (PM2) and (PM4) of Definition 2.3, $\mathcal{C} \subseteq \mathcal{B}$ be nonempty, closed and convex and $\varphi : \mathcal{B} \rightarrow \bar{\mathbb{R}}$ be proper, convex and l.s.c.. Assume that one of the following conditions holds:*

- (a) *The set \mathcal{C} is bounded.*
- (b) *There exists $x_0 \in \mathcal{C}$ such that $\varphi(x_0) < \infty$ and*

$$\inf_{F \in \mathcal{F}(x)} \frac{\langle F, x - x_0 \rangle + \varphi(x)}{\|x\|} \rightarrow \infty \text{ as } \|x\| \rightarrow \infty, \quad \forall x \in \mathcal{C}.$$

Then, for a given $f \in \mathcal{B}^$, there exist $y \in \mathcal{C}$ and $F \in \mathcal{F}(y)$ such that*

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x); \quad \forall x \in \mathcal{C}.$$

Proof. The proof is given in Theorem 4.1 and Proposition 4.1 in Kenmochi (1974).

We turn to the solvability of QVI (1). Let $v \in \Omega$ be arbitrary.

Consider the following Parametric Variational Inequality (for short, PVI)⁶: find $y \in \mathcal{K}(v)$ and $F \in \mathcal{F}(y)$ such that

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in \mathcal{K}(v). \tag{17}$$

Define a mapping (multivalued, in general) $\Gamma : \Omega \subset \mathcal{B} \rightrightarrows \mathcal{P}(\Omega)$ such that for each $v \in \Omega$, $\Gamma(v)$ is the set of all solutions to the PVI with parameter v .

We have the following:

⁶Here v is the parameter.

THEOREM 3.2 *Let $\Omega \subset \mathcal{B}$ be nonempty, convex and closed, $\Gamma : \Omega \rightrightarrows \mathcal{P}(\Omega)$ be such that $\forall u \in \Omega$, $\Gamma(u)$ is nonempty, closed, convex and the graph $\mathcal{G}(\Gamma)$ is weakly closed. Assume that either the set Ω is bounded or the set $\Gamma(\Omega)$ is bounded. Then the set $\mathcal{S}(\mathcal{QVI})$ is nonempty.*

Proof. Follows from Theorem 2.1 and the obvious observation that

$$y \in \mathcal{S}(\mathcal{QVI}) \quad \Longleftrightarrow \quad y \in \Gamma(y). \quad (18)$$

REMARK 3.3 *The above result is interesting in the sense that it does not impose conditions on the data $(\mathcal{F}, f, \varphi)$. However, Propositions 3.1-3.2 and Theorem 3.1 give conditions under which $\Gamma(u)$ is nonempty, closed and convex.*

REMARK 3.4 *In view of Theorem 3.1 it is easy to check that the coerciveness condition (b) assures the existence of a ball $B_R(0)$ with radius $R > 0$, such that no point outside the ball is a candidate for solution. This leads to the equivalence of the condition that $\Gamma(u)$ is bounded and the condition that a coerciveness condition analogous to (b) of Theorem 3.1 holds for all $x \in \Omega$.*

In the following result we discuss the weak-closedness of the graph $\mathcal{G}(\Gamma)$.

PROPOSITION 3.4 *Let for the operator $\mathcal{F} : \mathcal{B} \rightrightarrows \mathcal{B}^*$ the condition (PM2) of Definition 2.3 be valid, the functional $\varphi : \mathcal{B} \rightarrow \bar{\mathbb{R}}$ be proper, convex and l.s.c.. Assume that the following three conditions hold:*

- (i) *For each $x \in \Omega$ the set $\mathcal{F}(x)$ is bounded.*
- (ii) *For $\{v_n\}_{n=1}^\infty \subset \Omega$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$, the following relation holds:⁷*

$$w - \lim \mathcal{K}(v_n) \subseteq \mathcal{K}(v) \subseteq s - \lim \mathcal{K}(v_n). \quad (19)$$

- (iii) *For a sequence $\{z_n\}_{k=1}^\infty \rightarrow z$ as $n \rightarrow \infty$, in the sense of (19), the following relation holds:*

$$\limsup_{n \rightarrow \infty} \varphi(z_n) \leq \varphi(z). \quad (20)$$

Then, the graph $\mathcal{G}(\Gamma)$ is weakly closed.

Proof. Let $[y_n, v_n] \in \mathcal{G}(\Gamma)$ be such that $y_n \rightarrow y$ and $v_n \rightarrow v$ as $n \rightarrow \infty$. We claim that $[y, v] \in \mathcal{G}(\Gamma)$.

