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Professor Jakub Gutenbaum
on his 70th birthday

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Uniform stabilization of the quasi-linear Kirchhoff wave equation with a nonlinear boundary feedback[1]

by

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Abstract: An n-dimensional quasi-linear wave equation defined on bounded domain Ω with Neumann boundary conditions imposed on the boundary Γ and with a *nonlinear boundary feedback* acting on a portion of the boundary $\Gamma_1 \subset \Gamma$ is considered. Global existence, uniqueness and uniform decay rates are established for the model, under the assumption that the $H^1(\Omega) \times L_2(\Omega)$ norms of the initial data are sufficiently small. The result presented in this paper extends these obtained recently in Lasiecka and Ong (1999), where the Dirichlet boundary conditions are imposed on the *uncontrolled* portion of the boundary $\Gamma_0 = \Gamma \setminus \overline{\Gamma_1}$, and the two portions of the boundary are assumed disjoint, i.e. $\overline{\Gamma_1} \cap \overline{\Gamma_0} = \emptyset$. The goal of this paper is to remove this restriction. This is achieved by considering the "pure" Neumann problem subject to convexity assumption imposed on Γ_0.

Keywords: quasilinear Kirchhoff wave equation, global existence, a priori bounds, nonlinear damping, uniform decay rates.

1. Introduction

1.1. The model

Let Ω be a bounded, open domain in R^n with a smooth boundary Γ which consists of two parts: $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_0}$, where Γ_1, Γ_0 are open sets and Γ_1 is nonempty.

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The following model of the quasi-linear Kirchhoff wave equation with a boundary nonlinear feedback is considered:

$$\begin{aligned} &u_{tt} = (1+\alpha\int_\Omega |\nabla u|^2 d\Omega)\Delta u \quad \text{on} \quad \Omega\times(0,\infty) \\ &\frac{\partial}{\partial\nu}u = 0 \quad \text{on} \quad \Gamma_0\times(0,\infty) \\ &\frac{\partial}{\partial\nu}u = f(u_t,u) \quad \text{on} \quad \Gamma_1\times(0,\infty) \\ &u(0,\cdot)=u_0,\ u_t(0,\cdot)=u_1 \text{ in } \Omega \end{aligned} \tag{1}$$

Here the parameter α is positive and the nonlinear function $f(u_t,u)$ represents a boundary feedback control for the system. The vector ν denotes the unit outward normal to the boundary.

The Kirchhoff wave equation (1) provides a model well known in the literature, describing nonlinear oscillation of a stretched membrane where the nonlinear strain-displacement relation is accounted for. It was first introduced by Kirchhoff (1876) and it has been studied since then by many authors (see a very informative survey article of Arosio, 1993, for an account of results and references).

The main goal of this paper is to study global existence, uniqueness and uniform stabilization of solutions to (1) with an appropriate choice of feedback control $f(u_t,u)\in L_2((0,\infty)\times\Gamma_1)$.

Quasi-linear wave equations (and their abstract counterparts) have been the subject of intense studies, see Assila (2000), Arosio (1993), Arosio and Garavaldi (1991), Arosio and Spagnolo (1986), Brito (1982), Ehibara, Medeiros and Miranda (1986), Menzala (1979), Nishihara (1997), Yamada (1982), and a long list of references therein. The interesting feature in these problems (from the mathematical point of view) is the "discrepancy" between *weak* solutions ($u(t)\in H^1(\Omega), u_t(t)\in L_2(\Omega)$) and *regular* solutions ($u(t)\in H^2(\Omega), u_t(t)\in H^1(\Omega)$). Indeed, while global a priori bounds for *weak* solutions are easily obtainable by elementary energy methods, there is until now no general existence result for weak solutions. On the other hand, it is relatively straightforward to establish *local in time* existence of *regular* solutions. However, the problem is that of a priori bounds in higher norms. In fact, up to now, there are no general global existence results available for this type of quasi-linear problems. Without getting involved into a thorough review of the literature on this topic, which is very vast (see Arosio, 1993), we just mention that most of the results available in the literature assume existence of a *linear interior damping* in the model, along with smallness of the norms for the initial data. This raises an interesting question whether some other forms of a weaker and possibly nonlinear damping will produce the same effect. The first contribution in this direction is Tucsnak (1982) where a *one dimensional* model with *boundary linear damping* is considered. In this case, the author was able to obtain (i) global solvability of regular solutions for sufficiently small initial data and (ii) exponential decay rates for the weak energy function. Subsequently, the results in Tucsnak (1982) were generalized to n-dimensional models, but still with *linear* boundary damping and geometric "star shaped" conditions imposed on the domain Ω (Ong, 1997, Miranda and San Gil Jutuca, 1999). Regarding the case of a nonlinear (linearly bounded at the origin) dissipation, a rather recent paper Kouemou-Patcheu (1997) treats an *interior* but *nonlinear* dissipation. The dissipation considered in Kouemou-Patcheu (1997) is assumed to have a linear bound (from below) at the origin. The results obtained in Kouemou-Patcheu (1997) are: (i) global existence of regular solutions for small initial data and (ii) exponential decay rates for a weak energy function.

The case of a *nonlinear boundary damping* $f(u_t, u) \equiv -g(u_t)$ was recently treated in Lasiecka and Ong (1999). In Lasiecka and Ong (1999) it is assumed that the function $g(\cdot) \in C^1(R)$ is strictly increasing, zero at the origin and bounded linearly at infinity. A superlinear growth of $g(s)$ is allowed at the origin, which, in turn, provides uniform decay rates for solutions which are not exponential (polynomial, logarithmic, etc. – depending on behavior of $g(s)$ at zero). The results obtained in Lasiecka and Ong (1999) include: global well-posedness for the $H^2(\Omega) \times H^1(\Omega)$ solutions with "small" initial energy, uniform decay rates in $H^1(\Omega) \times L_2(\Omega)$ and $H^2(\Omega) \times H^1(\Omega)$ topologies. The model considered in Lasiecka and Ong (1999) assumes the Dirichlet boundary conditions imposed on the *uncontrolled* portion of the boundary. This forces, due to potential elliptic singularities, the assumption that the two portions of the boundary Γ_0 and Γ_1 be disjoint. On the other hand, it is well known that in many applications (e.g. structural acoustic problems), the requirement that the domain is not simply connected may be unrealistic. This motivates our present paper, whose goal is to reconsider the issue *without the assumption* that the two portions of the boundary are *disjoint*.

