Dedicated to
Professor Jakub Gutenbaum
on his 70th birthday

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Remarks on stability of positive linear systems

by

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Abstract: Spectral properties of nonnegative matrices are considered. Asymptotic stability and stabilisation problems of positive discrete-time and continuous-time linear systems by feedbacks are discussed. The electric RC-networks are presented as examples of positive systems. Numerical calculations were made using the MATLAB program.

Keywords: nonnegative matrices, positive linear systems, asymptotic stability.

1. Introduction

1.1. Nonnegative matrix

Let $A = [a_{ij}]$ be a real matrix. A real matrix is nonnegative if and only if $a_{ij} \geq 0$. For nonnegative A we will write $A \geq 0$. A vector with nonnegative real components is called nonnegative vector. In this case we will write $x \geq 0$.

Let $\lambda(A)$ be the spectrum of a square matrix A. Let $\lambda_i(A) \in \lambda(A)$ be an eigenvalue of A. The spectral radius of a matrix A is denote by $\rho(A) = \max_i |\lambda_i(A)|$ and the growth constant of A is respectively denoted by $\alpha(A) = \max_i Re\lambda_i(A)$.

Remark 1. For any matrix $A \geq 0$ there exists a real number $\lambda_{max} \in \lambda(A)$ such that $\lambda_{max} = \rho(A)$. See for example Gantmacher (1988, p. 334, 344).

Remark 2. Let $\eta \in R$ and $A = [a_{ij}] \geq 0$. A real number η is greater than the maximal eigenvalue $\lambda_{max} = \rho(A)$ of nonnegative matrix A, i.e. $\lambda_{max} = \rho(A) < \eta$, iff all principal minors of matrix $\eta \cdot I - A$ are greater than zero, i.e. $M_i[\eta \cdot I - A] > 0, i = 1, 2, ..., n$, where $M_1[\eta \cdot I - A] = \eta - a_{11}$, $M_2[\eta \cdot I - A] = \det \begin{bmatrix} \eta - a_{11} & -a_{12} \\ -a_{21} & \eta - a_{22} \end{bmatrix}$, ..., $M_n = \det[\eta \cdot I - A]$. See for example Gantmacher (1988, p. 349).

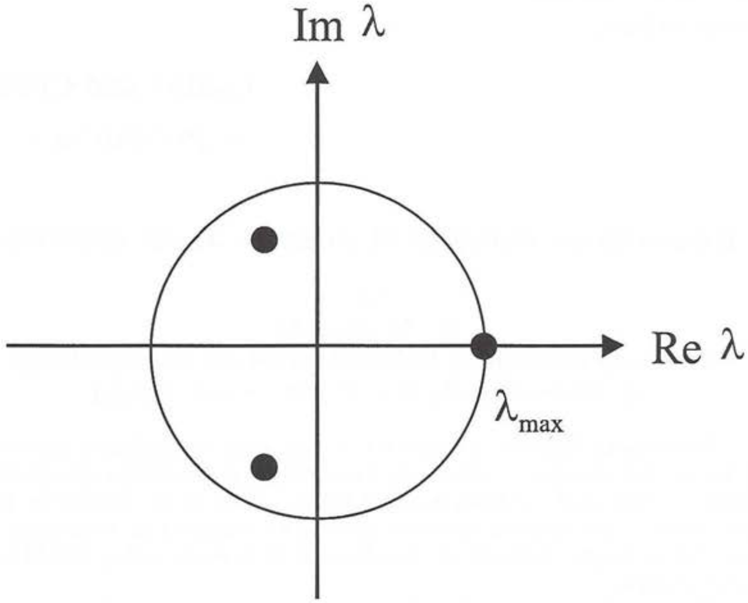

Figure 1. Spectrum of a nonnegative matrix

Example 1. Consider the nonnegative matrix (Frobenius matrix)

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/16 & 1/16 & 1/8 \end{bmatrix} \geq 0 \,. \tag{1}$$

In this case $\lambda(F) = \{-0.1875 + 0.2997i, -0.1875 + 0.2997i, 0.5\}$, $|\lambda(F)| = \{|\lambda_1(F)|, |\lambda_2(F)|, |\lambda_3(F)|\} = \{0.3536, 0.3536, 0.5\}$ and $\lambda_{max} = \rho(F) = 0.5$ (see Fig.1).

Remark 3. $G = [g_{ij}] \leq H = [h_{ij}]$ if and only if $g_{ij} \leq h_{ij}$. If $G \geq 0$, $H \geq 0$ and $G \leq H$ ($G \neq H$), then $\rho(G) \leq \rho(H)$. See for example Gantmacher (1988, p. 335 and 350).