Indeed, by the definition, $y_n \in \mathcal{K}(v_n)$ and there exists $F_n \in \mathcal{F}(y_n)$ such that

$$\langle F_n - f, z - y_n \rangle \geq \varphi(y_n) - \varphi(z) \quad \forall z \in \mathcal{K}(y_n). \quad (21)$$

In view of the hypothesis (19), and the condition that $y_n \in \mathcal{K}(v_n)$, we infer, firstly, that $y \in \mathcal{K}(v)$ and, secondly, that for each $w \in \mathcal{K}(v)$ there exists $w_n \in \mathcal{K}(v_n)$ such that $w_n \rightarrow w$ as $n \rightarrow \infty$.

⁷For a sequence of sets $\{\mathcal{K}_n\}_{n=1}^\infty$, we define:
 $w - \lim \mathcal{K}_n := \{y : y_k \rightarrow y, y_k \in \mathcal{K}_k, \text{ where } \{\mathcal{K}_k\}_{k=1}^\infty \text{ is a subsequence of } \{\mathcal{K}_n\}_{n=1}^\infty\};$
 $s - \lim \mathcal{K}_n := \{y : y_n \rightarrow y, \text{ where } y_n \in \mathcal{K}_n\}.$

Let $x_n \in \mathcal{K}(v_n)$ be such that $x_n \rightarrow y$ as $n \rightarrow \infty$.

Arranging $z := x_n$ in (21), we obtain

$$\langle F_n - f, x_n - y_n \rangle \geq \varphi(y_n) - \varphi(x_n),$$

which implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle F_n, y_n - x_n \rangle &\leq \limsup_{n \rightarrow \infty} [\langle f, y_n - y \rangle + \|f\| \|x_n - y\|] + \\ &\quad \limsup_{n \rightarrow \infty} [\varphi x_n - \varphi y_n] \\ &\leq \limsup_{n \rightarrow \infty} [\varphi x_n - \varphi y] + \\ &\quad \limsup_{n \rightarrow \infty} [\varphi y - \varphi y_n] \end{aligned} \quad (22)$$

From the hypothesis (20), we have

$$\limsup_{n \rightarrow \infty} [\varphi x_n - \varphi y] \leq 0,$$

and from the condition that $\varphi(\cdot)$ is l.s.c., we get

$$\limsup_{n \rightarrow \infty} [\varphi y - \varphi y_n] \leq 0.$$

Combining the above two estimates with (22), we obtain

$$\limsup_{n \rightarrow \infty} \langle F_n, y_n - x_n \rangle \leq 0,$$

which further leads to

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle F_n, y_n - y \rangle &\leq \limsup_{n \rightarrow \infty} \langle F_n, x_n - y \rangle \\ &\leq 0. \end{aligned}$$

From the condition (PM2) of Definition 2.3, we get to the conclusion that for an arbitrary $x \in \mathcal{K}(v) \subseteq \mathcal{B}$, there exists $F(x) \in \mathcal{F}(y)$ such that:

$$\liminf_{n \rightarrow \infty} \langle F_n, y_n - x \rangle \geq \langle F(x), y - x \rangle.$$

Since $\langle f, y_n - x \rangle \rightarrow \langle f, y - x \rangle$ as $n \rightarrow \infty$, we have

$$\liminf_{n \rightarrow \infty} \langle F_n - f, y_n - x \rangle \geq \langle F(x) - f, y - x \rangle. \quad (23)$$

For $x \in \mathcal{K}(v)$, it is always possible to find $x_n \in \mathcal{K}(v_n)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Therefore, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle F_n - f, y_n - x \rangle &\leq \limsup_{n \rightarrow \infty} \langle F_n - f, y_n - x \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle F_n - f, y_n - x_n \rangle + \end{aligned}$$

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \langle F_n - f, x_n - x \rangle \\
& \leq \limsup_{n \rightarrow \infty} [\varphi(x_n) - \varphi(y_n)] \\
& \leq \limsup_{n \rightarrow \infty} [\varphi(x_n) - \varphi(x)] + \\
& \quad \limsup_{n \rightarrow \infty} [\varphi(x) - \varphi(y_n)] \\
& \leq \varphi(x) - \varphi(y).
\end{aligned}$$

By substituting the above estimate into (23), we reach the conclusion that for an arbitrary $x \in \mathcal{K}(v)$, there exists $F(x) \in \mathcal{F}(y)$ such that:

$$\langle F(x) - f, x - y \rangle \geq \varphi(y) - \varphi(x).$$

Since the above estimate is derived for an arbitrary $x \in \mathcal{K}(v)$, we conclude that it is, indeed, valid $\forall x \in \mathcal{K}(v)$.