To achieve this, we impose the Neumann boundary conditions over the entire boundary, including the *uncontrolled* part. This, in turn, leads to an array of new technical difficulties. Indeed, it is well known that boundary controllability/stabilization of *linear waves* with *uncontrolled Neumann part* of the boundary, is an outstanding problem (see Lasiecka, Triggiani and Zhang, 2000, Isakov and Yamada, 2000, for recent results in this direction). Moreover, the Neumann boundary conditions prescribed on the full boundary with the feedback acting on velocity of the displacement only leads to unstable dynamics. Indeed, this is due to the fact that the spectrum of the linearized generator has unstable zero eigenvalue. To cope with the issue, it is customary to introduce a feedback control acting on the position as well. This motivates the following structure for the boundary feedback to be considered in this paper. For

$(x,t) \in \Gamma_1 \times (0,\infty)$ we set:

$$f(u_t(t,x),u(t,x)) \equiv -g(u_t(t,x)) - d(x)(1+\alpha\int_\Omega |\nabla u(t,x)|^2 dx)^{-1}u(t,x) \qquad (2)$$

where a nonlinear function $g(\cdot) \in C^1(R)$ is assumed strictly increasing and zero at the origin. The function $d(x)$ is continuous, nonnegative and such that *supp* $d \subset \Gamma_1$, *supp* $d \neq \emptyset$.

Our main results provide well-posedness and uniform decay rates for the "weak" and "strong" energies to quasi-linear Kirchhoff wave equation, subject to an additional convexity assumption imposed on the *uncontrolled* part of the boundary. It should be interesting to note that the role of the feedback control in this problem is two-fold. First, it guarantees global existence of solutions. Second, it provides uniform decay rates for the energy function. The presence of a non-negative function $d(x)$ in (2) is necessary to secure strong stability.

We finally mention that the results obtained here rely critically on a new unique continuation theorem, Tataru (1995), Hörmander (1997), for the wave equation with nonsmooth time dependent coefficients in the principal part.

1.2. Formulation of the results

We shall use the following notation

$$|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)}; \quad (u,v)_\Omega \equiv (u,v)_{L_2(\Omega}; \ \langle u,v\rangle_\Gamma \equiv (u,v)_{L_2(\Gamma)}.$$

$H^s(\Omega)$ stands for the usual Sobolev spaces when $s \geq 0$. For $s < 0$, $H^s(\Omega) \equiv [H^s(\Omega)]'$, where duality is with respect to pivot space $L_2(\Omega)$. The same notation will apply with Ω replaced by Γ. We note that the notation for Sobolev spaces H^s with $s < 0$ is not a standard one. Indeed, $H^s(\Omega), s < 0$, denotes, typically, (see Lions and Magenes, 1972) the dual spaces to $H_0^{-s}(\Omega)$.

The constant C will always denote a generic constant different in various circumstances. The symbol $C(s)$ will denote a function of s which is bounded for the bounded values of the arguments.

In order to formulate our results we introduce the following energy functions.

$$E_{0,u}(t) \equiv |u_t(t)|^2_{0,\Omega} + \frac{\alpha}{2}|\nabla u(t)|^4_{0,\Omega} + |\nabla u(t)|^2_{0,\Omega} + |\sqrt{d}u(t)|^2_{0,\Gamma_1}$$

$$E_{1,u,w}(t) \equiv |w_t(t)|^2_{0,\Omega} + (1+\alpha|\nabla u(t)|^2_{0,\Omega})|\nabla w(t)|^2_{0,\Omega} + |\sqrt{d}w(t)|^2_{0,\Gamma_1}$$

$$E_{1,u_0,u_1} \equiv (1+\alpha|\nabla u_0|^2_{0,\Omega})|\nabla u_1|^2_{0,\Omega} + (1+\alpha|\nabla u_0|^2_{0,\Omega})^2|\Delta u_0|^2_{0,\Omega} + |\sqrt{d}u_1|^2_{0,\Gamma_1}.$$

We note that when u is a regular solution to (1) with sufficiently regular initial data, which satisfy compatibility conditions on the boundary (see below), then:

$$E_{1,u,u_t}(0) = E_{1,u_0,u_1}.$$

Since the support of d is nonempty (in Γ_1), the boundedness of $E_{0,u}(t)$ (resp $E_{1,u,w}(t)$) implies the boundedness of $|u(t)|_{H^1(\Omega)}$ (resp. $|w(t)|_{H^1(\Omega)}$).

In order to present our results, we shall formulate the following Assumptions:

- Geometric condition valid on a portion of the boundary Γ_0 which is *not* subject to dissipation (Γ_0 may be empty, in which case there is no assumption):
ASSUMPTION 1 $(x-x_0)\cdot\nu \le 0$ *on* Γ_0 *where* $x_0 \in R^n$ *and* Γ_0 *is convex.*

- "Smallness" hypothesis imposed on the initial data:
ASSUMPTION 2 $E_{0,u}(0)E_{1,u,u_t}(0) \le \rho$, *where the constant* ρ *is sufficiently small.*
- Growth condition imposed on the nonlinear dissipation g:
ASSUMPTION 3 $m \le g'(s) \le M;\ |s| > 0,\ \ 0 < m \le M$.

THEOREM 1 **(Global existence and exponential decay rates)** *Consider equation (1) with (2) and any initial data such that* $u_0 \in H^2(\Omega); u_1 \in H^1(\Omega)$ *subject to compatibility conditions:* $\frac{\partial}{\partial\nu}u_0 = 0$ *on* Γ_0; $\frac{\partial}{\partial\nu}u_0 = -g(u_1) - d[1 + \alpha|\nabla u(0)|^2_{0,\Omega}]^{-1}$ *on* Γ_1. *The boundary* Γ_0 *is subject to Assumption 1 and the nonlinear function* $g(s)$ *satisfies Assumption 3. Then:*

1. *there exist* $\rho > 0$ *(depending on* Ω, α, m, M *) such that for all initial data subject to Assumption 2 there exist a unique, global, regular solution*
$$u \in C(0,\infty; H^2(\Omega)), u_t \in C(0,\infty; H^1(\Omega))$$
with $E_{1,u,u_t}(t) \le CE_{1,u_0,u_1}$.
2. *The following decay rates for the energy function* $E_{0,u}(t)$ *hold*
$$E_{0,u}(t) \le Ce^{-\omega t}E_{0,u}(0) \tag{3}$$
with $\omega > 0$, *possibly dependent on* $E_{0,u}(0), E_{1,u_0,u_1}$.

REMARK 1 *If we assume, in addition, that* Ω *is "star shaped", i.e.* $(x-x_0)\cdot\nu \ge 0$ *on* Γ_1 *where* x_0 *belongs to the hyperplane containing* Γ_0 *with* Γ_0 *being flat surface, and we replace the nonlinear feedback* $g(u_t)$ *by* $g(u_t)(x-x_0)\cdot\nu$, *then the decay rates given by (3) hold true with the constants* C, ω *depending only on the lower level of energy, i.e. only on* $E_{0,u}(0)$ *(and, of course, on* Ω, α, m, M*).*

REMARK 2 *The linear bounds for the function* $g(s)$ *at infinity are typical for all problems related to boundary stabilization of linear hyperbolic equations such as waves and plates (see Horn, 1992, Komornik, 1994, Lasiecka and Tataru, 1993, Lasiecka and Triggiani, 1998, Lions, 1988, Lagnese and Lions, 1988, and references therein). Thus, it is not reasonable to expect that the results of this paper can be easily generalized to superlinear (at infinity) boundary damping. This is in contrast with interior dissipation where the superlinearity of the damping can be handled by Sobolev's embeddings (Komornik, 1994, Kouemou-Patcheu, 1997).*

REMARK 3 *The case of a superlinear growth of* $g(s)$ *at the origin has been treated in Lasiecka and Ong (1999) for the Dirichlet boundary conditions on* Γ_0. *These results require some additional restrictions imposed on the geometry of the domain. While the extensions of these results to the superlinear case for the Neumann problem considered here are possible, the analysis in the Neumann* *uncontrolled case is more subtle and careful attention should be paid in carrying out the details.*

REMARK 4 *The role of the parameter $d(x)$ in equation (2) is to control the steady states. For this reason it is assumed that the support of d in Γ_1 is nonempty.*

Finally, we shall discuss the decay rates for higher norms of solutions u. The following result describes the situation.

THEOREM 2 (**Decay rates for** $E_{1,u,u_t}(t)$) *Under the Assumptions of Theorem 1, there exist $\rho > 0$, depending on α, Ω, m, M, such that for all initial data subject to Assumption 2 we have:*

$$E_{1,u,u_t}(t) \le Ce^{-\omega t} E_{1,u_0,u_1}$$

where the constant $\omega > 0$ depends on the size of $E_{0,u}(0)$ (but not on E_{1,u_0,u_1}).

REMARK 5 *We note that the results of Theorems 1 and 2 do not require any geometric conditions to be satisfied on the "dissipative" portion of the boundary. Thus, there is no need for geometric constraints whenever the dissipation is active. This is in agreement with physical intuition.*

REMARK 6 *One could consider more general forms of quasi-linear terms in equation (1) (like, e.g., in Arosio and Garavaldi, 1991, as long as there is no degeneracy). The appropriate modifications of the arguments should be reasonably straightforward.*

The remainder of this paper is devoted to the proofs of the main results.

2. Preliminaries

We begin with local existence and uniqueness result available for system (1).

PROPOSITION 1 *Let us assume that the initial data satisfy the regularity and compatibility requirements of Theorem 1. Then, there exists a unique solution*

$$u \in C(0, T_0; H^2(\Omega)), u_t \in C(0, T_0; H^1(\Omega)); u_{tt} \in C(0, T_0; L_2(\Omega))$$

for some value of $T_0 > 0$.

The proof of this Proposition follows the same line of arguments as in Lasiecka and Ong (1999), hence it is omitted.
We introduce the following notation

$$b(t) \equiv \alpha|\nabla u(t)|^2_{0,\Omega} + 1$$

The following energy estimate is standard for this problem:

LEMMA 1 *Let u be a regular solution to (1) defined on* $(0,t)$. *Then*

$$E_{0,u}(t) + 2\int_s^t \int_{\Gamma_1} b(t)g(u_t)u_t d\Gamma_1 dz = E_{0,u}(s);\ 0 \le s \le t \tag{4}$$

Proof: Equality in (4) is obtained by multiplying equation (1) by u_t, integrating over $\Omega\times(s,t)$ and applying Green's identity. Since the solutions display sufficient regularity, the "calculus" here is classical.

Let u be a regular (local) solution corresponding to the original problem (1). Denote $w \equiv u_t$. Then w satisfies the following linear equation

$$w_{tt} = b(t)\Delta w + b'(t)\Delta u = b(t)\Delta w + \frac{b'(t)}{b(t)} w_t \quad \text{on} \quad \Omega\times(0,\infty)$$
$$\frac{\partial}{\partial\nu} w = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty)$$
$$\frac{\partial}{\partial\nu} w = -g'(u_t)w_t - db(t)^{-1}w + db'(t)b(t)^{-2}u \quad \text{on} \quad \Gamma_1\times(0,\infty)$$
$$w(t=0) = u_1,\ w_t(t=0) = b(0)\Delta u_0 \text{ in } \Omega \tag{5}$$

The second energy inequality deals with the solution w:

LEMMA 2 *Let u be a regular solution of (1) and $w = u_t$. Then*

$$E_{1,u,w}(t) + 2\int_s^t b(z)\int_{\Gamma_1} g'(u_t)w_t^2 d\Gamma_1 dz \le E_{1,u,w}(s)$$
$$+ C_{E_{0,u}(0)}\int_s^t |b'(z)|E_{1,u,w}(z)dz \tag{6}$$

$$|E_{1,u,w}(t) - E_{1,u,w}(s)| \le C_{E_{0,u}(0)}[\int_s^t b(z)\int_{\Gamma_1} g'(u_t)w_t^2 d\Gamma_1 dz$$
$$+ \int_s^t |b'(z)|E_{1,u,w}(z)dz] \tag{7}$$

