Dedicated to
Professor Jakub Gutenbaum
on his 70th birthday

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An electric network chain with feedbacks

by

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Abstract: In the paper an R, L, C, G electric network chain with feedbacks is considered. The formulas for the transfer function, for its poles and for the integral square error value are given.

Keywords: electric network chain, poles, transfer function, Parseval's formula

1. Introduction

In this paper the chain composed of n equal elements of the R, L, C, G-type is considered. Each element of the chain is closed by feedback, dependent on voltage. The appropriate gain is denoted by K. The last port is loaded by a resistance R_0.

By applying Laplace transformation to the Kirchhoff's laws for the elementary system we obtain

$$(u_k - u_{k-1}) + i_k(R + sL) + K(u_k - u_{k-1}) = 0, \quad \text{for } k = 1, 2, ..., n \tag{1}$$

$$i_k - i_{k+1} - u_k(G + sC) = 0, \quad \text{for } k = 1, ..., n \tag{2}$$

$$i_{n+1} = \frac{u_n}{R_0} \tag{3}$$

where u_0 is a given value, and u_n is an output voltage. Evidently, the values u_k and i_k are the functions of the complex variable s.

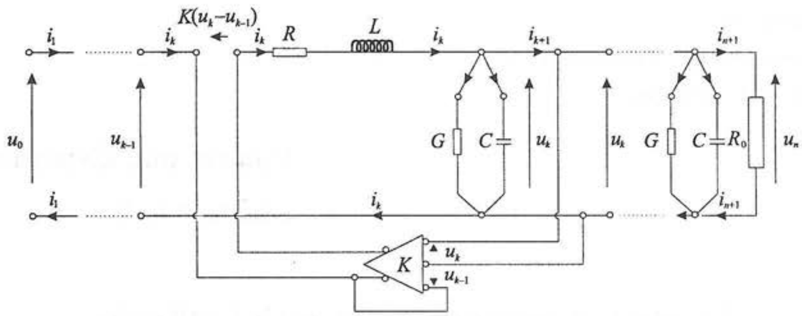

Figure 1.

2. The transfer function

By putting $k+1$ in place of k in the equation (1) we obtain

$$(u_{k+1}-u_k)+i_{k+1}(R+sL)+K(u_{k+1}-u_k)=0,\ \text{for } k=1,2,...,n-1 \quad (4)$$

Upon eliminating currents $i_k - i_{k+1}$ from equation (2) and using equations (1), (3) and (4), we obtain the following relation between the voltages:

$$-u_{k-1}+A(s)u_k-u_{k+1}=0, \quad k=0,1,2,...,n-1 \tag{5}$$

$$-u_{n-1}+\left[A(s)-1+\frac{R+sL}{R_0(1+K)}\right]u_n=0, \quad k=n \tag{6}$$

where

$$A(s)=\frac{(G+sC)(R+sL)}{1+K}+2\ . \tag{7}$$

For the unloaded chain the resistance $R_0=\infty$, and the current $i_{n+1}=0$. The recurrent relation (4) can be written in the matrix form:

$$\begin{bmatrix} A(s) & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & A(s) & -1 & ... & 0 & 0 & 0 \\ 0 & -1 & A(s) & ... & 0 & 0 & 0 \\ .... & & & & & & \\ 0 & 0 & 0 & ... & A(s) & -1 & 0 \\ 0 & 0 & 0 & ... & -1 & A(s) & -1 \\ 0 & 0 & 0 & ... & 0 & -1 & B(s) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \\ 0 \\ ... \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

where:

$$B=A(s)-1+\frac{R+sL}{R_0(1+K)}\ . \tag{9}$$

Let $T_n(s)$ denote the $n\times n$ matrix presented by (8).