An application of Proposition 3.3 yields that there exists $F \in \mathcal{F}(y)$ such that

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x); \quad \forall x \in \mathcal{K}(v).$$

This implies that $[y, v] \in \mathcal{G}(\Gamma)$. The proof is complete. \blacksquare

4. Regularization

In the present section, we assume that neither the set $\Omega \subset \mathcal{B}$ is bounded nor the operator \mathcal{F} satisfies any sort of coerciveness condition. In this situation, as it is evident from the discussion made in the previous section, the $QVI(1)$ may fail to have a solution. Also the existence theorems, available in the literature, become inefficient in this situation.

In order to handle the present situation, we intend to employ the so-called Browder-Tikhonov regularization method.

We assume that instead of the exact data $(\mathcal{F}, f, \varphi)$ only the *noisy* data $(\mathcal{F}_{\alpha_n}, f_{\beta_n}, \varphi_{\gamma_n})$ are available. Here $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences of positive reals.

Let $\{\epsilon_n\}_{n=1}^{\infty}$, $\epsilon_n > 0$, $n \in \mathcal{N}$ be a sequence of positive reals which is (strictly) decreasing and converging to zero.

The relationship between the exact data and the noisy data is given through the following assumptions:

Assumption 4.1. There exists a continuous function $\tau : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ such that

$$\sigma_{\mathcal{B}^*}(\mathcal{F}(x), \mathcal{F}_{\alpha_n}(x)) \leq \alpha_n \tau(\|x\|), \quad \forall x \in \mathcal{D}(\mathcal{F}) \cup \mathcal{D}(\mathcal{F}_{\alpha_n}),$$

where $\sigma_{\mathcal{B}^*}(Q_1, Q_2)$ is the Hausdorff distance between the sets Q_1 and Q_2 .

Assumption 4.2. For $f_{\beta_n} \in \mathcal{B}^*$, we have

$$\|f - f_{\beta_n}\| \leq \beta_n.$$

Assumption 4.3. There exists a continuous function $\kappa : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ such that

$$\|\varphi(x) - \varphi_{\gamma_n}(x)\| \leq \gamma_n \kappa(\|x\|), \quad \forall x \in \mathcal{D}(\varphi) \cup \mathcal{D}(\varphi_{\gamma_n}).$$

Assumption 4.4. The mappings $\tau, \kappa : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ satisfy the following estimates:

$$\limsup_{t \rightarrow \infty} \frac{\tau(t)}{t} < \infty, \quad \limsup_{t \rightarrow \infty} \frac{\kappa(t)}{t} < \infty, \quad \forall t \geq 0.$$

Assumption 4.5. For $n \rightarrow \infty$

$$\alpha_n, \beta_n, \gamma_n, \alpha_n/\epsilon_n, \beta_n/\epsilon_n, \gamma_n/\epsilon_n \rightarrow 0.$$

Consider the following Regularized Quasi-Variational Inequality (\mathcal{RQVI} for short): find $y_{\pi_n} \in \mathcal{K}(y_{\pi_n})$ and $\tilde{F}_{\pi_n} \in \mathcal{F}_{\alpha_n}(y_{\pi_n})$ such that

$$\langle \tilde{F}_{\pi_n} + \epsilon_n \mathcal{R}(y_{\pi_n}) - f_{\beta_n}, x - y_{\pi_n} \rangle \geq \varphi_{\gamma_n}(y_{\pi_n}) - \varphi_{\gamma_n}(x), \quad \forall x \in \mathcal{K}(y_{\pi_n}). \quad (24)$$

In the above \mathcal{RQVI} , the operator $\mathcal{R} : \mathcal{B} \rightarrow \mathcal{B}^*$ is the regularizing operator, ϵ_n the regularization parameter and y_{π_n} is the regularized solution to the $\mathcal{QVI}(1)$. Here the symbol $\pi_n := (\alpha_n, \beta_n, \gamma_n, \epsilon_n)$ shows the influence of the error parameters $\alpha_n, \beta_n, \gamma_n$ and the regularization parameter ϵ_n .

We denote by $\mathcal{S}_{\epsilon_n}(\mathcal{RQVI})$ the set of all solutions to the \mathcal{RQVI} (24) with regularization parameter ϵ_n .