Proof. We multiply equation (5) by w_t and integrate over $\Omega\times(s,t)$. By applying Green's identities and noting that $b(t)\Delta u = u_{tt} = w_t$, we obtain:

$$E_{1,u,w}(t) + 2\int_s^t b(z)\int_{\Gamma_1} g'(u_t)w_t^2 d\Gamma_1 dz = E_{1,u,w}(s)$$
$$+ \int_s^t b'(z)[|\nabla w|_{0,\Omega}^2 + 2(\Delta u, w_t)_\Omega + b(z)^{-1}\langle du, w_t\rangle_{\Gamma_1}]dz = E_{1,u,w}(s)$$
$$+ \int_s^t b'(z)[|\nabla w|_{0,\Omega}^2 + 2\frac{1}{b(z)}|w_t|_{0,\Omega}^2 + b(z)^{-1}\langle du, w_t\rangle_{\Gamma_1}]dz$$
$$= E_{1,u,w}(s) + \int_s^t \frac{b'(z)}{b(z)}[b(z)|\nabla w|_{0,\Omega}^2 + 2|w_t|_{0,\Omega}^2 + b(z)^{-1}\langle du, w_t\rangle_{\Gamma_1}]dz \tag{8}$$

Taking the advantage of coercivity Assumption 3 and the Trace Theorem we obtain:

$$|\langle du, w_t\rangle_{\Gamma_1}| \le \epsilon \int_{\Gamma_1} g'(u_t) w_t^2 d\Gamma_1 + C_\epsilon |u|^2_{1,\Omega} \tag{9}$$

Our next step is to estimate the H^1 norm of u in terms of energy E_{1,u,u_t}. This can be done by viewing the original equation as an elliptic problem (with a forcing term u_{tt}) and applying the standard energy method. This procedure gives the following equality:

$$b(t)|\nabla u(t)|^2_{0,\Omega} + |\sqrt{d}u(t)|^2_{0,\Gamma_1} + b(t)\langle g(u_t(t), u(t)\rangle_{\Gamma_1} = (u_{tt}(t), u(t))_\Omega \tag{10}$$

Noting that by (4) $1 \le b(t) \le 1 + \alpha E_{0,u}(0)$, and applying Assumption 3, Trace Theorem and Sobolev's embeddings, we obtain

$$\begin{aligned} &|u(t)|^2_{1,\Omega} \le \epsilon[|u(t)|^2_{0,\Omega} + |u(t)|^2_{0,\Gamma_1}] + C_{\epsilon,E_{0,u}(0)}[|u_{tt}(t)|^2_{0,\Omega} + |u_t(t)|^2_{1,\Omega}] \\ &\le \epsilon[|u(t)|^2_{1,\Omega} + C_{\epsilon,E_{0,u}(0)} E_{1.u.u_t}(t) \end{aligned} \tag{11}$$

Taking $\epsilon < 1/2$ provides the desired estimate for the H^1 norm of u

$$|u(t)|^2_{1,\Omega} \le C_{E_{0,u}(0)} E_{1.u.u_t}(t) \tag{12}$$

Collecting inequalities (9) and (12) yields

$$|\langle du(t), w_t(t)\rangle_{\Gamma_1}| \le \epsilon \int_{\Gamma_1} g'(u_t(t)) w_t^2(t) d\Gamma_1 + C_{\epsilon,E_{0,u}(0)} E_{1.u.u_t}(t) \tag{13}$$

Inequality (13) when combined with (8) produces (after selecting ϵ suitably small) the relation in Lemma 2. ■

The estimate in Lemma 2 gives the basic energy relation for the strong solutions. The main task in proving Theorem 1 is to show an a priori bound for the energy function $E_{1,u,w}$. This will imply, in a standard way, global existence of regular solutions. Upon inspecting the inequality in Lemma 2, it becomes clear that this task amounts to establishing a relation between the boundary energy represented by the integral term on the left of (6) and the interior energy represented by the integral term on the right of (7). The idea (motivated by the experience in a study of boundary stabilization, see, Bardos, Lebeau and Rauch, 1992, Komornik, 1994, Lagnese, 1989, Lasiecka and Triggiani, 1998, Lions, 1988) is to show that the boundary damping "controls" a potential increase of internal energy represented by the integral term on the right of (7). This is achieved by a combination of multipliers and microlocal analysis estimates. Details of this argument are the same as given in Section 3 of Lasiecka and Ong (1999), and here we just present the final inequality.

LEMMA 3 *Let u be a regular solution of (1) defined on $(0,T)$ and $w = u_t$. Then, $\forall 0 < t < T$*

$$\int_0^t E_{1,u,w}(z)dz \le CE_{1,u,w}(0) + C\int_0^t [|b'(z)| + |b'(z)|^2] E_{1,u,w}(z)dz +$$

$$C(E_{0,u}(0)+1)\int_0^t [|w_t|^2_{0,\Gamma_1} + |\frac{\partial}{\partial\nu}w|^2_{0,\Gamma_1}]]dz + C\int_0^t E_{0,u}(z)dz \tag{14}$$

where the constants do not depend on T.

We shall show that Lemma 2 and Lemma 3 imply the following estimate

LEMMA 4

$$\int_0^t E_{1,u,w}(z)dz + E_{1,u,w}(t) \leq CE_{1,u,w}(0)$$
$$+C_{E_{0,u}(0)}[\int_0^t [|b'(z)| + |b'(z)|^2]E_{1,u,w}(z)ds + \int_0^t E_{0,u}(z)dz] \tag{15}$$

Proof. The result of Lemma 4 follows from the estimates in Lemma 2 and Lemma 3, once we provide an appropriate upper-bound for the boundary terms in (14). By reading off the boundary conditions in the equation for w, recalling Assumption 3 and applying the estimate in (12) along with the Trace Theorem, we obtain:

$$\int_0^t [|w_t|^2_{0,\Gamma_1} + |\frac{\partial}{\partial\nu}w|^2_{0,\Gamma_1}]]dz$$
$$\leq C\int_0^t [\langle g'(u_t)w_t, w_t\rangle_{\Gamma_1} + |w|^2_{0,\Gamma_1} + |b'(z)|^2|u|^2_{0,\Gamma_1}]dz$$
$$\leq C_{E_{0,u}(0)}\int_0^t [\langle g'(u_t)w_t, w_t\rangle_{\Gamma_1} + |w|^2_{0,\Gamma_1} + |b'(z)|^2 E_{1,u,u_t}(z)]dz \tag{16}$$