The determinant

$$M_n(s) = \det\, T_n(s) \tag{10}$$

is the polynomial of the $2n$-th degree.
Applying Cramers method to the matrix equation (8) we find that

$$u_k(s) = \frac{M_{n-k}(s)}{M_n(s)} u_0(s) \quad \text{for } k = 1, 2, ..., n \tag{11}$$

(which may be proved by inspection), where

$$M_0(s) \stackrel{def}{=} 1,\ M_1 = B(s),\ M_2(s) = A(s) \cdot B(s) - 1\ . \tag{12}$$

The transfer function of the chain is equal to

$$H(s) = \frac{u_n(s)}{u_0(s)} = \frac{1}{M_n(s)} \tag{13}$$

3. The poles of the transfer function

It can be observed that for the determinants of the matrix $T_n(s)$ the following recurrent relation is fulfilled

$$M_n(s) - A(s)M_{n-1}(s) + M_{n-2}(s) = 0, \quad n = 2, 3, 4, ... \tag{14}$$

The polynomial $M_n(s)$ satisfying equation (14) can be presented as a function of the index n in the following form

$$M_n(s) = c_1 r_1^n(s) + c_2 r_2^n(s)\ , \tag{15}$$

where $r_1(s)$ and $r_2(s)$ are the roots of the characteristic equation

$$r^2 - A(s)r + 1 = 0\ , \tag{16}$$

and the coefficients $c_1(s)$ and $c_2(s)$ satisfy the initial conditions

$$\left.\begin{array}{l} c_1 + c_2 = M_0(s) = 1 \\ c_1 r_1 + c_2 r_2 = M_1(s) = B(s) \end{array}\right\} . \tag{17}$$

It is convenient to introduce the notation:

$$A(s) = 2\cos\varphi(s)\ . \tag{18}$$

Using (18) we can write the roots of the equation (16) in the form:

$$r_{1,2} = \cos[\varphi(s)] \pm i\ \sin[\varphi(s)]\ . \tag{19}$$

The substitution of (18) and (19) into (17) gives

$$\left.\begin{aligned} c_1 &= \frac{\cos\varphi + i\,\sin\varphi - 1 + \dfrac{R+sL}{R_0(1+K)}}{2i\,\sin\varphi} \\ c_2 &= -\frac{\cos\varphi - i\,\sin\varphi - 1 + \dfrac{R+sL}{R_0(1+K)}}{2i\,\sin\varphi} \end{aligned}\right\} . \tag{20}$$

We can assume $\sin\varphi \neq 0$ because the roots of the equation (16) are different in general. Returning to (15) with substitution (19) and (20) we obtain that

$$M_n = \frac{\sin[(n+1)\varphi(s)] - \sin[n\varphi(s)] + \dfrac{R+sL}{R_0(1+K)}\sin[n\varphi(s)]}{\sin\varphi(s)} . \tag{21}$$

In what follows we will consider the particular case, when the chain is unloaded, i.e. $R_0 = \infty$.

3.1. Unloaded chain: $R_0 = \infty$

In this case the formula (21) has the simpler form:

$$M_{n_0} = \frac{\sin[(n+1)\varphi(s)] - \sin[n\varphi(s)]}{\sin\varphi(s)} = \frac{\cos\left[(2n+1)\dfrac{\varphi(s)}{2}\right]}{\cos\left[\dfrac{\varphi(s)}{2}\right]} . \tag{22}$$

The complex number s is a solution of equation $M_{n_0}(s) = 0$ if and only if there exists an integer $k = 1, 2, ..., n$ such that

$$\varphi(s) = \frac{2k-1}{2n+1}\pi \quad \text{for } k = 1, 2, ..., n . \tag{23}$$

Substituting (23) into (18) and taking into account (7) we obtain the set of equations

$$\begin{gathered} LCs^2 + (GL + RC)s + GR + 4(1+K)\sin^2\frac{2k-1}{2n+1}\frac{\pi}{2} = 0 , \\ \text{for } k = 1, 2, ..., n , \end{gathered} \tag{24}$$

for all roots of the polynomial $M_{n_0}(s)$. For every $k = 1, 2, ..., n$ we have two roots - s_{1k} and s_{2k}.