In the present study, we use the following potential operator as the regularization operator:

$$\mathcal{R}(x) = \nabla \|x\|^m, \quad x \neq 0, \quad \mathcal{R}(0) = 0, \quad m > 1.$$

Since

$$\frac{d}{dt} \|x + th\|^m = m \|x + th\|^{m-1} \frac{d}{dt} \|x + th\|,$$

it immediately follows that

$$\mathcal{R}(x) = \nabla \|x\|^m = m \|x\|^{m-1} \nabla \|x\|. \quad (25)$$

We claim that the operator \mathcal{R} is monotone. Indeed, using the equality

$$\langle \nabla \|x\|, x \rangle = \|x\|, \quad (26)$$

we obtain

$$\begin{aligned} \langle \mathcal{R}x - \mathcal{R}z, x - z \rangle &= \langle \mathcal{R}x, x \rangle + \langle \mathcal{R}z, z \rangle - \langle \mathcal{R}x, z \rangle + \langle \mathcal{R}z, x \rangle \\ &\geq m[\|x\|^m + \|z\|^m] - \|\mathcal{R}x\| \|z\| - \|\mathcal{R}z\| \|x\| \\ &= m[\|x\|^m + \|z\|^m - \|x\|^{m-1} \|z\| - \|z\|^{m-1} \|x\|] \\ &= m(\|x\| - \|z\|) (\|x\|^{m-1} - \|z\|^{m-1}) \\ &\geq 0. \end{aligned} \quad (27)$$

Thus, the operator $\mathcal{R} : \mathcal{B} \rightarrow \mathcal{B}^*$ is monotone for $m > 1$.

In view of the relations (25) and (26), it is also evident that \mathcal{R} is coercive as well. For other interesting properties of this operator the reader is referred to Vainberg (1973).

In light of the inequality (27), we get the following relations

$$\langle \mathcal{R}x - \mathcal{R}z, x - z \rangle \geq (m-1)\|x\|^{m-2}(\|x\| - \|z\|)^2; \quad 1 < m \leq 2. \quad (28)$$

$$\langle \mathcal{R}x - \mathcal{R}z, x - z \rangle \geq (m-1)\|z\|^{m-2}(\|x\| - \|z\|)^2; \quad 2 \leq m. \quad (29)$$

In order to get the above inequalities, we have used the following standard relationships:

$$A^n - B^n \geq nB^{n-1}(A - B); \quad n \geq 1, \quad A, B \geq 0.$$

$$A^n - B^n \geq nA^{n-1}(A - B); \quad 0 < n \leq 1 \quad A, B \geq 0.$$

It is well known (see Zeidler, 1990) that every reflexive Banach space (and its topological dual space) can be renormed so that in the new norms the space and the dual become locally uniformly convex and that the new norms are differentiable in the sense of Fréchet. Therefore, without any loss to generality, we shall assume henceforth that the spaces \mathcal{B} and \mathcal{B}^* are locally uniformly convex and the norm $\|\cdot\|$ is Fréchet differentiable. Therefore, the regularizing operator \mathcal{R} is well defined.

THEOREM 4.1 *Let $\mathcal{F}_{\alpha_n} : \mathcal{B} \rightrightarrows \mathcal{P}(\mathcal{B}^*)$ be maximal monotone, $\Omega \subset \mathcal{B}$ be nonempty, convex and closed, $\mathcal{K} : \Omega \rightrightarrows \mathcal{P}(\Omega)$ be such that $\forall u \in \Omega, \mathcal{K}(u) \neq \emptyset$, closed and convex, $\varphi_{\gamma_n} : \mathcal{B} \rightarrow \bar{\mathbb{R}}$ be proper, convex and l.s.c.. Assume that the following three conditions hold:*

(i) *For $\{v_n\}_{n=1}^{\infty} \subset \Omega$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$, the following relation holds:*

$$w - \lim \mathcal{K}(v_n) \subseteq \mathcal{K}(v) \subseteq s - \lim \mathcal{K}(v_n). \quad (30)$$

(ii) *For a sequence $\{z_n\}_{n=1}^{\infty} \rightarrow z$ as $n \rightarrow \infty$, in the sense of (30), the following relation holds:*

$$\limsup_{n \rightarrow \infty} \varphi_{\gamma_n}(z_n) \leq \varphi_{\gamma_n}(z). \quad (31)$$

(iii) *There exists $x_0 \in \bigcap_{v \in \Omega} \mathcal{K}(v) \cap \text{int } \mathcal{D}(\varphi_{\gamma_n})$.*

Then, for $n \in \mathcal{N}$ and given $f_{\beta_n} \in \mathcal{B}^$ the RQVI (24) has a nonempty solution set, that is $\mathcal{S}_{\epsilon_n}(\text{RQVI}) \neq \emptyset$.*

Proof. In view of Theorem 3.2, it will be enough to show that $\forall u \in \Omega, \Gamma_{\epsilon_n}(u) \neq \emptyset$ and $\Gamma_{\epsilon_n}(\Omega)$ is bounded.

For this, we show that there exists $x_0 \in \Omega$ such that $\varphi_{\gamma_n}(x_0) < \infty$ and

$$\inf_{F \in \mathcal{F}_{\alpha_n}(x)} \frac{\langle F + \epsilon_n \mathcal{R}(x), x - x_0 \rangle + \varphi_{\gamma_n}(x)}{\|x\|} \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty, \quad \forall x \in \Omega.$$

Since $\varphi_{\gamma_n}(\cdot)$ is proper, convex and l.s.c. functional, for each $z \in \text{int } \mathcal{D}(\varphi_{\gamma_n})$, the subdifferential of φ_{γ_n} at z , denoted by $\partial \varphi_{\gamma_n}(z) \neq \emptyset$.