1.2. The Metzler matrix

A matrix $A = [a_{ij}] \in R^{n \times n}$ is called the Metzler matrix if its all off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0, i \neq j$.

Remark 4. It has been shown (Minc, 1988; Kaczorek, 1997) that $e^{At} \geq 0$ iff $A \in R^{n \times n}$ is a Metzler matrix.

Example 2. The matrix $M = F - I$, where F is given by (1), is a Metzler matrix. In this case we have

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1/16 & 1/16 & -7/8 \end{bmatrix} \tag{2}$$

and $\lambda(M) = \{-1.1875 + 0.2997i, -1.1875 + 0.2997i, -0.5\}$. Let $e^{Mt} = [e_{ij}(t)]$. For M given by (2) we obtain

$$e_{11}(t) = \frac{2}{9}e^{-\frac{t}{2}} + e^{-\frac{19t}{16}}[\frac{5\sqrt{23}}{207}\sin(t\sqrt{23}/16) + \frac{7}{9}\cos(t\sqrt{23}/16)]$$

$$e_{21}(t) = \frac{1}{9}e^{-\frac{t}{2}} - e^{-\frac{19t}{16}}[\frac{11\sqrt{23}}{207}\sin(t\sqrt{23}/16) + \frac{1}{9}\cos(t\sqrt{23}/16)]$$

$$e_{31}(t) = \frac{1}{18}e^{-\frac{t}{2}} + e^{-\frac{19t}{16}}[\frac{7\sqrt{23}}{414}\sin(t\sqrt{23}/16) - \frac{1}{18}\cos(t\sqrt{23}/16)]$$

$$e_{12}(t) = \frac{2}{3}e^{-\frac{t}{2}} + e^{-\frac{19t}{16}}[\frac{26\sqrt{23}}{69}\sin(t\sqrt{23}/16) - \frac{2}{3}\cos(t\sqrt{23}/16)]$$

$$e_{22}(t) = \frac{1}{3}e^{-\frac{t}{2}} - e^{-\frac{19t}{16}}[\frac{2\sqrt{23}}{69}\sin(t\sqrt{23}/16) - \frac{2}{3}\cos(t\sqrt{23}/16)]$$

$$e_{32}(t) = \frac{1}{6}e^{-\frac{t}{2}} - e^{-\frac{19t}{16}}[\frac{5\sqrt{23}}{138}\sin(t\sqrt{23}/16) + \frac{1}{6}\cos(t\sqrt{23}/16)]$$

$$e_{13}(t) = \frac{16}{9}e^{-\frac{t}{2}} - e^{-\frac{19t}{16}}[\frac{176\sqrt{23}}{207}\sin(t\sqrt{23}/16) + \frac{16}{9}\cos(t\sqrt{23}/16)]$$

$$e_{23}(t) = \frac{8}{9}e^{-\frac{t}{2}} + e^{-\frac{19t}{16}}[\frac{56\sqrt{23}}{207}\sin(t\sqrt{23}/16) - \frac{8}{9}\cos(t\sqrt{23}/16)]$$

$$e_{33}(t) = \frac{4}{9}e^{-\frac{t}{2}} + e^{-\frac{19t}{16}}[\frac{\sqrt{23}}{207}\sin(t\sqrt{23}/16) + \frac{5}{9}\cos(t\sqrt{23}/16)]$$

It is easy to show that the elements of matrix $e^{Mt} = [e_{ij}(t)]$ are nonnegative, i.e. $e_{ij}(t) \geq 0$.

LEMMA 1 *For any Metzler matrix M there exists a real number $\lambda_{max} \in \lambda(M)$ such that $\lambda_{max} = \alpha(M)$, where $\alpha(M) = \max_i Re\lambda_i(M)$, $i = 1, 2, ..., n$, is the growth constant of M.*

Proof. For every matrix M there exists a real number $\eta \geq 0$ such that matrix $\eta \cdot I + M = A$ is nonnegative. Let $s \in \lambda(A)$. Then $s - \eta = \lambda \in \lambda(M)$ and (see Remark 1) real number $\rho(A) - \eta = \alpha(M) \in \lambda(M)$. ■

Consequently, from Remark 3 and Lemma 1 we have the following theorem:

THEOREM 1 *Let M_{min}, M, M_{max} be Metzler matrices. If $M_{min} \leq M \leq M_{max}$, then $\alpha(M_{min}) \leq \alpha(M) \leq \alpha(M_{max})$.*

2. Asymptotic stability of positive linear systems

2.1. The discrete-time system

Let us consider a discrete-time linear system described by equations

$$\begin{aligned} & x(k+1) = Ax(k) + Bu(k), \quad x(0) \in R^n, \\ & y(k) = Cx(k), \quad y(k) \in R^m, \quad u(k) \in R^r, \\ & k = 0, 1, 2, ... \end{aligned} \tag{3}$$

where $x(k)$, $u(k)$ and $y(k)$ are the state vector, input vector and output vector respectively, A, B, C are real matrices of appropriate dimensions.