It is evident that the whole system is always stable for $K \geq -(1 + \frac{1}{4}GR)$. From (13), (22) and (24) we find that the transfer function of the unloaded chain $R_0 = \infty$ is

$$H(s) = \prod_{k=1}^{n} \frac{1+K}{LC(s - s_{1k})(s - s_{2k})} . \tag{25}$$

4. Integral square error

From the Parseval's formula we can calculate the integral

$$J_2 = \int_0^{\infty} \varepsilon^2(t)dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} H(s) \cdot H(-s)ds \tag{26}$$

where $\varepsilon(t)$ is the impulse response of the chain.

4.1. Unloaded chain: $R_0 = \infty$

Since $H(s)H(-s)$ is a rational function, we can apply the method of residues. It is well known that if $N(s)$ and $P(s)$ are analytic functions in a neighbourhood of a point s_* and if $N(s_*) \neq 0$, $P(s_*) = 0$ and $P'(s_*) \neq 0$ then

$$\operatorname*{Res}_{s_*} \left(\frac{N(s)}{P(s)} \right) = \frac{N(s_*)}{P'(s_*)} \tag{27}$$

Let us assume that all the poles of $H(s)$ are single, or, equivalently, that for each $k = 1, 2, .., n$ the equation (24) has two different solutions $s_{1,k} \neq s_{2,k}$. We use formula (27) for $H(s) \cdot H(-s) = N(s)/P(s)$, where

$$N(s) = H(-s) \cos \left[\frac{\varphi(s)}{2} \right] , \tag{28}$$

$$P(s) = \cos \left[(2n+1) \frac{\varphi(s)}{2} \right] , \tag{29}$$

and $\cos[\varphi(s)] = \frac{1}{2}A(s)$. We calculate the derivative:

$$\begin{aligned} P'(s) &= -\frac{2n+1}{2} \sin \left[(2n+1) \frac{\varphi(s)}{2} \right] \frac{d\,\varphi(s)}{d\,s} = \\ &= \frac{d\,A(s)}{d\,s} \frac{(2n+1) \sin \left[\frac{2n+1}{2} \varphi(s) \right]}{4 \sin \varphi(s)} . \end{aligned} \tag{30}$$

We use the equalities $\varphi(s_{1,k}) = \varphi(s_{2,k}) = \frac{2k-1}{2n+1}\pi$. Finally, if the system is stable (for all poles $\mathrm{Res}_n < 0$), we can write

$$\begin{aligned} J_2 &= \sum_{l=1}^{2} \sum_{k=1}^{n} \operatorname*{Res}_{s_{lk}} [H(s)H(-s)] = \\ &= 4\frac{1+K}{2n+1} \sum_{l=1}^{2} \sum_{k=1}^{n} \frac{(-1)^{2k-1} H(-s_{lk})}{2LCs_{lk} + GL + RC} \cdot \sin \frac{2k-1}{2n+1}\pi \cos \frac{2k-1}{2n+1} \frac{\pi}{2} , \end{aligned} \tag{31}$$

provided that $s_{1,k} \neq s_{2,k}$.

4.2. Chain with load R_0

In this case it is not possible to obtain the solution of equation $M_n(s) = 0$ in the closed analytical form.
For calculation of the integral square error it is not necessary to know the roots of polynomial (21).
We present now a method which is described in Górecki (1993) and based on the knowledge of the coefficients of the polynomial $M_n(s)$.
It is easy to prove by induction (see Jeśmanowicz and Łoś, 1972) that

$$\begin{aligned} M_n(s) &= \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2\cos[\varphi(s)])^{n-2k} + \\ &+ \left[\frac{R+sL}{R_0(1+K)} - 1\right] \sum_{k=0}^{[n-1/2]} (-1)^k \binom{n-k-1}{k} (2\cos[\varphi(s)])^{n-2k-1} \end{aligned} \tag{32}$$

where $[x]$ denotes integer of x.
Using relation (18) we can write

$$\begin{aligned} M_n(s) &= \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} A^{n-2k}(s) + \\ &+ \left[\frac{R+sL}{R_0(1+K)} - 1\right] \sum_{k=0}^{[n-1/2]} (-1)^k \binom{n-k-1}{k} A^{n-2k-1}(s) \ . \end{aligned} \tag{33}$$