Therefore, with $z := x_0$ as in the hypothesis, we obtain

$$\varphi_{\gamma_n}(x) \geq \varphi_{\gamma_n}(x_0) + \langle x_0^*, x - x_0 \rangle, \quad \forall x_0^* \in \partial \varphi_{\gamma_n}(x_0), \quad \forall x \in \Omega \subseteq B$$

and then:

$$\varphi_{\gamma_n}(x) \geq \varphi_{\gamma_n}(x_0) - \|x_0^*\| \|x - x_0\|. \tag{32}$$

From the monotonicity of $\mathcal{F}_{\alpha_n}(\cdot)$, $\forall F \in \mathcal{F}_{\alpha_n}(x)$, $\forall \bar{F} \in \mathcal{F}_{\alpha_n}(x_0)$, we have

$$\langle F, x - x_0 \rangle \geq -\|\bar{F}\| \|x - x_0\|. \tag{33}$$

For the regularizing operator \mathcal{R} , we have

$$\begin{aligned} \langle \mathcal{R}(x), x - x_0 \rangle &= \langle \mathcal{R}(x), x \rangle - \langle \mathcal{R}(x), x_0 \rangle \\ &\geq m [\|x\|^m - \|x\|^{m-1} \|x_0\|]. \end{aligned} \tag{34}$$

Combining (32), (33) and (34), yields

$$\begin{aligned} \langle F + \epsilon_n \mathcal{R}(x), x - x_0 \rangle + \varphi_{\gamma_n}(x) &\geq m\epsilon_n \{ \|x\|^{m-1} - \|x\|^{m-2} \|x_0\| \} \|x\| \\ &\quad + \{ \varphi_{\gamma_n}(x_0) \|x\|^{-1} \} \|x\| \\ &\quad + \{ \|x_0^*\| - \|\bar{F}\| \} \|x - x_0\|. \end{aligned}$$

Since we are interested in the behaviour of the above inequality as $\|x\| \rightarrow \infty$, replacement of the term $\|x - x_0\|$ by $\|x\|$ will create no trouble, and hence, it is enough to study the behaviour of the estimate

$$\begin{aligned} \langle F + \epsilon_n \mathcal{R}(x), x - x_0 \rangle + \varphi_{\gamma_n}(x) &\geq m\epsilon_n \{ \|x\|^{m-1} - \|x\|^{m-2} \|x_0\| \} \|x\| \\ &\quad + \{ \varphi_{\gamma_n}(x_0) \|x\|^{-1} \} \|x\| \\ &\quad + \{ -\|x_0^*\| - \|\bar{F}\| \} \|x\|. \end{aligned}$$

From the fact the above estimate is valid $\forall F \in \mathcal{F}(x)$, we deduce that

$$\inf_{F \in \mathcal{F}_{\alpha_n}(x)} \frac{\langle F + \epsilon_n \mathcal{R}(x), x - x_0 \rangle + \varphi_{\gamma_n}(x)}{\|x\|} \rightarrow \infty \text{ as } \|x\| \rightarrow \infty, \quad \forall x \in \Omega.$$

The proof is complete. ■

THEOREM 4.2 *Along with the hypotheses of Theorem 4.1, assume that $\mathcal{F} : B \rightrightarrows \mathcal{P}(B^*)$ be maximal monotone, φ be proper, convex and l.s.c., Assumptions 4.1-4.5 hold and $\mathcal{S}(QVI) \neq \emptyset$. Then, the sequence $\{y_{\pi_n}\}_{n=1}^\infty$, where $y_{\pi_n} \in \mathcal{S}_{\epsilon_n}(\mathcal{R}QVI)$ is chosen arbitrarily, is uniformly bounded.*

Proof. The validity of the hypotheses of Theorem 4.1 implies that $\forall n \in \mathcal{N}$, $\mathcal{S}_{\epsilon_n}(\mathcal{R}QVI) \neq \emptyset$. Let $y_{\pi_n} \in \mathcal{S}_{\epsilon_n}(\mathcal{R}QVI)$, that is

$$y_{\pi_n} \in \Gamma_{\epsilon_n}(y_{\pi_n}),$$

where $\Gamma_{\epsilon_n} : \Omega \rightrightarrows \mathcal{P}(\Omega)$ is such that it assigns to each $v \in \Omega$ the set of all solutions to the following Regularized Parametric Variational Inequality (for short, \mathcal{RPVI}): find $y_{\pi_n} \in \mathcal{K}(v)$ and $\tilde{F}_{\pi_n} \in \mathcal{F}(y_{\pi_n})$ such that

$$\langle \tilde{F}_{\pi_n} + \epsilon_n \mathcal{R}(y_{\pi_n}) - f_{\beta_n}, x - y_{\pi_n} \rangle \geq \varphi_{\gamma_n}(y_{\pi_n}) - \varphi_{\gamma_n}(x), \quad \forall x \in \mathcal{K}(v). \quad (35)$$