The system (3) is called positive if for any $x(0) \geq 0$ and $u(k) \geq 0$ we have $x(k) \geq 0$ and $y(k) \geq 0$ for $k > 0$. The discrete-time system (3) is positive if and only if $A \geq 0, B \geq 0, C \geq 0$ (see Kaczorek, 1997, p. 35).

Remark 5. It is well known that the system (3) is asymptotically stable if and only if $\rho(A) = \max_i |\lambda_i(A)| < 1$, where $\rho(A)$ is the spectral radius of A . For $A \geq 0$ $\rho(A) < 1$ if and only if $M_i[I - A] > 0, \ \ i = 1, 2, ..., n$ (see Remark 2 with $\eta = 1$).

Example 3. For F given by (1) we have

$$I - F = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -0.0625 & -0.0625 & 0.875 \end{bmatrix} \tag{4}$$

and $M_1[I - F] = 1 > 0, \ M_2[I - F] = 1 > 0, \ M_3[I - F] = 0.75 > 0$. Thus $|\lambda_i(F)| < 1, \ \ i = 1, 2, 3$ (see Example 1).

2.2. The continuous-time system

Let us consider a continuous-time linear system described by the equations

$$\begin{aligned} & \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in R^n, \quad t \geq 0, \\ & y(t) = Cx(t), \quad y(t) \in R^m, \quad u(t) \in R^r, \end{aligned} \tag{5}$$

where $x(t)$, $u(t)$ and $y(t)$ are the state vector, input vector and output vector respectively, A, B, C are real matrices of appropriate dimensions.

The system (5) is called positive if for any $x(0) \geq 0$ and $u(t) \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for $t > 0$. The continuous-time system (5) is positive iff A is a Metzler matrix and $B \geq 0, C \geq 0$ (see Kaczorek, 1997, p. 35).

Remark 6. It is well known that the system (5) is asymptotically stable iff $Re\lambda_i(A) < 0, \ \ i = 1, 2, ..., n$. When A is a Metzler matrix $Re\lambda_i(A) < 0$ iff $M_i[-A] > 0, \ \ i = 1, 2, ..., n$ (there exists a real number η such that matrix $\eta \cdot I + A$ is nonnegative matrix and see Remark 2 with $\eta = 0$). We can notice that $M_i[-A] > 0, \ \ i = 1, 2, ..., n$ if and only if $M_1[A] < 0, \ M_2[A] > 0, ..., (-1^n)M_n[A] > 0$ (Turowicz, 1995, p. 195). We can also notice that $Re\lambda_i(A) < 0$ if $\alpha(A) < 0$.

Example 4. For M given in (2) we have $M_1[M] = -1 < 0,\ M_2[M] = 1 > 0,\ M_3[M] = -0.75 < 0$. Thus $Re\lambda_i(M) < 0,\ i = 1, 2, 3$ (see Example 2), i.e. M given by (2) is an asymptotically stable Metzler matrix.

LEMMA 2 *Let $A = [a_{ij}]$ be a Metzler matrix, i.e. $a_{ij} \geq 0,\ i \neq j$. Let* $\det[\lambda \cdot I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$. *In this case $\alpha(A) < 0$ (or $Re\lambda_i(A) < 0,\ i = 1, 2, ..., n$) iff $a_i > 0,\ i = 0, 1, 2, ..., n-1$ (Kaczorek, 1999 B, p. 18).*

Proof. If $Re\lambda_i(A) < 0,\ i = 1, 2, ..., n$, then it is clear that $a_i > 0,\ i = 0, 1, 2, ..., n-1$. Now let $a_i > 0,\ i = 0, 1, 2, ..., n-1$. If A is a Metzler matrix, then the real number $\alpha(A) = \max_i Re\lambda_i(A),\ i = 1, 2, ..., n$, is an eigenvalue of A . For $a_i > 0$ and real $\lambda \geq 0$ the characteristic polynomial $\det[\lambda \cdot I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 > 0$. Thus, the characteristic polynomial has no real nonnegative roots. But a Metzler matrix has a real eigenvalue $\alpha(A)$. Consequently, we have $\alpha(A) < 0$.

This proof is similar to Kaczorek's (1999b, p. 18), but it does not assume that a Metzler matrix has real eigenvalues only.

3. Examples of positive electric RC-networks

3.1. Discrete electric RC-network

Let us consider an electric RC-network (see Kaczorek, 1997, p. 34) shown in Fig. 2. The parameters of the network, $R_i > 0$ and $C_i > 0$, are known. