

*Dedicated to  
Professor Jakub Gutenbaum  
on his 70th birthday*

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## **Lipschitz stability of solutions to parametric optimal control for elliptic equations**

by

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**Abstract:** A family of parameter dependent elliptic optimal control problems with nonlinear boundary control is considered. The control function is subject to amplitude constraints. A characterization of conditions is given under which solutions to the problems exist, are locally unique and Lipschitz continuous in a neighborhood of the reference value of the parameter.

**Keywords:** parametric optimal control, elliptic equation, nonlinear boundary control, Lipschitz stability of the solutions, generalized equations.

### **1. Introduction**

This paper belongs to a series of works, where the local Lipschitz stability of solutions to parametric optimal control problems for nonlinear systems is analyzed. Due to the presence of inequality type constraints, the problems are non-smooth. The main tool in stability analysis for such problems is Robinson's implicit function theorem for generalized equations (Robinson, 1980). Using this theorem it can be shown that a *sufficient* condition of Lipschitz stability for nonlinear systems is that the stationary points of the linear-quadratic accessory problems are Lipschitz continuous with respect to additive perturbations. Usually, this last problem is much easier to investigate than the original one. This approach was applied to parametric mathematical programs in finite and infinite dimensions as well as to optimal control problems. However, Robinson's theorem does

not provide information on the gap between the obtained sufficient and necessary conditions of Lipschitz stability. To get this information, an extension of Robinson's theorem obtained in Dontchev (1995) can be used. It allows to get necessary conditions of Lipschitz stability, provided that the dependence of data on the parameter is sufficiently strong. Using this approach, a characterization of the Lipschitz stability property was recently obtained for mathematical programs (Dontchev and Rockafellar, 1996), as well as for optimal control problems for systems described by nonlinear ordinary (Dontchev and Malanowski, 1999) and parabolic (Malanowski and Tröltzsch, 1999) equations, subject to control constraints.

The present paper concerns a nonlinear boundary control problem for an elliptic equation. We follow the approach of Malanowski and Tröltzsch (1999) and characterize local Lipschitz stability with respect to the parameter for the solutions of this problem. Due to similarities with Malanowski and Tröltzsch (1999) we do not always work out all details and refer to that paper.

The main technical difference is that we eliminate the state, by introducing the solution map of the state equation, and we treat the problem as depending on the control alone. This approach is especially useful in the proof of necessity, since, in comparison with Dontchev and Malanowski (1999) and Malanowski and Tröltzsch (1999), it allows to weaken the required *strong dependence* on the data. Namely, we are able to avoid an additive perturbation of the state equation, which was needed in Dontchev and Malanowski (1999) and in Malanowski and Tröltzsch (1999). The approach can be applied, not only to the considered problem, but in general, to control constrained problems with different dynamics.

As in Malanowski and Tröltzsch (1999), the crucial point in the stability analysis is to derive conditions of  $L^\infty$ -Lipschitz stability of the solutions to the accessory linear-quadratic optimal control problems. Here we use the result obtained in the thesis Unger (1997).

We should mention that the stability and sensitivity of solutions to elliptic optimal control problems were analyzed by J.F. Bonnans. In Theorem 3.1 of Bonnans (1998) he obtained a sufficient condition of  $L^2$ -Lipschitz stability for such problems in the case of distributed control and bounded dimension of the domain.

The organization of the paper is the following. In Section 2 we introduce the considered optimal control problem, as well as the basic assumptions, and we recall some regularity results for the solution of the state equation. In Section 3 we recall abstract implicit function theorems and we use Robinson's theorem to derive *sufficient* conditions of Lipschitz stability. In Section 4 we use Dontchev's theorem to show that these conditions are also *necessary*, provided that the dependence of the data on the parameter is sufficiently strong.

## 2. Preliminaries

Let  $\Omega \subset \mathbb{R}^n$  denote a bounded domain with boundary  $\Gamma$ . As usually, by  $\Delta y$  and  $\partial_\nu y$  we denote the Laplace operator and the co-normal derivative of  $y$  at  $\Gamma$ , respectively. Moreover, let  $H$  be a Banach space of parameters and  $G \subset H$  an open and bounded set of feasible parameters.

For any  $h \in G$  consider the following elliptic optimal control problem:

$$(O_h) \quad \text{Find } (y_h, u_h) \in Z^\infty := (C(\bar{\Omega}) \cap W^{1,2}(\Omega)) \times L^\infty(\Gamma) \text{ such that}$$

$$F(y_h, u_h, h) = \min\{F(y, u, h) := \int_\Omega \varphi(y(x), h) dx + \int_\Gamma \psi(y(x), u(x), h) dS_x\} \quad (1)$$

subject to

$$\begin{aligned} -\Delta y(x) + y(x) &= 0 && \text{in } \Omega \\ \partial_\nu y(x) &= b(y(x), u(x), h) && \text{on } \Gamma, \end{aligned} \quad (2)$$

$$u \in \mathcal{U}^{ad} := \{v \in L^\infty(\Gamma) \mid q \leq v(x) \leq r \text{ a.e. in } \Gamma\}. \quad (3)$$

In this setting,  $q < r$  are fixed real numbers,  $dS_x$  denotes the surface measure induced on  $\Gamma$ , and the subscript  $x$  indicates that the integration is performed with respect to  $x$ . We assume:

- (A1) The domain  $\Omega$  has  $C^{1,1}$ -boundary  $\Gamma$ .  
 (A2) For any  $h \in G$ , the functions  $\varphi(\cdot, h) : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\psi(\cdot, \cdot, h) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $b(\cdot, \cdot, h) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are of class  $C^2$ . Moreover, for any fixed  $u \in \mathbb{R}$  and  $h \in G$ ,  $b(\cdot, u, h) : \mathbb{R} \rightarrow \mathbb{R}$  is monotonically decreasing.

There is a bound  $c_G > 0$  such that

$$|b(0, 0, h)| + |D_{(y,u)} b(0, 0, h)| + |D_{(y,u)}^2 b(0, 0, h)| \leq c_G \quad \forall h \in G.$$

Moreover, for any  $K > 0$  a constant  $l(K)$  exists such that

$$|D_{(y,u)}^2 b(y_1, u_1, h) - D_{(y,u)}^2 b(y_2, u_2, h)| \leq l(K)(|y_1 - y_2| + |u_1 - u_2|)$$

for all  $y_i, u_i$  such that  $|y_i| \leq K, |u_i| \leq K$ , and all  $h \in G$ . The same conditions as above are also satisfied by  $\varphi$  and  $\psi$ .

- (A3) For all fixed real  $y$  and  $u$  with  $|y| \leq K, |u| \leq K$ , there is a constant  $l_H(K)$  such that

$$|b(y, u, h_1) - b(y, u, h_2)| \leq l_H(K) \|h_1 - h_2\|_H \quad \forall h_i \in G, \quad i = 1, 2.$$

The same estimate holds also for  $\varphi$  and  $\psi$ .

**DEFINITION 2.1** For any  $u \in L^\infty(\Gamma)$  a function  $y \in W^{1,2}(\Omega)$  is said to be a weak solution of (2), if for all  $z \in W^{1,2}(\Omega)$  the following equation holds:

$$\int_\Omega \left( \sum_{i=1}^n \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_i} + yz \right) dx = \int_\Gamma b(y, u, h) z dS_x. \quad (4)$$

■

We will need the following standard regularity result for linear elliptic equations (see, e.g., Casas, 1993).

LEMMA 2.2 Consider the following linear equation

$$\begin{aligned} -\Delta z(x) + z(x) &= f_1(x) && \text{in } \Omega, \\ \partial_\nu z(x) + \beta(x)z(x) &= f_2(x) && \text{on } \Gamma. \end{aligned} \quad (5)$$

Suppose that  $\beta \in L^\infty(\Gamma)$  is nonnegative. If  $f_1 \in L^2(\Omega)$  and  $f_2 \in L^2(\Gamma)$ , then (5) has a unique weak solution  $z \in W^{1,2}(\Omega)$ . If  $f_1 \in L^q(\Omega)$ ,  $q > n/2$ ,  $f_2 \in L^p(\Gamma)$ ,  $p > n - 1$ , then  $z \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$  and there is a constant  $c > 0$ , independent of  $\beta$ , such that

$$\|z\|_{W^{1,2}(\Omega)} + \|z\|_{C(\bar{\Omega})} \leq c(\|f_1\|_{L^q(\Omega)} + \|f_2\|_{L^p(\Gamma)}). \quad (6)$$

By the following lemma, problem  $(O_h)$  is well posed.