Since the mapping \mathcal{R} , due to the strict convexity of the space \mathcal{B} , is strictly monotone, we infer from Proposition 2.3 of Giannesi (1996) that $\forall v \in \Omega$, the set $\Gamma_{\epsilon_n}(v)$ is single-valued.

Since all fixed points of the single-valued mapping $\Gamma_{\epsilon_n}(\cdot)$ belong to the set $\{\Gamma_{\epsilon_n}(v) : v \in \Omega\}$, it will be enough to show that: for an arbitrary $v \in \Omega$, the image $\Gamma_{\epsilon_n}(v)$ is uniformly bounded.

From the assumption that $\mathcal{S}(\mathcal{QVI}) \neq \emptyset$, it is clear that for every $v \in \Omega$, $\Gamma(v)$ is nonempty, i.e. there exist at least one $y \in \mathcal{K}(v)$ and $F \in \mathcal{F}(y)$ such that

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x); \quad \forall x \in \mathcal{K}(v). \quad (36)$$

Let us choose the same $v \in \Omega$ in (35) and (36).

Arranging $x := y$ in (35) and $x := y_{\pi_n}$ in (36) and summing-up the resulting inequalities side-by-side, we obtain

$$\begin{aligned} \langle \tilde{F}_{\pi_n} + \epsilon_n \mathcal{R}(y_{\pi_n}) - f_{\beta_n}, y - y_{\pi_n} \rangle + \langle F - f, y_{\pi_n} - y \rangle \\ \geq \varphi_{\gamma_n}(y_{\pi_n}) - \varphi_{\gamma_n}(y) + \varphi(y) - \varphi(y_{\pi_n}), \end{aligned}$$

where $\tilde{F}_{\pi_n} \in \mathcal{F}_{\alpha_n}(y_{\pi_n})$ and $F \in \mathcal{F}(y)$.

The above inequality can be expressed as

$$\begin{aligned} \alpha_n \tau(\|y\|) \|y_{\pi_n} - y\| + \gamma_n [\kappa(\|y_{\pi_n}\|) + \kappa(\|y\|)] + \beta_n \|y_{\pi_n} - y\| \\ - \langle \tilde{F}_{\pi_n} - \tilde{F}, y_{\pi_n} - y \rangle + \epsilon_n \langle \mathcal{R}y_{\pi_n}, y \rangle \geq \epsilon_n \langle \mathcal{R}y_{\pi_n}, y_{\pi_n} \rangle, \end{aligned}$$

where $\tilde{F} \in \mathcal{F}_{\alpha_n}(y)$.

Assumption 4.4 assures the existence of constants L_i and M_i , $i = 1, 2$, such that

$$\tau(t) \leq L_1 t + M_1, \quad \text{and} \quad \kappa(t) \leq L_2 t + M_2, \quad \forall t \geq 0.$$

Therefore, using these estimates in the previous inequality, we obtain

$$\begin{aligned} \frac{\beta_n}{\epsilon_n} \|y_{\pi_n} - y\| + \frac{\alpha_n}{\epsilon_n} [L_1 \|y\| + M_1] + \frac{\gamma_n}{\epsilon_n} [L_2 \|y_{\pi_n}\| + L_2 \|y\| + 2M_2] \\ + \|y_{\pi_n}\|^m \|y\| \geq \|y_{\pi_n}\|^{m+1} \end{aligned}$$

The above inequality, in view of Assumption 4.5, confirms the existence of a constant K , such that

$$\|\Gamma_{\epsilon_n}(v)\| = \|y_{\pi_n}\| \leq K.$$

This completes the proof. ■

THEOREM 4.3 *Assume that the hypotheses of Theorem 4.2 hold. Then, every weak limit point of a subsequence $\{y_{\Delta_n}\}_{n=1}^\infty$ of $\{y_{\pi_n}\}_{n=1}^\infty$ is a solution to $QVI(1)$. Furthermore, if $QVI(1)$ is uniquely solvable, then the whole sequence $\{y_{\pi_n}\}_{n=1}^\infty$ converges to the solution.*

Proof. Since the sequence $\{y_{\pi_n}\}_{n=1}^\infty$ is uniformly bounded, it is weakly compact by the reflexivity of the space B . Therefore, it is always possible to extract a subsequence $\{y_{\Delta_n}\}_{n=1}^\infty$ from $\{y_{\pi_n}\}_{n=1}^\infty$ such that $y_{\Delta_n} \rightharpoonup y \in B$ as $n \rightarrow \infty$. It is evident by the definition of the sequence $\{y_{\Delta_n}\}_{n=1}^\infty$ that $\forall n \in \mathcal{N}, y_{\Delta_n} \in \Omega$. Therefore, in view of the weak closedness of the set Ω we conclude that $y \in \Omega$.