For calculation of powers of $A^n(s)$ we can use (7) and the well known formula

$$\begin{aligned} A^n(s) &= \left[\frac{LCs^2 + (GL+RC)s + (GR+2+2K)}{(1+K)}\right]^n = \\ &= \frac{1}{(1+K)^n} \sum_{\alpha_1+\alpha_2+\alpha_3=n} \frac{n!}{\alpha_1!\alpha_2!\alpha_3!} Q(s) \ , \end{aligned} \tag{34}$$

where:

$$Q(s) = (LCs^2)^{\alpha_1}[(GL+RC)s]^{\alpha_2}(GR+2+2K)^{\alpha_3}$$

and α_1, α_2, α_3 are nonnegative integers which fulfil the condition $\alpha_1+\alpha_2+\alpha_3 = n$.
Assuming as before that $H(s)$ is a rational function

$$H(s) = \frac{N(s)}{P(s)} \tag{35}$$

where

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \tag{36}$$

$$N(s) = b_1 s^n + b_2 s^{n-1} + \cdots + b_{n-1} s + b_n \ , \tag{37}$$

the coefficients a_i and b_i are real, and a_0 is non-zero, we can calculate the integral square error (26) using a method described in Górecki (1993).
The integral J_2 can be calculated from the formula

$$J_2 = (-1)^{n-1} \frac{\Delta_{n-1}}{a_0 \Delta_n} \ , \tag{38}$$

where

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 & 0 & 0 \\ \dots . & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & a_n & a_{n-1} & a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_n \end{vmatrix} \tag{39}$$

is the Hurwitz determinant, which for a stable system is positive and

$$\Delta_{n-1} = \begin{vmatrix} N_1 & a_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ N_3 & a_2 & a_1 & a_0 & \dots & 0 & 0 & 0 \\ N_5 & a_4 & a_3 & a_2 & \dots & 0 & 0 & 0 \\ \dots . & & & & & & & \\ N_{n-1} & 0 & 0 & 0 & \dots & a_n & a_{n-1} & a_{n-2} \\ N_n & 0 & 0 & 0 & \dots & 0 & 0 & a_n \end{vmatrix} \tag{40}$$

$$N_r = b_1 b_{2r-1} - b_2 b_{2r-2} + \cdots + (-1)^n b_r b_{r+1} + (-1)^{r+1} \frac{b_r^2}{2} \qquad \text{for } r = 1, 2, ..., n \ . \tag{41}$$

Remark 1
In our case the formula (37) is very simple

$$N(s) = 1, \ \text{because } b_1 = b_2 = ... = b_{n-1} = 0 \text{ and } \ b_n = 1 \ .$$

For that reason

$$N_1 = N_3 = ... = N_{n-1} = 0 \text{ and } N_n = (-1)^{n+1} \frac{1}{2} \ .$$

According to this the first column of the determinant (40) is equal $(0, 0, ..., 0, 1)^T$.
Remark 2
In a very special case with the adjusted load impedance, when this impedance is equal

$$R_0 + sL_0 = R + sL$$

we obtain from formula (9) that

$$B(s) = A(s) \ .$$

According to this, formula (20) takes the form

$$c_1 = -\frac{\cos\varphi + i\sin\varphi}{-2i\sin\varphi}$$

$$c_2 = \frac{\cos\varphi - i\sin\varphi}{-2i\sin\varphi}$$

and formula (21):

$$M_n = -\frac{\sin[(n+1)\varphi(s)]}{\sin[\varphi(s)]}$$

$$\varphi(s) = \frac{k\pi}{N+1}, \qquad k = 1, 2, ..., n$$

with obvious further consequences.

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