LEMMA 2.3 If (A1) – (A3) hold, then for any  $u \in U^{ad}$  and any  $h \in G$  there exists a unique weak solution  $y(u, h) \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$  of (2). Moreover, there exists  $c > 0$  such that

$$\|y(u', h') - y(u'', h'')\|_{C(\bar{\Omega})} \leq c(\|u' - u''\|_{L^\infty(\Gamma)} + \|h' - h''\|_H). \quad (7)$$

*Proof* For all  $u \in U_{ad}$  and  $h \in G$ , the existence of a unique solution  $y$  to (2) follows from Theorem 2.2 in Casas and Tröltzsch (1993). In particular, for fixed  $h$ , the function  $b = b(y, u, h)$  is a real function satisfying all assumptions stated in Casas and Tröltzsch (1993). Moreover, there is a constant  $K > 0$  such that  $|y(x)| \leq K$  on  $\bar{\Omega}$ , independently of the concrete choice of  $u$  and  $h$ . To see this, let us use the expansion

$$b(y, u, h) = b(0, u, h) + D_y b(y^\vartheta, u, h) y,$$

where  $y^\vartheta(x) = \vartheta(x)y(x)$ , with  $\vartheta(x) \in [0, 1]$ . Hence

$$\partial_\nu y + \beta y = b(0, u, h), \quad (8)$$

where in view of (A2),  $\beta = -D_y b(y^\vartheta, u, h) \in L^\infty(\Gamma)$  is nonnegative. By (A2) and (A3), the right hand side of (8) is uniformly bounded, and Lemma 2.2 implies a uniform bound  $K$  for  $y$ . Let now  $(u_i, h_i) \in U_{ad} \times G$ ,  $i = 1, 2$ , be given, with the associated states  $y_i$ . Then

$$\begin{aligned} &b(y_1, u_1, h_1) - b(y_2, u_2, h_2) \\ &= D_y b(y^\vartheta, u_1, h_1)(y_1 - y_2) + b(y_2, u_1, h_1) - b(y_2, u_2, h_2), \end{aligned}$$

where now  $y^\vartheta(x) = y_1(x) + \vartheta(x)(y_2(x) - y_1(x))$ . Put  $\beta = -D_y b(y^\vartheta, u_1, h_1) \geq 0$  and  $z = y_1 - y_2$ . Then  $z$  solves

$$-\Delta z + z = 0, \quad \partial_\nu z + \beta z = f_2,$$

where

$$\begin{aligned} f_2 &= b(y_2, u_1, h_1) - b(y_2, u_2, h_2) \\ &= b(y_2, u_1, h_1) - b(y_2, u_2, h_1) + b(y_2, u_2, h_1) - b(y_2, u_2, h_2). \end{aligned}$$

According to **(A2)** and **(A3)**,  $|f_2(x)| \leq c(|u_1(x) - u_2(x)| + \|h_1 - h_2\|_H)$ . Hence Lemma 2.2 yields  $\|z\|_{C(\bar{\Omega})} \leq c(\|u_1 - u_2\|_{L^\infty(\Gamma)} + \|h_1 - h_2\|_H)$ .  $\square$

Let  $h_0 \in G$  be a given reference value of the parameter. We assume:

**(A4)** There exists a local solution  $(y_0, u_0)$  of  $(O_{h_0})$ .

Let us denote by  $s_h(\cdot) : L^\infty(\Gamma) \rightarrow C(\bar{\Omega}) \cap W^{1,2}(\Omega)$  the mapping which, for a fixed  $h \in G$ , to a given control  $u$  assigns the weak solution of the state equation (2). Then problem  $(O_h)$  can be reformulated as the following problem of optimization with respect to the control alone:

$$\begin{aligned} (O'_h) \quad & \text{Find } u_h \in \mathcal{U}^{ad} \text{ such that} \\ & \mathcal{F}(u_h, h) = \min_{u \in \mathcal{U}^{ad}} \mathcal{F}(u, h), \end{aligned}$$

where

$$\mathcal{F}(u, h) = F(s_h(u), u, h), \quad (9)$$

$\mathcal{F} : L^\infty(\Gamma) \times G \rightarrow \mathbb{R}$ . In the sequel, we will use both equivalent formulations  $(O_h)$  and  $(O'_h)$  of the problem. A standard first order necessary optimality condition for  $(O'_{h_0})$  is given in the form of the following variational inequality:

$$D_u \mathcal{F}(u_0, h_0)(u - u_0) \geq 0 \quad \text{for all } u \in \mathcal{U}^{ad}. \quad (10)$$

Denote by  $\mathcal{S}_0 := (\mathcal{S}_0^\Omega, \mathcal{S}_0^\Gamma) : L^2(\Gamma) \rightarrow L^2(\Omega) \times L^2(\Gamma)$  the mapping given by  $\mathcal{S}_0 : v \mapsto (z_0, z_0|_\Gamma)$ , where  $z_0 = z_0(v)$  is the weak solution of the linearized boundary value problem

$$\begin{aligned} -\Delta z(x) + z(x) &= 0 && \text{in } \Omega, \\ \partial_\nu z(x) &= D_y b(y_0, u_0, h_0)z(x) + D_u b(y_0, u_0, h_0)v(x). && \text{on } \Gamma. \end{aligned} \quad (11)$$

Standard calculations show that the adjoint mapping  $\mathcal{S}_0^* : L^2(\Omega) \times L^2(\Gamma) \rightarrow L^2(\Gamma)$  is given by

$$\mathcal{S}_0^* \begin{pmatrix} r \\ s \end{pmatrix} = (\mathcal{S}_0^\Omega)^* r + (\mathcal{S}_0^\Gamma)^* s = D_u b(y_0, u_0, h_0)p(r, s)|_\Gamma, \quad (12)$$

where  $p(r, s)$  is the weak solution of the adjoint equation

$$\begin{aligned} -\Delta p(x) + p(x) &= r(x) && \text{in } \Omega, \\ \partial_\nu p(x) &= D_y b(y_0, u_0, h_0)p(x) + s(x) && \text{on } \Gamma. \end{aligned} \quad (13)$$

Define the following Hamiltonian  $\mathcal{H} : \mathbb{R}^3 \times G \rightarrow \mathbb{R}$ :

$$\mathcal{H}(y, u, p, h) := \psi(y, u, h) + pb(y, u, h). \quad (14)$$

Let us write for short  $\varphi_0 := \varphi(y_0, h_0)$  and  $\psi_0 := \psi(y_0, u_0, h_0)$ . Using (11)-(13) as well as (1) and (14), we get

$$\begin{aligned}
 D_u \mathcal{F}(u_0, h_0) u &= D_y F(y_0, u_0, h_0) S_0 u + D_u F(y_0, u_0, h_0) u \\
 &= \int_{\Omega} D_y \varphi_0 S_0^{\Omega} u \, dx + \int_{\Gamma} (D_y \psi_0 S_0^{\Gamma} u + D_u \psi_0 u) \, dS_x \\
 &= (D_y \varphi_0, S_0^{\Omega} u)_{L^2(\Omega)} + (D_y \psi_0, S_0^{\Gamma} u)_{L^2(\Gamma)} + (D_u \psi_0, u)_{L^2(\Gamma)} \\
 &= ((S_0^{\Omega})^* D_y \varphi_0 + (S_0^{\Gamma})^* D_y \psi_0 + D_u \psi_0, u)_{L^2(\Gamma)} \\
 &= (D_u \mathcal{H}(y_0, u_0, p_0, h_0), u)_{L^2(\Gamma)},
 \end{aligned} \tag{15}$$

where  $p_0$  is the solution of the following adjoint equation

$$\begin{aligned}
 -\Delta p_0(x) + p_0(x) &= D_y \varphi(y_0, h_0) && \text{in } \Omega, \\
 \partial_\nu p_0(x) &= D_y b(y_0, u_0, h_0) p_0 + D_y \psi(y_0, u_0, h_0) && \text{on } \Gamma.
 \end{aligned} \tag{16}$$