We proceed to show that $y \in \mathcal{S}(QVI)$.

As $y_{\Delta_n} \in \mathcal{S}_{\bar{\epsilon}_n}(\mathcal{R}QVI)$, it satisfies the following two conditions, namely:

$$y_{\Delta_n} \in \mathcal{K}(y_{\Delta_n}) \tag{37}$$

and: for $\tilde{F}_{\Delta_n} \in \mathcal{F}_{\bar{\alpha}_n}(y_{\Delta_n})$ we have

$$\langle \tilde{F}_{\Delta_n} + \bar{\epsilon}_n \mathcal{R}y_{\Delta_n} - f_{\bar{\beta}_n}, x - y_{\Delta_n} \rangle \geq \varphi_{\bar{\gamma}_n}(y_{\Delta_n}) - \varphi_{\bar{\gamma}_n}(x), \quad \forall x \in \mathcal{K}(y_{\Delta_n}). \tag{38}$$

In light of the hypothesis (30), the above relation (37) implies

$$y \in \mathcal{K}(y). \tag{39}$$

Repeating the use of the hypothesis (30), we get the existence of a sequence $\{z_n\}_{n=1}^\infty$ such that $z_n \rightarrow y$ and $z_n \in \mathcal{K}(y_{\Delta_n})$.

Arranging $x := z_n$ in (38), we obtain

$$\langle \tilde{F}_{\Delta_n} + \bar{\epsilon}_n \mathcal{R}y_{\Delta_n} - f_{\bar{\beta}_n}, z_n - y_{\Delta_n} \rangle \geq \varphi_{\bar{\gamma}_n}(y_{\Delta_n}) - \varphi_{\bar{\gamma}_n}(z_n),$$

which implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \tilde{F}_{\Delta_n} + \bar{\epsilon}_n \mathcal{R}y_{\Delta_n} - f_{\bar{\beta}_n}, y_{\Delta_n} - z_n \rangle &\leq \limsup_{n \rightarrow \infty} [\varphi_{\bar{\gamma}_n}(z_n) - \varphi_{\bar{\gamma}_n}(y_{\Delta_n})] \\ &\leq \limsup_{n \rightarrow \infty} [\varphi(z_n) - \varphi(y_{\Delta_n})] + \\ &\quad \limsup_{n \rightarrow \infty} \bar{\gamma}_n[\kappa(\|z_n\|)] + \\ &\quad \limsup_{n \rightarrow \infty} \bar{\gamma}_n[\kappa(\|y_{\Delta_n}\|)] \\ &= \limsup_{n \rightarrow \infty} [\varphi(z_n) - \varphi(y_{\Delta_n})] \\ &\leq \limsup_{n \rightarrow \infty} [\varphi(z_n) - \varphi(y)] + \\ &\quad \limsup_{n \rightarrow \infty} [\varphi(y) - \varphi(y_{\Delta_n})] \end{aligned} \tag{40}$$

Following the same arguments as in Proposition 3.4, we deduce that

$$\limsup_{n \rightarrow \infty} \langle \tilde{F}_{\Delta_n} + \bar{\epsilon}_n \mathcal{R}y_{\Delta_n} - f_{\bar{\beta}_n}, y_{\Delta_n} - z_n \rangle \leq 0.$$

⁶Here, if necessary, we replace $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty, \{\epsilon_n\}_{n=1}^\infty$ by the corresponding subsequences $\{\bar{\alpha}_n\}_{n=1}^\infty, \{\bar{\beta}_n\}_{n=1}^\infty, \{\bar{\gamma}_n\}_{n=1}^\infty, \{\bar{\epsilon}_n\}_{n=1}^\infty$, respectively.

This further implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle F_{\Delta_n}, y_{\Delta_n} - z_n \rangle &\leq \limsup_{n \rightarrow \infty} [\bar{\epsilon}_n \|\mathcal{R}(y_{\Delta_n})\| + \bar{\beta}_n] \|y_{\Delta_n} - z_n\| + \\ &\quad \limsup_{n \rightarrow \infty} [\bar{\alpha}_n \tau(\|y_{\Delta_n}\|)] \|y_{\Delta_n} - z_n\| + \\ &\quad \langle f, y_{\Delta_n} - z_n \rangle \\ &\leq 0, \end{aligned}$$

since all the terms $\{\bar{\epsilon}_n, \bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n, \langle f, y_{\Delta_n} - y \rangle, \|z_n - y\|\} \rightarrow 0$ as $n \rightarrow \infty$ and the remaining terms are bounded.