The estimate for $|w|_{0,\Gamma_1}$ is obtained from the original equation and energy dissipation relation (4). Indeed,

$$\int_0^t |w|^2_{0,\Gamma_1}dz = \int_0^t |u_t|^2_{0,\Gamma_1}dz \leq C\int_0^t \langle g(u_t), u_t\rangle_{\Gamma_1}dz \leq CE_{0,u}(0) \tag{17}$$

The two last estimates when combined yield

$$\int_0^t [|w_t|^2_{0,\Gamma_1} + |\frac{\partial}{\partial\nu}w|^2_{0,\Gamma_1}]]dz$$
$$\leq C_{(E_{0,u}(0)}[\int_0^t [\langle g'(u_t)w_t, w_t\rangle_{\Gamma_1} + |b'(z)|^2 E_{1,u,u_t}(z)]dz + E_{0,u}(0)] \tag{18}$$

The results of Lemma 2, Lemma 3 and inequality (18) provide the final conclusion in Lemma 4. ■

The inequality in Lemma 4 can be reiterated on any subinterval (s,t)

$$\int_s^t E_{1,u,w}(z)dz + E_{1,u,w}(t) \leq CE_{1,u,w}(s)+$$
$$C_{E_{0,u}(0)}[\int_s^t [|b'(z)| + |b'(z)|^2]E_{1,u,w}(z)ds + \int_s^t E_{0,u}(z)dz] \tag{19}$$

By using the inequality in (19) together with the "energy barrier" argument presented in Section 4 of Lasiecka and Ong (1999), one obtains the following a priori bounds valid for the higher level energy function E_{1,u,u_t}.

LEMMA 5 *Let u be a regular solution of (1) defined on $(0,T)$. Assume that Assumption 3 and Assumption 1 are in force.*

There exists a $\rho > 0$, depending on α, Ω, m, M, such that for all initial data complying with Assumption 2 we obtain for all $0 \le t \le T$

$$|u(t)|_{2,\Omega} + |u_t(t)|_{1,\Omega} + |u_{tt}(t)|_{0,\Omega} \le C(|u_0|_{2,\Omega}, |u_1|_{1,\Omega}) \tag{20}$$

where the constant C does not depend on T.

Lemma 5 together with Proposition 1 imply the result stated in the first part of Theorem 1.

3. Uniform decay rates for the u -Proof of part 2 in Theorem 1

We begin with establishing decay rates for solutions of our original problem. The key Lemma in this direction is the following inequality:

LEMMA 6 *Assume that Assumption 3 and Assumption 1 are satisfied. There exists $T > 0$, sufficiently large (depending on α, Ω) such that any regular solution u corresponding to (1) and defined on $(0,T]$ satisfies the inequality.*

$$p_T E_{0,u}(T) + E_{0,u}(T) \le E_{0,u}(0) \tag{21}$$

where the constant $p_T \equiv C(sup_{t\in[0,T]} E_{1,u.u_t}(t) + 1)$ and C is a continuous function depending on Ω, α, M, m.

COROLLARY 1 *Under the Assumptions of Lemma 5 the decay rates proclaimed in Theorem 1 are valid for all solutions to (1) of bounded energy E_{1,u,u_t} and for all $t \ge 0$.*

Proof: The result of this Corollary follows at once from Lemma 5, Lemma 6 and Lemma 3.1 in Lasiecka and Tataru (1993). Indeed, the solution u can be continued for all times and by a priori bounds in Lemma 5 we obtain that the constant p_T can be made independent on T (it will depend only on E_{1,u_0,u_1}). This allows to reiterate the inequality in (21) on each subinterval $(mT, (m+1)T)$ to obtain

$$p(E_{0,u}((m+1)T)) + E_{0,u}((m+1)T) \le E_{0,u}(mT); \quad m = 0, 1 \ldots \tag{22}$$

The application of Lemma 3.3 in Lasiecka and Tataru (1993) provides the desired exponential decay rates stated in the second part of Theorem 1. ■

To complete the argument, it suffices to establish the validity of Lemma 6.

Proof of Lemma 6

STEP 1: We begin with boundary observability estimate obtained by the multipliers method applied to the original equation (1).

PROPOSITION 2 *Let u be a regular solution of (1) defined on* $(0, T)$. *Then*

$$\int_0^T E_{0,u}(z)dz \leq CE_{0,u}(T)$$
$$+C(E_{0,u}(0)+1)\int_0^T \left[|u_t|^2_{0,\Gamma_1} + \left|\frac{\partial}{\partial\nu}u\right|^2_{0,\Gamma_1} + \left|\frac{\partial}{\partial\tau}u\right|^2_{0,\Gamma_1} + |u|^2_{0,\Omega}\right] dz$$

Here $\frac{\partial}{\partial\tau}$ *denotes the tangential derivatives on the boundary* Γ_1. *The constant* C *depends on intrinsic quantities in the equation such as* Ω *and* α.

Proof. Follows by applying new multipliers $h \cdot \nabla u$ and $div\ hu$, to the original equation (1), where the vector field $h \in C^2(\Omega)$ has the following properties:

1. $h(x) \cdot \nu = 0,\ on \Gamma_0$
2. Jacobian (h) $\equiv J(h) \geq c_0 > 0$ in Ω.

A construction of the vector field with the above properties, under the conditions stated in Assumption 1 is given in Lasiecka and Lebiedzik (2000) (see also Lasiecka, Triggiani and Zhang, 2000, for more general domains).

Application of the first multiplier gives

$$\int_0^T [(u_t^2, divh)_\Omega - b(z)(|\nabla u|^2, divh)_\Omega + b(z)(\nabla u, J(h)\nabla u)_\Omega]dz$$
$$= (u_t, h\nabla u)_\Omega|_0^T$$
$$-1/2\int_0^T\int_\Gamma [1/2[|u_t|^2 - b(z)|\nabla u|^2]h\cdot\nu + b(z)\frac{\partial}{\partial\nu}uh\nabla u]d\Gamma dz \tag{23}$$

An application of the second multiplier gives:

$$\int_0^T [(u_t^2, divh)_\Omega - b(z)(|\nabla u|^2, divh)_\Omega]dz = (u_t, divhu)_\Omega|_0^T$$
$$-\int_0^t\int_\Gamma b(z)\frac{\partial}{\partial\nu}u\ udivh\ d\Gamma dz + \int_0^t b(z)(\nabla u, u\nabla divh)_\Omega dz \tag{24}$$

Combining the two estimates

$$(c_0-\epsilon)\int_0^T b(z)|\nabla u|^2_{0,\Omega}dz - 1/2\int_0^T\int_{\Gamma_0} b(z)|\nabla u|^2 h\cdot\nu d\Gamma_0 dz$$
$$\leq C[E_{0,u}(0) + E_{0,u}(T)] + \epsilon\int_0^T\int_\Gamma b(z)u^2 d\Gamma dz +$$
$$C_\epsilon\int_0^T b(z)[\int_{\Gamma_1}[|\frac{\partial}{\partial\nu}u|^2 + |u_t|^2 + |\frac{\partial}{\partial\tau}u|^2]d\Gamma_1 + |u|^2_{0,\Omega}]dz \tag{25}$$

Taking ϵ suitably small, applying Trace Theorem, once more the equality in (24), and noting that

$$1 \leq b(t) \leq \alpha E_{0,u}(0) + 1$$

yields:

$$\int_0^T E_{0,u}(z)dz - 1/2\int_0^T \int_{\Gamma_0} |\nabla u|^2 h\cdot \nu d\Gamma_0 dz \leq C[E_{0,u}(0) + E_{0,u}(T)]$$
$$+C(E_{0,u}(0)+1)\int_0^T [\int_{\Gamma_1} [|\frac{\partial}{\partial \nu}u|^2 + |u_t|^2 + |\frac{\partial}{\partial \tau}u|^2]d\Gamma_1 + |u|^2_{0,\Omega}]dz \tag{26}$$

Recalling Assumption 1, and the energy estimate in Lemma 2 implies the result stated in Proposition 2. ■

STEP 2: By estimating the tangential derivatives, in the same manner as in Lasiecka and Ong (1999) (i.e., by applying Lemma 7.1 in Lasiecka and Triggiani, 1998, to the original equation (1)) we obtain the following result. Let u be a regular solution of (1) defined on $(0,T)$. Then

$$\int_0^T E_{0,u}(z)dz \leq CE_{0,u}(0) + C(E_{0,u}(0)+1)\int_0^T |u(z)|^2_{0,\Omega}dz$$
$$+C(E_{0,u}(0)+1)\int_0^T [|u_t|^2_{0,\Gamma_1} + |\frac{\partial}{\partial \nu}u|^2_{0,\Gamma_1} + b(z)\langle g(u_t), u_t\rangle_{\Gamma_1}]dz \tag{27}$$

Using the boundary conditions and energy identity (4) we obtain from (27)

$$TE_{0,u}(T) + \int_0^T E_{0,u}(z)dz \leq CE_{0,u}(T)$$
$$+C(E_{0,u}(0)+1)\int_0^T [|u_t|^2_{0,\Gamma_1} + |g(u_t)|^2_{0,\Gamma_1} + |u|^2_{0,\Omega}]dz \tag{28}$$

One more application of the energy identity (4), *taking T large enough, so that* $T > C$ (this is a point where we need to know that the solutions can be continued for time which is large enough) gives:

$$\int_0^T E_{0,u}(z)dz + E_{0,u}(T) + E_{0,u}(0) \leq$$
$$C(E_{0,u}(0)+1)\int_0^T [|u_t|^2_{0,\Gamma_1} + |g(u_t)|^2_{0,\Gamma_1} + |u|^2_{0,\Omega}]dz \tag{29}$$

STEP 4: Our next step is to absorb the lower order terms in the inequality (29). This is done by the compactness/uniqueness argument. The new (with respect to the literature) features of this step, in the context of the present problem are: (i) due to the presence of the nonlinearity in the equation, the passage through the limit requires the bounds on the higher norms of the initial data; thus, control of $sup_{t\in(0,T)}E_{1,u,u_t}(t)$ is necessary; (ii) since the uniqueness argument will be applied to the equation (5) (which is the wave equation with nonconstant coefficients in the principal part), one has to make sure that the zero overdetermined traces on the boundary Γ_1 imply the nullity of the solution.

This, however, follows from the fact that the coefficients in the (linear) equation are only time dependent with the $C^1(0,T)$ regularity in the principal part and $C^0(0,T)$ regularity in the lower order terms. This is enough to apply the new uniqueness result, Tataru (1995), Theorem 3, and also Hörmander (1997) (Theorem 4.1 and Remark after Corollary 4.7). Thus, our main aim is to prove the following inequality:

PROPOSITION 3 *Let $T > 0$ be sufficiently large (depending on Ω, α). Then for all solutions such that $sup_{t\in(0,T)} E_{1,u,u_t}(t) \leq R$ we have:*

$$\int_0^T |u(z)|^2_{0,\Omega} dz \leq C_{R,T} \int_0^T [|u_t|^2_{0,\Gamma_1} + |g(u_t)^2|_{0,\Gamma_1}]] dz \tag{30}$$

Proof: By contradiction, assume that the inequality in Proposition 3 does not hold. Then, there exists a sequence of initial data $u_{0,n}, u_{1,n}$ and the corresponding solutions $u_n(t)$ of (1) with $E_{1,u_n,u_{nt}}(t) \leq R$ (uniformly bounded in n) over $(0,T)$ such that

$$\frac{\int_0^T |u_n(z)|^2_{0,\Omega} dz}{\int_0^T [|u_{nt}|^2_{0,\Gamma_1} + |g(u_{nt})^2|_{0,\Gamma_1}]] dz} \to \infty \tag{31}$$

This, in particular, implies (on a subsequence denoted by the same symbol)

$$\begin{aligned}
&\int_0^T [|u_{nt}|^2_{0,\Gamma_1} + |g(u_{nt})^2|_{\Gamma_1}]] dz \to 0 \\
&u_n \to u \text{ weakly}^* \text{ in } L_\infty(0,T; H^2(\Omega)) \\
&u_{nt} \to u_t \text{ weakly}^* \text{ in } L_\infty(0,T; H^1(\Omega)) \\
&u_n \to u \text{ strongly in } C(0,T; H^1(\Omega))
\end{aligned} \tag{32}$$