Hence we have

$$D_u \mathcal{F}(u_0, h_0) u = \int_{\Gamma} D_u \mathcal{H}(y_0, u_0, p_0, h_0) u \, dS_x, \tag{17}$$

and we can identify  $D_u \mathcal{F}(u_0, h_0)$  with the real function  $D_u \mathcal{H}(y_0, u_0, p_0, h_0)$ . This function belongs to  $L^\infty(\Gamma)$ . To see this, we apply Lemma 2.2 to (16) to obtain the regularity  $p_0 \in C(\bar{\Omega})$ . By (17), optimality condition (10) takes the form

$$\int_{\Gamma} D_u \mathcal{H}(y_0, u_0, p_0, h_0) (u - u_0) \, dS_x \geq 0 \quad \text{for all } u \in \mathcal{U}^{ad}. \tag{18}$$

On the other hand, since  $D_u \mathcal{H}(y_0, u_0, p_0, h_0) \in L^\infty(\Gamma)$ , then in view of (17), we can treat  $D_u \mathcal{F}(u_0, h_0)$  as an element of  $L^\infty(\Gamma)$  and rewrite condition (10) in the form of the following *generalized equation*:

$$0 \in D_u \mathcal{F}(u_0, h_0) + \mathcal{N}(u_0), \tag{19}$$

where

$$\mathcal{N}(u) := \begin{cases} \lambda \in \{L^\infty(\Gamma) \mid \int_{\Gamma} \lambda(v - u) \, dS_x \leq 0 \quad \forall v \in \mathcal{U}^{ad}\} & \text{if } u \in \mathcal{U}^{ad}, \\ \emptyset & \text{if } u \notin \mathcal{U}^{ad} \end{cases}$$

is a multivalued mapping with closed graph.

### 3. Application of an abstract implicit function theorem

We are interested in the following problem:

*Find conditions under which a neighborhood  $G_0 \subset H$  of  $h_0$  exists such that for all  $h \in G_0$  there exists a locally unique solution  $(y_h, u_h)$  of  $(O_h)$ , which is a Lipschitz continuous function of  $h$ .*

First, we will find conditions of existence, local uniqueness and Lipschitz continuity of stationary points of  $(O'_h)$ , i.e., of the solutions to the generalized equation

$$0 \in D_u \mathcal{F}(u_h, h) + \mathcal{N}(u_h) \quad (20)$$

analogous to (19). Afterwards we will show that these stationary points correspond to the solutions of  $(O'_h)$ . In a standard way (see, e.g., Dontchev and Hager, 1993; Dontchev, Hager, Poor and Yang, 1995; Malanowski, 1992; Malanowski and Tröltzsch, 1999) we apply to (20) the implicit function theorem for generalized equations. To this end, along with (20) we consider the following generalized equation obtained by linearization and perturbation of (20) at the reference point:

$$\delta \in D_u \mathcal{F}(u_0, h_0) + D_{uu}^2 \mathcal{F}(u_0, h_0)(v - u_0) + \mathcal{N}(v), \quad (21)$$

where  $\delta \in L^\infty(\Gamma)$  is a perturbation. In the sequel, a solution to (21) will be interpreted as a stationary point of an accessory linear-quadratic optimal control problem.

We shall explain later that  $D_{uu}^2 \mathcal{F}(u_0, h_0)(v - u_0)$  can be identified with a measurable and essentially bounded function. In the sequel by

$$\mathcal{B}_\rho^X(x_0) = \{x \in X \mid \|x - x_0\|_X \leq \rho\}$$

we will denote the closed ball of radius  $\rho$  centred at  $x_0 \in X$ . Moreover, to simplify the notation, we will write  $U = L^\infty(\Gamma)$ .

Our sufficiency analysis is based on the following Robinson's abstract implicit function theorem (see Robinson (1980) : Theorem 2.1 and Corollary 2.2).

**THEOREM 3.1** *Suppose that  $D_u \mathcal{F}(u, \cdot)$  is Lipschitz continuous in  $h$ , uniformly with respect to  $u$  in a neighborhood of  $u_0$ . If there is a constant  $\widehat{\ell}$  such that*

- (i) *for any  $\epsilon > 0$ , there exist  $\rho_1 > 0$  and  $\rho_2 > 0$  such that, for each  $\delta \in \mathcal{B}_{\rho_1}^U(0)$  there is a unique in  $\mathcal{B}_{\rho_2}^U(u_0)$  solution to the linearized generalized equation (21), which is Lipschitz continuous in  $\delta$  with the constant  $\widehat{\ell} + \epsilon$ ,*

*then*

- (ii) *for any  $\epsilon > 0$ , there exist  $\sigma_1 > 0$  and  $\sigma_2 > 0$  such that, for each  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$  there is a unique in  $\mathcal{B}_{\sigma_2}^U(u_0)$  solution to the nonlinear generalized equation (20), which is Lipschitz continuous in  $h$  with the constant  $\widehat{\ell} + \epsilon$ .* ■

Theorem 3.1 allows to deduce existence, local uniqueness and Lipschitz continuity of solutions to (20) from the same properties of the solutions to the linear generalized equation (21). Usually, this last problem is much easier than the original one.

In verifying the necessity of the derived sufficient conditions of Lipschitz stability, we will consider a special situation, where the dependence of data upon the parameter is *strong* in the following sense:

$$D_u \mathcal{F}(u, h) = D_u \mathcal{F}^0(u) + h, \text{ where } h \in H = L^\infty(\Gamma). \quad (22)$$

**REMARK 3.2** As in Dontchev and Malanowski (1999) and in Malanowski and Tröltzsch (1999), instead of (22), we can consider a more general situation, where

$$H = H^0 \times L^\infty(\Gamma), \text{ with } H^0 \text{ being an arbitrary Banach space, and}$$

$$D_u \mathcal{F}(u, h) = D_u \mathcal{F}^0(u, h^0) + h^1, \text{ where } h^0 \in H^0 \text{ and } h^1 \in L^\infty(\Gamma).$$

■

The next theorem follows from Theorem 3 in Dontchev (1995).

**THEOREM 3.3** *If (22) holds, then (ii) implies (i).*

■

In order to apply Theorem 3.1 we have to find the concrete expressions of the derivatives in the linear generalized equation (21). Using (16) together with (11) and (15), we find (21) in the following form

$$\delta \in D_{uu}^2 \mathcal{F}(u_0, h_0) v + \delta_0 + \mathcal{N}(v). \quad (23)$$

The expression of  $D_{uu}^2 \mathcal{F}(u_0, h_0)$  is obtained by standard computations (see, e.g. (6.6) and (6.7) in Malanowski and Tröltzsch, 1999):

$$D_{uu}^2 \mathcal{F}(u_0, h_0) v = \mathcal{K}(u_0, h_0) v + D_{uu}^2 \mathcal{H}(y_0, u_0, p_0, h_0) v, \quad (24)$$

with

$$\begin{aligned} \mathcal{K}(u_0, h_0) &= (\mathcal{S}_0^\Omega)^* D_{yy}^2 \varphi(y_0, h_0) \mathcal{S}_0^\Omega + (\mathcal{S}_0^\Gamma)^* D_{yy}^2 \mathcal{H}(y_0, u_0, p_0, h_0) \mathcal{S}_0^\Gamma \\ &\quad + (\mathcal{S}_0^\Gamma)^* D_{yu}^2 \mathcal{H}(y_0, u_0, p_0, h_0) + D_{uy}^2 \mathcal{H}(y_0, u_0, p_0, h_0) \mathcal{S}_0^\Gamma, \end{aligned} \quad (25)$$

and

$$\delta_0 = D_u \mathcal{F}(u_0, h_0) - D_{uu}^2 \mathcal{F}(u_0, h_0) u_0. \quad (26)$$

By Lemma 2.2 we find that, for a fixed  $v \in L^\infty(\Gamma)$ ,  $D_{uu}^2 \mathcal{F}(u_0, h_0) v \in L^\infty(\Gamma)$ . Certainly,  $v_0 = u_0$  is a solution of (23) for  $\delta = 0$ . To simplify the notation, by  $(\cdot, \cdot)$  we will understand the inner product in  $L^2(\Gamma)$ .