Therefore, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle F_{\Delta_n}, y_{\Delta_n} - y \rangle &\leq \limsup_{n \rightarrow \infty} \langle F_{\Delta_n}, z_n - y \rangle \\ &\leq 0. \end{aligned}$$

As the operator \mathcal{F} is maximal monotone and $\mathcal{D}(\mathcal{F}) \equiv \mathcal{B}$, it is pseudo-monotone as well.

Therefore, from the condition (PM2) of Definition 2.3, we conclude that for an arbitrary $x \in \mathcal{K}(y) \subseteq \mathcal{B}$, there exists $F(x) \in \mathcal{F}(y)$ such that:

$$\liminf_{n \rightarrow \infty} \langle F_{\Delta_n}, y_{\Delta_n} - x \rangle \geq \langle F(x), y - x \rangle.$$

From the fact that $\langle f, y_{\Delta_n} - x \rangle \rightarrow \langle f, y - x \rangle$, we can express the above inequality in the form

$$\liminf_{n \rightarrow \infty} \langle F_{\Delta_n} - f, y_{\Delta_n} - x \rangle \geq \langle F(x) - f, y - x \rangle. \quad (41)$$

Since $x \in \mathcal{K}(y)$, it is always possible to find $x_n \in \mathcal{K}(y_{\Delta_n})$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$.

In order to get an estimate for the term on the left hand side in (41), we consider

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle F_{\Delta_n} - f, y_{\Delta_n} - x \rangle &\leq \limsup_{n \rightarrow \infty} \langle F_{\Delta_n} - f, y_{\Delta_n} - x \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle \tilde{F}_{\Delta_n} - f, y_{\Delta_n} - x_n \rangle + \\ &\quad \limsup_{n \rightarrow \infty} \langle F_{\Delta_n} - f, x_n - x \rangle + \\ &\quad \limsup_{n \rightarrow \infty} \langle F_{\Delta_n} - \tilde{F}_{\Delta_n}, y_{\Delta_n} - x_n \rangle \\ &\leq \limsup_{n \rightarrow \infty} \bar{\alpha}_n \tau(\|y_{\Delta_n}\|) \|y_{\Delta_n} - x_n\| + \\ &\quad \limsup_{n \rightarrow \infty} \bar{\epsilon}_n \|\mathcal{R}(y_{\Delta_n})\| \|y_{\Delta_n} - x_n\| + \\ &\quad \limsup_{n \rightarrow \infty} [\varphi_{\bar{\gamma}_n}(x_n) - \varphi_{\bar{\gamma}_n}(y_{\Delta_n})] + \\ &\quad \limsup_{n \rightarrow \infty} \bar{\beta}_n \|y_{\Delta_n} - x_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{n \rightarrow \infty} [\varphi(x_n) - \varphi(y_{\Delta_n})] + \\
&\quad \bar{\gamma}_n [\kappa(\|x_n\|) + \kappa(\|y_{\Delta_n}\|)] \\
&\leq \limsup_{n \rightarrow \infty} [\varphi(x_n) - \varphi(y_{\Delta_n})] \\
&\leq \limsup_{n \rightarrow \infty} [\varphi(x_n) - \varphi(x)] + \\
&\quad \limsup_{n \rightarrow \infty} [\varphi(x) - \varphi(y_{\Delta_n})] \\
&\leq \varphi(x) - \varphi(y).
\end{aligned}$$

On substituting the above estimate to (41), we reach the conclusion that, for an arbitrary $x \in \mathcal{K}(y)$, there exists $F(x) \in \mathcal{F}(y)$ such that:

$$\langle F(x) - f, x - y \rangle \geq \varphi(y) - \varphi(x).$$

Since the above estimate is derived for an arbitrary $x \in \mathcal{K}(y)$, we conclude that it is, indeed, valid $\forall x \in \mathcal{K}(y)$.

An application of Proposition 3.3, yields that there exists $F \in \mathcal{F}(y)$ such that:

$$\langle F - f, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in \mathcal{K}(y).$$

This together with (39) implies that $y \in \mathcal{S}(QVI)$. Furthermore if QVI (1) is uniquely solvable, then clearly y is the unique limit of any weakly convergent subsequence of $\{y_{\pi_n}\}_{n=1}^{\infty}$. Therefore we have the (weak) convergence of the whole sequence $\{y_{\pi_n}\}_{n=1}^{\infty}$ to y . The proof is complete. ■

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