The above convergence allows us to pass to the limit in the original equation (note that it is this point where the strong convergence in $H^1(\Omega)$ is needed and the bound on $E_{1,u_n,u_{nt}}(t)$ is necessary) and leads to the following limit problem:

$$\begin{aligned}
&u_{tt} = (1 + \alpha \int_\Omega |\nabla u|^2 d\Omega) \Delta u \quad \text{on} \quad \Omega \times (0,T) \\
&\frac{\partial}{\partial \nu} u = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) \\
&\frac{\partial}{\partial \nu} u = -b(t)^{-1} du;; \; u_t = 0 \quad \text{on} \quad \Gamma_1 \times (0,T)
\end{aligned} \tag{33}$$

Let $w \equiv u_t$. Then

$$\begin{aligned}
&w_{tt} = b(t)\Delta w + \frac{b'(t)}{b(t)} w_t \quad \text{on} \quad \Omega \times (0,T) \\
&\frac{\partial}{\partial \nu} w = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) \\
&\frac{\partial}{\partial \nu} w = -b'(t) b^{-2}(t) du, \quad w = 0 \quad \text{on} \quad \Gamma_1 \times (0,T)
\end{aligned} \tag{34}$$

where, we recall, $b(t) \equiv \alpha|\nabla u(t)|^2_{0,\Omega}+1$. Since $b \in C^1(0,T), b \geq 1$, T is assumed sufficiently large (depending on Ω, α) and

$$\frac{\partial}{\partial\nu}w = 0; \text{ on } \Gamma_1 \times (0,T) \text{ outside the support of } d$$

the unique continuation result from Hörmander (1997), Tataru (1995) applies and gives $w \equiv 0$.

This, in turn, implies $u \equiv 0$ (here we have used the fact that *supp* $d \neq \emptyset$), and consequently by (32)

$$c_n^2 \equiv \int_0^T |u_n(z)|^2_{0,\Omega}dz \to 0 \tag{35}$$

With $\tilde{u}_n \equiv \frac{u_n}{c_n}$ and recalling (32) we obtain

$$\int_0^T |\tilde{u_n}(z)|^2_{0,\Omega}dz = 1 \quad , \int_0^T [|\tilde{u_{nt}}|^2_{0,\Gamma_1} + \frac{|g(u_{nt})|^2_{0,\Gamma_1}}{c_n^2}]dz \to 0 \tag{36}$$

On the other hand, from (29), after the division by c_n^2, we infer

$$\begin{aligned} &E_{0,\tilde{u}_n}(t) \leq C(E_{0,u_{n0}}(0)+1)[1+\int_0^T [|\tilde{u_{nt}}|^2_{0,\Gamma_1} + \frac{|g(u_{nt})|^2_{0,\Gamma_1}}{c_n^2}]dz] \\ &\leq C(E_{0,u_{n0}}(0)) \end{aligned} \tag{37}$$

The above implies

$$\begin{aligned} &\tilde{u}_n \to \tilde{u} \text{ weakly* in } L_\infty(0,T;H^1(\Omega)); \\ &\tilde{u}_{nt} \to \tilde{u}_t \text{ weakly*} L_\infty(0,T;L_2(\Omega)) \\ &\tilde{u}_n \to \tilde{u} \text{ strongly in } C(0,T;L_2(\Omega)) \end{aligned} \tag{38}$$

Moreover, with $b_n(t) \equiv (1+\alpha\int_\Omega |\nabla u_n|^2 d\Omega)$, $\tilde{u_n}$ satisfies

$$\begin{aligned} &\tilde{u}_{ntt} = b_n(t)\Delta\tilde{u_n} \quad \text{on} \quad \Omega \times (0,T) \\ &\frac{\partial}{\partial\nu}\tilde{u}_n = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) \\ &\frac{\partial}{\partial\nu}\tilde{u}_n = -\frac{g(u_{nt})}{c_n} - db_n(t)^{-1}\tilde{u}_n \quad \text{on} \quad \Gamma_1 \times (0,T) \end{aligned} \tag{39}$$

Now, passing to the limit in (39) (note that $u_n(t) \to 0$ strongly in $H^1(\Omega)$ so that the quasi-linear term in the limit equation disappears, as $b_n(t) \to 1, \ n \to \infty$)) gives

$$\begin{aligned} &\tilde{u}_{tt} = \Delta\tilde{u} \quad \text{on} \quad \Omega \times (0,T) \\ &\frac{\partial}{\partial\nu}\tilde{u} = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) \\ &\frac{\partial}{\partial\nu}\tilde{u} = -d\tilde{u}; \quad \tilde{u}_t = 0 \quad \text{on} \quad \Gamma_1 \times (0,T) \end{aligned} \tag{40}$$

A well known Holmgren's unique continuation argument together with the uniqueness of elliptic equations imply that $\tilde{u} \equiv 0$, contradicting (36) and (38). This completes the proof of the Proposition. ■

STEP 5: Combining the result in Proposition 3 with (29) gives

$$\int_0^T E_{0,u}(z)dz + E_{0,u}(T) \le C_{T,R} \int_0^T [|u_t|^2_{0,\Gamma_1} + |g(u_t)^2|_{0,\Gamma_1}]]dz \tag{41}$$

where $C_T = C_T(E_{0,u}(0), R)$ is defined in Lemma 6 and T is sufficiently large.

STEP 6: Our final step is to obtain the inequality in Lemma 6. This is done by expressing the boundary terms by means of the feedback. Indeed, by applying Assumption 3 we get

$$\int_0^T E_{0,u}(z)dz + E_{0,u}(T) \le C_{T,R} \int_0^T b(t) \int_{\Gamma_1} g(u_t)u_t d\Gamma_1 dz \tag{42}$$

From (42) and energy identity (4) we obtain

$$E_{0,u}(T) \le C_{T,R} \int_0^T b(t) \int_{\Gamma_1} g(u_t)u_t d\Gamma_1 dz \le C_{T,R}[E_{0,u}(0) - E_{0,u}(T)]$$

Hence

$$\frac{E_{0,u}(T)}{C_{T,R}} + E_{0,u}(T) \le E_{0,u}(0)$$

this inequality giving the final result in Part 1 of Lemma 6. ■

4. Exponential decay rates for $w = u_t$ – proof of Theorem 2

Let $T > 0$ be an arbitrary constant to be determined later. Our starting point is inequality (15) in Lemma 4 which gives

$$\begin{aligned}
&\int_0^T E_{1,u,u_t}(z)dz \le CE_{1,u,u_t}(0) \\
&+C_{E_{0,u}(0)} \int_0^T [|b'(z)| + |b'(z)|^2]E_{1,u,u_t}(z)dz + C\int_0^T E_{0,u}(z)dz \\
&\le CE_{1,u,u_t}(0) + C_{E_{0,u}(0)} \int_0^T [E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z) + E^2_{1,u,u_t}(z)E_{0,u}(z)]dz \\
&+C\int_0^T E_{0,u}(z)dz
\end{aligned} \tag{43}$$

An important point is that by using the exponential decay rates already established in Theorem 1

$$\int_0^T E_{0,u}(z)dz \le C[1 - e^{-\omega T}]E_{0,u}(0) \le CE_{0,u}(0) \le CE_{1,u,u_t}(0), \tag{44}$$

where the last inequality follows by the direct comparison of the norms involved, after accounting for compatibility relations satisfied by the initial data.

From (43) and (44) and estimating $E_{1,u,u_t}(0)$ in terms of $E_{1,u,u_t}(T)$ from (7) gives

$$\int_0^T E_{1,u,u_t}(z)dz \le CE_{1,u,u_t}(T) + C_{E_{0,u}(0)}[\int_0^T E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z) + E^2_{1,u,u_t}(z)E_{0,u}(z)dz + \int_0^T |w_t|^2_{0,\Gamma_1}dz] \tag{45}$$

On the other hand from Lemma 2 and Assumption 3 we also obtain the estimates

$$\begin{aligned}&|E_{1,u,u_t}(t) - E_{1,u,u_t}(s)|\\ &\le C_{E_{0,u}(0)}[\int_s^t b(z)|w_t|^2_{0,\Gamma_1}dz + \int_s^t |b'(z)|E_{1,u,u_t}(z)dz]\\ &\le C_{E_{0,u}(0)}[\int_s^t |w_t|^2_{0,\Gamma_1}dz + \int_s^t E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z)]dz\end{aligned} \tag{46}$$

and

$$2m\int_0^T |w_t|^2_{0,\Gamma_1} \le E_{1,u,u_t}(0) - E_{1,u,u_t}(T) + C_{E_{0,u}(0)}\int_0^T E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z)dz \tag{47}$$

From (45) and (46), applied with s replaced by t and t replaced by T

$$\begin{aligned}&1/2\int_0^T E_{1,u,u_t}(z)dz + 1/2TE_{1,u,u_t}(T)\\ &\le CE_{1,u,u_t}(T) + C_{E_{0,u}(0)}[\int_0^T [E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z) + E^2_{1,u,u_t}(z)E_{0,u}(z)]dz\\ &+C\int_0^T\int_t^T E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z)dzdt + [CE_{0,u}(0) + C + T]\int_0^T |w_t|^2_{0,\Gamma_1}dz\end{aligned} \tag{48}$$

By applying (47) we obtain

$$\begin{aligned}&1/2\int_0^T E_{1,u,u_t}(z)dz + 1/2TE_{1,u,u_t}(T) \le CE_{1,u,u_t}(T)\\ &+\frac{1}{2m}[C_{E_{0,u}(0)} + T](E_{1,u,u_t}(0) - E_{1,u,u_t}(T))\\ &+C_{E_{0,u}(0)}[T+1]\int_0^T E^{3/2}_{1,u,u_t}(z)E^{1/2}_{0,u}(z) + E^2_{1,u,u_t}(z)E_{0,u}(z)dz\end{aligned} \tag{49}$$

By using the a priori bounds for $E_{1,u,u_t}(t)$ we obtain the estimate

$$1/2\int_0^T E_{1,u,u_t}(z)dz + 1/2TE_{1,u,u_t}(T) \le CE_{1,u,u_t}(T)$$

$$+(C_{E_{0,u}(0)}+T)(E_{1,u,u_t}(0)-E_{1,u,w}(T))+$$
$$C_{E_{0,u}(0)}[T+1][E^{1/2}_{1,u,u_t}(0)E^{1/2}_{0,u}(0)+E_{1,u,u_t}(0)E_{0,u}(0)]\int_0^T E_{1,u,u_t}(z)dz \quad (50)$$

Since $E_{1,u,u_t}(0)$ is bounded in terms of the initial data, by Assumption 2 we obtain:

$$C_{E_{0,u}(0)}[E^{1/2}_{1,u,u_t}(0)E^{1/2}_{0,u}(0)+E_{1,u,u_t}(0)E_{0,u}(0)]<\rho_1$$

with ρ_1 sufficiently small. (Note that $C_{E_{0,u}(0)}$ is an increasing function of $E_{0,u}(0)$). Hence

$$(1/2-C[T+1]\rho_1)\int_0^T E_{1,u,u_t}(z)dz+1/2TE_{1,u,u_t}(T)$$
$$\leq CE_{1,u,u_t}(T)+[C_{E_{0,u}(0)}+T](E_{1,u,u_t}(0)-E_{1,u,u_t}(T)) \quad (51)$$

Now, selecting ρ_1 sufficiently small, taking T large enough so that $T>4C$ (note that T does not depend on the size of E_{1,u_0,u_1}) yields:

$$1/4TE_{1,u,w}(T)\leq[C_{E_{0,u}(0)}+T](E_{1,u,u_t}(0)-E_{1,u,u_t}(T)) \quad (52)$$

Hence

$$E_{1,u,u_t}(T)\leq\frac{C_{E_{0,u}(0)}+T}{1/4T+C_{E_{0,u}(0)}+T}E_{1,u,u_t}(0)\leq\gamma E_{1,u,u_t}(0);\ \gamma<1 \quad (53)$$

Since $\gamma<1$ and it is independent of $E_{1,u,u_t}(0)$, the inequality in (53) implies uniform decay rates for the energy function $E_{1,u,u_t}(t)$, as required for the conclusion in Theorem 2 ■

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