The generalized equation (23) constitutes the first order optimality condition for the following linear-quadratic problem.

$$\begin{aligned} (\text{LO}'_\delta) \quad &\text{Find } v_\delta \in \mathcal{U}^{ad} \text{ such that} \\ &\frac{1}{2}(v_\delta, D_{uu}^2 \mathcal{F}(u_0, h_0) v_\delta) + (\delta_0 - \delta, v_\delta) \\ &= \min_{v \in \mathcal{U}^{ad}} \left\{ \frac{1}{2}(v, D_{uu}^2 \mathcal{F}(u_0, h_0) v) + (\delta_0 - \delta, v) \right\}. \end{aligned}$$

In view of (24) and (25), problem  $(\text{LO}'_\delta)$  is equivalent to the following linear-quadratic optimal control problem.

$$\begin{aligned} (\text{LO}_\delta) \quad &\text{Find } (z_\delta, v_\delta) \in Z^\infty \text{ such that} \\ &\mathcal{I}(z_\delta, v_\delta, \delta) = \min \mathcal{I}(z, v, \delta) \end{aligned}$$



subject to

$$\begin{aligned} -\Delta z(x) + z(x) &= 0 && \text{in } \Omega \\ \partial_\nu z(x) &= D_y b(y_0(x), u_0(x), h_0) z(x) && \\ &+ D_u b(y_0(x), u_0(x), h_0) v(x) && \text{on } \Gamma, \end{aligned} \quad (27)$$

$$v \in \mathcal{U}^{ad},$$

where

$$\mathcal{I}(z, v, \delta) := \frac{1}{2} \mathcal{J}_0[(z, v), (z, v)] + \int_{\Gamma} (\delta_0(x) - \delta(x)) v(x) dS_x,$$

with the quadratic form

$$\begin{aligned} \mathcal{J}_0[(z, v), (z, v)] &= \int_{\Omega} D_{yy}^2 \varphi(y_0, h_0) z^2 dx \\ &+ \int_{\Gamma} [z, v] \begin{bmatrix} D_{yy}^2 \mathcal{H}(y_0, u_0, p_0, h_0) & D_{yu}^2 \mathcal{H}(y_0, u_0, p_0, h_0) \\ D_{uy}^2 \mathcal{H}(y_0, u_0, p_0, h_0) & D_{uu}^2 \mathcal{H}(y_0, u_0, p_0, h_0) \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} dS_x. \end{aligned} \quad (28)$$

To verify assumption (i) of Theorem 3.1, we have to show that, for all sufficiently small perturbations  $\delta$ , problem  $(LO_\delta)$  (or equivalently  $(LO'_\delta)$ ) has a locally unique stationary point, which is Lipschitz continuous in  $\delta$ . For this purpose, we will need a coercivity assumption. To introduce it, for any  $\alpha \geq 0$  define the sets

$$\begin{aligned} I^\alpha &= \{x \in \Gamma \mid D_u \mathcal{H}(y_0, u_0, p_0, h_0)(x) > \alpha\}, \\ J^\alpha &= \{x \in \Gamma \mid -D_u \mathcal{H}(y_0, u_0, p_0, h_0)(x) > \alpha\}. \end{aligned} \quad (29)$$

We assume:

**(AC)** (*coercivity*) There exist  $\alpha > 0$  and  $\gamma > 0$  such that

$$\mathcal{J}_0[(z, v), (z, v)] \geq \gamma \|v\|_{L^2(\Gamma)}^2 \quad (30)$$

for all pairs  $(z, v)$  satisfying (27) and such that  $v \in V_\alpha^2$ , where

$$V_\alpha^q := \{v \in L^q(\Gamma) \mid v(x) = 0 \text{ for a.a. } x \in I^\alpha \cup J^\alpha\}, \quad q \in [2, \infty]. \quad (31)$$

In view of definitions (24) and (28), condition (30) is equivalent to

$$(v, D_{uu}^2 \mathcal{F}(u_0, h_0) v) \geq \gamma \|v\|_{L^2(\Gamma)}^2 \quad \text{for all } v \in V_\alpha^2. \quad (32)$$

Note that in the situation where  $\text{meas}(I^0 \cup J^0) = 0$ , the set (31) becomes the whole space  $L^q(\Gamma)$  and the analysis is much simpler. Therefore, we do not consider this case. By Satz 18 in Unger (1997) we have

**LEMMA 3.4** *If (AC) holds, then there exist constants  $\rho_1 > 0$  and  $\rho_2 > 0$  such that, for all  $\delta \in \mathcal{B}_{\rho_1}^U(0)$  there is a unique solution  $(z_\delta, v_\delta)$  of  $(LO_\delta)$  such that  $v_\delta \in \mathcal{B}_{\rho_2}^U(0)$ . Moreover, there exists a constant  $\ell > 0$  such that*

$$\|z_{\delta'} - z_{\delta''}\|_{C(\bar{\Omega})}, \|v_{\delta'} - v_{\delta''}\|_{L^\infty(\Gamma)} \leq \ell \|\delta' - \delta''\|_{L^\infty(\Gamma)} \quad \text{for all } \delta', \delta'' \in \mathcal{B}_{\rho_1}^U(0).$$

■

Theorem 3.1 and Lemma 3.4 imply

**PROPOSITION 3.5** *If conditions (A1)-(A4) and (AC) hold, then there exist constants  $\sigma_1 > 0$ ,  $\sigma_2 > 0$  and  $\ell > 0$  such that, for each  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$  there exists a unique in  $\mathcal{B}_{\sigma_2}^U(u_0)$  solution  $u_h$  of (20) and*

$$\|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \ell \|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}_{\sigma_1}^H(h_0). \quad (33)$$

■

It follows from (7) and (33) that there exists a constant  $\ell' > 0$  such that

$$\|y_{h'} - y_{h''}\|_{C(\bar{\Omega})} \leq \ell' \|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}_{\sigma_1}^H(h_0), \quad (34)$$

where  $y_h$  is the solution to the state equation (2) corresponding to the control  $u_h$ . By (13), a similar estimate holds for  $p$ ,

$$\|p_{h'} - p_{h''}\|_{C(\bar{\Omega})} \leq \ell' \|h' - h''\|_H \quad \text{for all } h', h'' \in \mathcal{B}_{\sigma_1}^H(h_0). \quad (35)$$

In view of Proposition 3.5, for  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$ , we can define the maps  $\mathcal{S}_h^\Omega: v \mapsto z_h(v)$ ,  $\mathcal{S}_h^\Gamma: v \mapsto z_h(v)|_\Gamma$ , where  $z_h(v)$  is the solution of the linearized equation (11), with  $D_y b$  and  $D_u b$  evaluated at  $(y_h, u_h, h)$ . Analogously to the definition of  $S_0$ , we put  $S_h = (\mathcal{S}_h^\Omega, \mathcal{S}_h^\Gamma)$  and view  $S_h$  as an operator from  $L^2(\Gamma)$  to  $L^2(\Omega) \times L^2(\Gamma)$ . The adjoint operator  $S_h^*: L^2(\Omega) \times L^2(\Gamma) \rightarrow L^2(\Gamma)$  is given by (12), (13), with the subscript 0 substituted by  $h$ . The following lemma summarizes regularity properties of the mappings  $\mathcal{S}_h$  and  $\mathcal{S}_h^*$ , which will be used in the sequel.

**LEMMA 3.6** *The operators  $S_h$  and  $S_h^*$  are compact in the  $L^2$ -spaces defined above. Moreover, the operators are continuous between the following spaces:*

$$\begin{aligned} S_h &: L^r(\Gamma) \rightarrow L^s(\Omega) \times L^s(\Gamma) \\ S_h^* &: L^r(\Omega) \times L^r(\Gamma) \rightarrow L^s(\Gamma) \quad \text{for all } s \in [2, \infty), \end{aligned} \quad (36)$$

if  $n = 2$  and  $r \geq 2$ ,

$$\begin{aligned} S_h &: L^r(\Gamma) \rightarrow L^{r+\tau}(\Omega) \times L^{r+\tau}(\Gamma) \\ S_h^* &: L^r(\Omega) \times L^r(\Gamma) \rightarrow L^{r+\tau}(\Gamma) \quad \text{where } \tau = \frac{2}{n-1} > 0, \end{aligned} \quad (37)$$

if  $n > 2$  and  $r \geq 2$ .

■

*Proof* Define  $\Lambda_h$  by  $\Lambda_h: (u, v) \mapsto (z, z|_\Gamma)$ , where

$$\begin{aligned} -\Delta z + z &= u \\ \partial_\nu z &= D_y b(y_h, u_h, h) z + D_u b(y_h, u_h, h) v. \end{aligned}$$

To simplify notation, let us write  $L^p = L^p(\Omega) \times L^p(\Gamma)$ . We will show that  $\Lambda_h$  is continuous either from  $L^r$  to  $L^s$  for all  $s \in [2, \infty)$ , or from  $L^r$  to  $L^{r+\tau}$ , if

$n = 2$  or  $n > 2$ , respectively. This includes the statement of the lemma. It is known that  $\Lambda_h : L^2 \rightarrow W^{1,2}(\Omega) \times W^{1/2,2}(\Gamma)$ , is continuous. The embeddings  $W^{1,2}(\Omega) \subset L^\alpha(\Omega)$  and  $W^{1/2,2}(\Gamma) \subset L^\beta(\Gamma)$  are compact for all  $\alpha < 2n/(n-2)$  and all  $\beta < 2(n-1)/(n-2) < 2n/(n-1)$ . Therefore,

$$\Lambda_h : L^2 \rightarrow L^\beta \text{ is compact for } \beta \leq \frac{2n}{n-1} = 2 + \frac{2}{n-1}. \quad (38)$$

In particular,  $\Lambda_h$  is compact in  $L^2$ , which implies in turn the compactness of  $S_h$  and  $S_h^*$ , as stated in the lemma.

On the other hand, since by the Sobolev embedding theorem  $W^{1,2}(\Omega) \times W^{1/2,2}(\Gamma) \subset L^s(\Omega) \times L^s(\Gamma)$  for any  $s < \infty$ , if  $n = 2$ , we find that the mappings

$$\Lambda_h : L^2 \rightarrow \begin{cases} L^s & \text{for any } s < \infty, \text{ if } n = 2, \\ L^{\frac{2n}{n-1}} & \text{if } n > 2 \end{cases} \quad (39)$$

are continuous. Thus, (36) follows immediately. To prove (37), note that by Lemma 2.2,  $\Lambda_h : L^q(\Omega) \times L^p(\Gamma) \rightarrow L^\infty$  is continuous if  $q > n/2$  and  $p > n-1$ . Since  $n-1 \geq n/2$  for  $n \geq 2$ , the mapping  $\Lambda_h : L^p \rightarrow L^\infty$  is continuous for  $p > n-1$ . So, in particular

$$\Lambda_h : L^n \rightarrow L^\infty \text{ is continuous.} \quad (40)$$

Applying the interpolation theorem to (39) and (40), we find that

$$\Lambda_h : L^{r_\theta} \rightarrow L^{s_\theta}$$

is continuous, where  $\theta \in [0, 1]$  and  $r_\theta, s_\theta$  are given by

$$\frac{1}{r_\theta} = \frac{1-\theta}{2} + \frac{\theta}{n}, \quad \frac{1}{s_\theta} = \frac{1-\theta}{(2n/(n-1))} + \frac{\theta}{\infty} = \frac{(n-1)(1-\theta)}{2n}.$$

Hence

$$s_\theta - r_\theta = \frac{2n}{(n-1)(1-\theta)} - \frac{2n}{n(1-\theta) + 2\theta} \geq \frac{2}{n-1} \text{ for all } \theta \in [0, 1].$$

If  $\theta$  varies from 0 to 1, then  $r_\theta$  moves from 2 to  $n$ , thus we have proven (37) for all  $r \in [2, n]$ . For  $r > n$ , (37) holds trivially by (40).  $\square$

We should mention that  $\tau$  in (37) is not the optimal one, since we did not exploit the optimal regularity of  $S_h$ , but it is enough for our purpose.

The following theorem is our principal *sufficiency* result.

**THEOREM 3.7** *If conditions (A1)–(A4) and (AC) hold, then there exist constants  $\sigma_1 > 0$ ,  $\sigma_2 > 0$  and  $\lambda > 0$  such that, for each  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$  there exists a unique in  $\mathcal{B}_{\sigma_2}^{Z^\infty}(y_0, u_0)$  solution  $(y_h, u_h)$  of  $(O_h)$  and*

$$\|y_{h'} - y_{h''}\|_{C(\bar{\Omega})}, \|p_{h'} - p_{h''}\|_{C(\bar{\Omega})}, \|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \lambda \|h' - h''\|_H$$

for all  $h', h'' \in \mathcal{B}_{\sigma_1}^H(h_0)$ .  $\blacksquare$

*Proof* In view of (33) – (35), to prove the theorem, it is enough to show that  $u_h$  satisfying (20) is a locally unique solution to  $(O'_h)$ . We have already pointed out that, for  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$ , the derivatives  $D_u \mathcal{F}(u_h, h)$  and  $D_{uu}^2 \mathcal{F}(u_h, h)$  can be represented as in (15) and (24), with all terms evaluated at  $h$ , rather than at  $h_0$ . Thus  $D_u \mathcal{F}(u_h, h)$  can be treated as an element of  $L^\infty(\Gamma)$ . In view of Theorem 3.7 we find that the mapping

$$h \mapsto D_u \mathcal{F}(u_h, h) : \mathcal{B}_{\sigma_1}^H(h_0) \rightarrow L^\infty(\Gamma) \quad (41)$$

is continuous. Moreover, the quadratic form associated with  $D_{uu}^2 \mathcal{F}(u_h, h)$  depends continuously on  $h$ , since  $u_h$ ,  $y_h$ , and  $p_h$  depend continuously on  $h$ , and therefore the estimate

$$(u_1, [D_{uu}^2 \mathcal{F}(u_{h'}, h') - D_{uu}^2 \mathcal{F}(u_{h''}, h'')] u_2) \leq c \|h' - h''\|_H \|u_1\|_{L^2(\Gamma)} \|u_2\|_{L^2(\Gamma)} \quad (42)$$

can be derived with some effort. By the continuity of  $D_u \mathcal{F}(u_h, h) = D_u \mathcal{H}(y_h, u_h, p_h, h)$  it follows from (29) that, for  $\sigma_1 > 0$  sufficiently small, we have

$$|D_u \mathcal{F}(u_h, h)(x)| \geq \frac{\alpha}{2} \quad \text{for a.a. } x \in I^\alpha \cup J^\alpha \text{ and all } h \in \mathcal{B}_{\sigma_1}^H(h_0). \quad (43)$$

On the other hand, using (42), and shrinking  $\sigma_1 > 0$  if necessary, we obtain from (32) that

$$(v, D_{uu}^2 \mathcal{F}(u_h, h)v) \geq \frac{\gamma}{2} \|v\|_{L^2(\Gamma)}^2 \quad \text{for all } v \in V_\alpha^2 \text{ and all } h \in \mathcal{B}_{\sigma_1}^H(h_0). \quad (44)$$

The estimates (43) and (44) constitute sufficient optimality conditions for  $(O_h)$ . Indeed, expanding  $\mathcal{F}(u, h)$  into the Taylor series at  $u_h$ , we get

$$\begin{aligned} \mathcal{F}(u, h) - \mathcal{F}(u_h, h) &= (D_u \mathcal{F}(u_h, h), u - u_h) \\ &+ \frac{1}{2}((u - u_h), D_{uu}^2 \mathcal{F}(u_h, h)(u - u_h)) + r(u - u_h), \end{aligned} \quad (45)$$

where

$$\frac{|r(v)|}{\|v\|_{L^2(\Gamma)}} \rightarrow 0 \quad \text{as } \|v\|_{L^\infty(\Gamma)} \rightarrow 0. \quad (46)$$

By (17), the optimality condition (10) evaluated at  $h$  yields

$$D_u \mathcal{F}(u_h, h)(x)(u(x) - u_h(x)) \geq 0 \quad \text{for a.a. } x \in \Gamma \text{ and all } u \in \mathcal{U}^{ad}. \quad (47)$$

On the other hand, in view of (42), there exists a constant  $c > 0$  such that

$$(v, D_{uu}^2 \mathcal{F}(u_h, h)v) \leq c \|v\|_{L^2(\Gamma)}^2. \quad (48)$$

For any  $u \in L^2(\Gamma)$ , let us denote by  $u^1$  and  $u^2$  the projections onto  $V_\alpha^2$  and onto its orthogonal complement, respectively, i.e.,

$$u^1(x) = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ u(x) & \text{on } \Gamma \setminus I^\alpha \cup J^\alpha, \end{cases} \quad u^2(x) = \begin{cases} u(x) & \text{on } I^\alpha \cup J^\alpha, \\ 0 & \text{on } \Gamma \setminus I^\alpha \cup J^\alpha. \end{cases}$$

Using (44), (48) and Young's inequality we obtain

$$\begin{aligned}
& ((u - u_h), D_{uu}^2 \mathcal{F}(u_h, h)(u - u_h)) \\
&= ((u^1 - u_h^1), D_{uu}^2 \mathcal{F}(u_h, h)(u^1 - u_h^1)) \\
&+ 2((u^1 - u_h^1), D_{uu}^2 \mathcal{F}(u_h, h)(u^2 - u_h^2)) \\
&+ ((u^2 - u_h^2), D_{uu}^2 \mathcal{F}(u_h, h)(u^2 - u_h^2)) \\
&\geq \frac{\gamma}{2} \|u^1 - u_h^1\|_{L^2(\Gamma)}^2 - 2c \|u^1 - u_h^1\|_{L^2(\Gamma)} \|u^2 - u_h^2\|_{L^2(\Gamma)} \\
&- c \|u^2 - u_h^2\|_{L^2(\Gamma)}^2 \\
&\geq \frac{\gamma}{4} \|u^1 - u_h^1\|_{L^2(\Gamma)}^2 - c(1 + \frac{8c}{\gamma}) \|u^2 - u_h^2\|_{L^2(\Gamma)}^2 \\
&= \frac{\gamma}{4} \int_{\Gamma \setminus (I^\alpha \cup J^\alpha)} (u^1(x) - u_h^1(x))^2 dS_x - c_1 \int_{(I^\alpha \cup J^\alpha)} (u^2(x) - u_h^2(x))^2 dS_x
\end{aligned} \tag{49}$$

for all  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$ , where  $c_1 = c(1 + \frac{8c}{\gamma})$ . Combining (47) through (49) and using (43) together with (45) we obtain

$$\begin{aligned}
& \mathcal{F}(u, h) - \mathcal{F}(u_h, h) \\
&\geq \int_{I^\alpha \cup J^\alpha} \left( \frac{\alpha}{2} |u(x) - u_h(x)| - \frac{c_1}{2} |u(x) - u_h(x)|^2 \right) dS_x \\
&+ \frac{\gamma}{8} \int_{\Gamma \setminus (I^\alpha \cup J^\alpha)} |u(x) - u_h(x)|^2 dS_x + r(u - u_h) \quad \text{for all } u \in \mathcal{U}^{ad}.
\end{aligned} \tag{50}$$

Choosing  $\sigma = \frac{4\alpha}{\gamma + 4c_1}$  we get

$$\mathcal{F}(u, h) - \mathcal{F}(u_h, h) \geq \frac{\gamma}{16} \|u - u_h\|_{L^2(\Gamma)}^2 + r(u - u_h) \quad \text{for all } u \in \mathcal{U}^{ad} \cap \mathcal{B}_\sigma^U(u_h).$$

By (46), for  $\sigma > 0$  sufficiently small, we obtain

$$\mathcal{F}(u, h) - \mathcal{F}(u_h, h) \geq \frac{\gamma}{32} \|u - u_h\|_{L^2(\Gamma)}^2 \quad \text{for all } u \in \mathcal{U}^{ad} \cap \mathcal{B}_\sigma^U(u_h),$$

i.e.,  $u_h$  is a solution of  $(O_h)$ , unique in  $\mathcal{B}_\sigma^U(u_h)$ . Choosing  $\sigma_2 = \frac{\sigma}{2}$  and  $\sigma_1 = \frac{\sigma}{2\ell}$  we complete the proof of the theorem.  $\blacksquare$

#### 4. Lipschitz stability: necessity

In this section we are going to show that **(AC)** is not only a sufficient, but also a *necessary* condition of local Lipschitz stability of the solutions to  $(O_h)$ , provided that the dependence of data upon the parameter  $h$  is *sufficiently strong*, in the sense that (22) holds. In view of definition (9), condition (22) is satisfied if

**(SD)** (*strong dependence*)

$$F(y, u, h) = \int_{\Omega} \varphi(y(x)) dx + \int_{\Gamma} [\psi^0(y(x), u(x)) + h(x)u(x)] dS_x, \tag{51}$$

$$b(y, u, h) = b(y, u),$$

where  $h \in L^\infty(\Gamma)$ .

Note that if **(SD)** holds, then the Hamiltonian (14) takes the form

$$\mathcal{H}(y, u, p, h) = \mathcal{H}^0(y, u, p) + hu, \quad (52)$$

where  $\mathcal{H}^0(y, u) = \psi(y, u) + pb(y, u)$ .

We assume that condition **(ii)** in Theorem 3.1 is satisfied and we will show that, if (51) holds, then **(AC)** is satisfied with some  $\alpha > 0$  and  $\gamma > 0$ . The proof is based on the same idea as in Dontchev and Malanowski (1999) and in Malanowski and Tröltzsch (1999), but technically it is simplified. Namely, we introduce small variations  $(\widehat{h}, u_{\widehat{h}})$  of the reference values, such that in a neighborhood of  $(\widehat{h}, u_{\widehat{h}})$ , problems  $(O'_h)$  with *inequality* constraints can be locally treated as problems with *equality* constraints, which are much easier to analyze. First, we derive necessary coercivity conditions for  $(O'_{\widehat{h}})$  and then deduce similar conditions for the reference problem  $(O'_{h_0})$ . To this end, let us choose any  $\alpha < \frac{\sigma_1}{2}$  and  $\epsilon < \min\{\sigma_2, \frac{1}{2}(r - q)\}$ , where  $q$  and  $r$  are the bounds in the inequality constraints (3), while  $\sigma_1$  and  $\sigma_2$  are given in Theorem 3.1/**(ii)**.

Define the set

$$K = \{x \in \Gamma \mid u_0(x) \leq \frac{1}{2}(q + r)\}$$

and introduce the following increment  $\Delta u$  of the reference values  $u_0$ :

$$\Delta u(x) = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ +\epsilon & \text{on } K \setminus I^\alpha, \\ -\epsilon & \text{on } [\Gamma \setminus K] \setminus J^\alpha. \end{cases} \quad (53)$$

Define  $\widehat{u} = u_0 + \Delta u$ . It follows from (53) that, for  $\widehat{u}$ , the control constraints (3) are active on the set  $I^\alpha \cup J^\alpha$  and *non-active with the margin*  $\epsilon > 0$  on the complement of this set:

$$\widehat{u}(x) \begin{cases} = q & \text{on } I^\alpha, \\ = r & \text{on } J^\alpha, \\ \in [q + \epsilon, r - \epsilon] & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha]. \end{cases} \quad (54)$$

(see Fig.1). Let  $\widehat{y}$  denote the solution of the state equation (2) for  $u = \widehat{u}$ . Similarly, let  $\widehat{p}$  be the solution of the adjoint equation (16) corresponding to  $u = \widehat{u}$  and  $y = \widehat{y}$ . Note that in view of (51) and (52),  $\widehat{y}$  and  $\widehat{p}$  do not depend on  $h$ . Using this fact we introduce a variation  $\widehat{h}$  of  $h_0$ , such that  $(\widehat{y}, \widehat{u}, \widehat{p})$  is a stationary point of  $(O'_{\widehat{h}})$ , i.e.,  $(\widehat{y}, \widehat{u}, \widehat{p}) = (y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}})$ . In order to get  $D_u \mathcal{H}(\widehat{y}, \widehat{u}, \widehat{p}, \widehat{h})(x) = D_u \mathcal{H}^0(\widehat{y}, \widehat{u}, \widehat{p})(x) + \widehat{h}(x) = 0$  on  $\Gamma \setminus (I^\alpha \cup J^\alpha)$ , we put  $\widehat{h} = h_0 + \Delta h$ , where

$$\Delta h(x) = \begin{cases} 0 & \text{on } I^\alpha \cup J^\alpha, \\ -D_u \mathcal{H}^0(\widehat{y}, \widehat{u}, \widehat{p})(x) - h_0(x) & \text{on } \Gamma \setminus (I^\alpha \cup J^\alpha). \end{cases} \quad (55)$$

Note that by the construction of  $\widehat{u}$ ,  $\widehat{y}$ ,  $\widehat{p}$ ,

$$\|D_u \mathcal{H}^0(\widehat{y}, \widehat{u}, \widehat{p}) - D_u \mathcal{H}^0(y_0, u_0, p_0)\|_{L^\infty(\Gamma)} \rightarrow 0 \quad \text{as } \|\Delta u\|_{L^\infty(\Gamma)} \rightarrow 0. \quad (56)$$

Using (29), (53) and (56), we find that, for  $\epsilon > 0$  sufficiently small

$$D_u \mathcal{H}(\hat{y}, \hat{u}, \hat{p}, \hat{h})(x) \begin{cases} > 0 & \text{on } I^\alpha, \\ < 0 & \text{on } J^\alpha, \\ = 0 & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha], \end{cases} \quad (57)$$

which, together with (54), shows that the variational inequality (18) is satisfied at  $(\hat{y}, \hat{u}, \hat{p}, \hat{h})$ . On the other hand, for  $x \in \Gamma \setminus (I^\alpha \cup J^\alpha)$  we have

$$\begin{aligned} & |D_u \mathcal{H}^0(\hat{y}, \hat{u}, \hat{p})(x) + h_0(x)| \leq |D_u \mathcal{H}(y_0, u_0, p_0, h_0)(x)| \\ & + |D_u \mathcal{H}^0(\hat{y}, \hat{u}, \hat{p})(x) - D_u \mathcal{H}^0(y_0, u_0, p_0)(x)| \\ & \leq \alpha + |D_u \mathcal{H}^0(\hat{y}, \hat{u}, \hat{p})(x) - D_u \mathcal{H}^0(y_0, u_0, p_0)(x)|. \end{aligned} \quad (58)$$

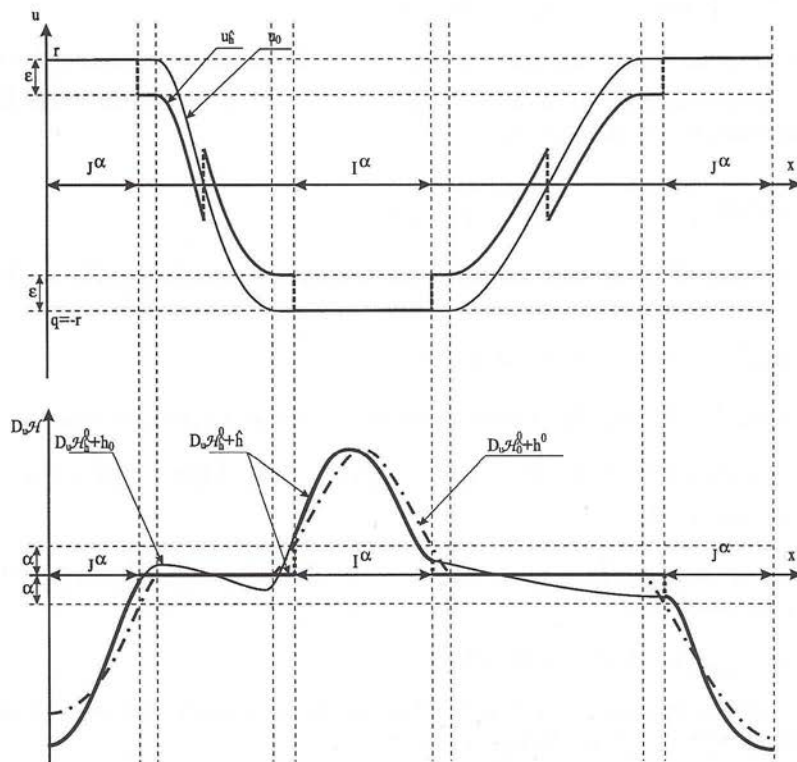


Figure 1. Construction of  $u_{\hat{h}}$  and  $\hat{h}$  ( $\mathcal{H}_0^0 := \mathcal{H}^0(y_0, u_0, p_0)$ ,  $\mathcal{H}_{\hat{h}}^0 := \mathcal{H}^0(y_{\hat{h}}, u_{\hat{h}}, p_{\hat{h}})$ ).

In view of (53), (55), (56) and (58), for  $\alpha > 0$  and  $\epsilon > 0$  small enough, we obtain  $\|\Delta h\|_{L^\infty(\Gamma)} < \sigma_1$ , i.e.,  $\widehat{h} \in \mathcal{B}_{\sigma_1}^H(h_0)$ . Hence, by our assumption,  $\widehat{u}$  is a unique in  $\mathcal{B}_{\sigma_2}^U(u_0)$  solution to (20), i.e.,  $\widehat{u} = u_{\widehat{h}}$ .

LEMMA 4.1 For  $\alpha > 0$  and  $\epsilon > 0$  sufficiently small

$$(u, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u) \geq 0 \quad \text{for all } u \in V_\alpha^2. \quad (59)$$

*Proof* In view of (54) and (57), locally around  $u_{\widehat{h}}$ , the inequality constraints (3) can be treated as equalities:

$$u_{\widehat{h}}(x) \begin{cases} = q & \text{on } I^\alpha, \\ = r & \text{on } J^\alpha, \\ \text{free} & \text{on } \Gamma \setminus [I^\alpha \cup J^\alpha]. \end{cases} \quad (60)$$

Indeed, in order that a feasible variation  $u_{\widehat{h}} + \Delta u$  satisfies the first order optimality condition at  $\widehat{h}$ , we must have  $\Delta u = 0$  on  $I^\alpha \cup J^\alpha$ . On the other hand, any variation  $u_{\widehat{h}} + \Delta u$  such that

$$|\Delta u(x)| \begin{cases} = 0 & \text{on } I^\alpha \cup J^\alpha, \\ \leq \epsilon & \text{on } \Gamma \setminus (I^\alpha \cup J^\alpha), \end{cases}$$

is feasible for  $(O_{\widehat{h}})$ . Hence, the first order optimality condition (20) at  $(\widehat{h}, u_{\widehat{h}})$  reduces to the equation

$$(D_u \mathcal{F}(u_{\widehat{h}}, \widehat{h}), u) = 0 \quad \text{for all } u \in V_\alpha^\infty. \quad (61)$$

Expanding  $\mathcal{F}(\cdot, \widehat{h})$  into the Taylor series at  $u_{\widehat{h}}$  and using (61), we obtain

$$0 \leq \mathcal{F}(u, \widehat{h}) - \mathcal{F}(u_{\widehat{h}}, \widehat{h}) = \frac{1}{2}((u - u_{\widehat{h}}), D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})(u - u_{\widehat{h}})) + r(u - u_{\widehat{h}}) \\ \text{for all } u \in V_\alpha^\infty.$$

Dividing by  $\|u - u_{\widehat{h}}\|_{L^2(\Gamma)}^2$ , passing to the limit with  $\|u - u_{\widehat{h}}\|_{L^\infty(\Gamma)} \rightarrow 0$  and using (46) we obtain

$$(v, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})v) \geq 0 \quad \text{for all } v \in V_\alpha^\infty.$$

In view of the linearity of  $V_\alpha^\infty$  as well as of the continuity and density of the embedding  $V_\alpha^\infty \subset V_\alpha^2$ , we arrive at (59).  $\square$

We are going to show now that the quadratic form (59) is actually *coercive*, with a constant independent of  $\widehat{h}$ . To this end we will use Theorem 3.3.

LEMMA 4.2 If (ii) holds, then

$$\|D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u\|_{L^\infty(\Gamma)} \geq \ell^{-1} \|u\|_{L^\infty(\Gamma)} \quad \text{for all } u \in V_\alpha^\infty. \quad (62)$$



*Proof* Let us introduce the linear generalized equation analogous to (21) but evaluated at  $(u_{\widehat{h}}, \widehat{h})$ :

$$\delta \in D_u \mathcal{F}(u_{\widehat{h}}, \widehat{h}) + D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})(v - u_{\widehat{h}}) + \mathcal{N}(v). \quad (63)$$

By (SD) and Theorem 3.3, (i) holds, i.e., for  $\delta$  sufficiently small, (63) has a locally unique solution  $v_\delta$ , which is locally Lipschitz in  $\delta$ , with the Lipschitz constant  $\ell > 0$  independent of the choice of  $\widehat{h} \in \mathcal{B}_{\sigma_1}^H(h_0)$ . Note that  $v_0 = u_{\widehat{h}}$  is the solution of (63) for  $\delta = 0$ . Hence, in the same way as in (61), we deduce from the construction of  $\widehat{h}$  and  $u_{\widehat{h}}$ , that for all  $\delta$ , in a small neighborhood of zero, (63) reduces to the equation

$$(D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})v_\delta + (D_u \mathcal{F}(u_{\widehat{h}}, \widehat{h}) - D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u_{\widehat{h}}) - \delta, v) = 0 \text{ for all } v \in V_\alpha^\infty.$$

In view of (61), choosing  $\delta \in V_\alpha^\infty$ , we obtain from the above equation

$$D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})(v_\delta - v_0) + \delta = 0.$$

Thus, for any sufficiently small  $\delta \in V_\alpha^\infty$ , this equation has a unique solution  $w_\delta := v_\delta - v_0 \in V_\alpha^\infty$ , which is Lipschitz in  $\delta$  with Lipschitz constant  $\ell > 0$ , and we arrive at (62).  $\square$

To show the coercivity of the quadratic form (59), note that, as in (24),  $D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h}) = \mathcal{K}(u_{\widehat{h}}, \widehat{h}) + D_{uu}^2 \mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})$ , where  $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$  is given in (25) with indices 0 substituted by  $\widehat{h}$ . By (25) and (36),  $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$  is a continuous mapping from  $L^p(\Gamma)$  into  $C(\Gamma)$ , for  $p > n - 1$ . Hence, by a known argument (see, e.g., Lemma 6.3 in Malanowski and Tröltzsch, 1999), (24) and (62) imply

$$D_{uu}^2 \mathcal{H}(y_{\widehat{h}}(x), u_{\widehat{h}}(x), p_{\widehat{h}}(x), \widehat{h}) \geq \ell^{-1} \quad \text{for a.a. } x \in \Gamma \setminus (I^\alpha \cup J^\alpha). \quad (64)$$

On the other hand, by Lemma 3.6,  $\mathcal{S}_h : L^2(\Gamma) \rightarrow L^2(\Omega) \times L^2(\Gamma)$  is compact. Therefore, it follows from (25) that  $\mathcal{K}(u_{\widehat{h}}, \widehat{h}) : L^2(\Gamma) \rightarrow L^2(\Gamma)$  is compact.

**LEMMA 4.3** *If (62) holds, then*

$$(u, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u) \geq \ell^{-1} \|u\|_{L^2(\Gamma)}^2 \quad \text{for all } u \in V_\alpha^2. \quad (65)$$

■

*Proof* Denote by  $\mathcal{P} : L^2(\Gamma) \rightarrow V_\alpha^2$  the orthogonal projection in  $L^2(\Gamma)$  onto the closed subspace  $V_\alpha^2$ . Then we have

$$(u, D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u) = (u, \mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u)_{V_\alpha^2} \quad \text{for all } u \in V_\alpha^2.$$

Notice that, in contrast to  $D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})$ , the operator  $\mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})$  is a mapping in  $V_\alpha^2$ . By a well known property of the spectrum of self-adjoint operators in a Hilbert space (see, e.g., Theorem 2, p.320 in Yosida, 1980) we have

$$\begin{aligned} & \min\{\mu \in \mathbb{R} \mid \mu \in \sigma\} \\ & = \inf\{(u, \mathcal{P} D_{uu}^2 \mathcal{F}(u_{\widehat{h}}, \widehat{h})u)_{V_\alpha^2} \mid u \in V_\alpha^2 \text{ with } \|u\|_{V_\alpha^2} = 1\}, \end{aligned}$$

where  $\sigma$  is the spectrum of

$$\mathcal{P}D_{uu}^2\mathcal{F}(u_{\widehat{h}}, \widehat{h}) := \mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) + \mathcal{P}D_{uu}^2\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) : V_{\alpha}^2 \rightarrow V_{\alpha}^2.$$

Hence, in view of (59), condition (65) will be satisfied if the operator

$$\mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) + (\mathcal{P}D_{uu}^2\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu)\mathcal{J} : V_{\alpha}^2 \rightarrow V_{\alpha}^2 \quad (66)$$

is invertible for all  $\mu \in [0, \ell^{-1})$ ,

where  $\mathcal{J}$  denotes the identity in  $V_{\alpha}^2$ . Note that by (64) the real function

$$(\mathcal{P}D_{uu}^2\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})(x) - \mu)^{-1}$$

is bounded, measurable and positive on  $\Gamma \setminus (I^{\alpha} \cup J^{\alpha})$  for any  $\mu \in [0, \ell^{-1})$ . Multiplying the operator in (66) by this function, we see that this operator is invertible if and only if  $\mathcal{R}_{\mu} + \mathcal{J} : V_{\alpha}^2 \rightarrow V_{\alpha}^2$  is invertible for all  $\mu \in [0, \ell^{-1})$ , where

$$\mathcal{R}_{\mu} := (\mathcal{P}D_{uu}^2\mathcal{H}(y_{\widehat{h}}, u_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu)^{-1}\mathcal{P}\mathcal{K}(u_{\widehat{h}}, \widehat{h}) : V_{\alpha}^2 \rightarrow V_{\alpha}^2 \quad (67)$$

So it remains to prove that the operator  $\mathcal{R}_{\mu} + \mathcal{J}$  is invertible. Note, that in view of compactness of  $\mathcal{K}(u_{\widehat{h}}, \widehat{h})$ , the operator  $\mathcal{R}_{\mu}$  is compact. It follows from the definition of  $\mathcal{R}_{\mu}$  and from (36), (37) that

$$\begin{aligned} \mathcal{R}_{\mu}V_{\alpha}^r &\subset V_{\alpha}^{r+\tau} && \text{for all } r \in [2, n-1], \\ \mathcal{R}_{\mu}V_{\alpha}^r &\subset V_{\alpha}^{\infty} && \text{for all } r > n-1, \end{aligned} \quad (68)$$

where the second property follows from Lemma 2.2. Consider the homogeneous equation in  $V_{\alpha}^2$

$$(\mathcal{R}_{\mu} + \mathcal{J})u = 0. \quad (69)$$

Let us apply in (69) a bootstrapping procedure. Assume that  $u \in V_{\alpha}^2$  is a solution of (69). Then  $u = -\mathcal{R}_{\mu}u$ , and (68) implies  $u \in V_{\alpha}^{2+\tau}$ . Continuing this procedure, after a finite number of steps, we find that  $u = -\mathcal{R}_{\mu}u \in V_{\alpha}^{\infty}$ , which in view of (62), shows that  $u = 0$  is the only solution to (69). By a known property of compact operators (see, e.g., Theorem 2, Chapter XIII, Sec.1 in Kantorovich and Akilov, 1977) the uniqueness of the solution of the homogeneous equation (69) implies that the operator  $(\mathcal{R}_{\mu} + \mathcal{J}) : V_{\alpha}^2 \rightarrow V_{\alpha}^2$  has a bounded inverse. Hence (66) holds and the proof of the lemma is completed.  $\square$

We can formulate now the principal result of this paper, i.e., a characterization of the Lipschitz stability property for the solutions to problems  $(O_h)$ .

**THEOREM 4.4** *If conditions (A1)-(A4) hold, then (AC) is a sufficient condition in order that*

(LC) There exist constants  $\sigma_1 > 0$ ,  $\sigma_2 > 0$  and  $\lambda > 0$  such that for each  $h \in \mathcal{B}_{\sigma_1}^H(h_0)$  there exists a unique in  $\mathcal{B}_{\sigma_2}^{Z^\infty}(y_0, u_0)$  solution  $(y_h, u_h)$  of  $(O_h)$  and

$$\|y_{h'} - y_{h''}\|_{C(\bar{\Omega})}, \|u_{h'} - u_{h''}\|_{L^\infty(\Gamma)} \leq \lambda \|h' - h''\|_H \text{ for all } h', h'' \in \mathcal{B}_{\sigma_1}^H(h_0). \quad (70)$$

If in addition, condition (SD) holds, then (AC) is also necessary for (LC) to be satisfied.  $\square$

*Proof* Sufficiency is given in Theorem 3.7. To prove necessity, note that from (65) we have

$$\begin{aligned} (u, D_{uu}^2 \mathcal{F}(u_0, h_0)u) &= (u, D_{uu}^2 \mathcal{F}(u_{\hat{h}}, \hat{h})u) \\ &+ (u, [D_{uu}^2 \mathcal{F}(u_0, h_0) - D_{uu}^2 \mathcal{F}(u_{\hat{h}}, \hat{h})]u) \geq \ell^{-1} \|u\|_{L^2(\Gamma)}^2 \\ &+ (u, [D_{uu}^2 \mathcal{F}(u_0, h_0) - D_{uu}^2 \mathcal{F}(u_{\hat{h}}, \hat{h})]u) \quad \text{for all } u \in V_\alpha^2. \end{aligned} \quad (71)$$

By (42) and (70), choosing  $\alpha > 0$  and  $\epsilon > 0$  sufficiently small, we get

$$|(u, [D_{uu}^2 \mathcal{F}(u_0, h_0) - D_{uu}^2 \mathcal{F}(u_{\hat{h}}, \hat{h})]u)| \leq \frac{\ell^{-1}}{2} \|u\|_{L^2(\Gamma)}^2. \quad (72)$$

Conditions (71) and (72) show that (AC) is satisfied with  $\gamma = \frac{\ell^{-1}}{2}$ .  $\square$

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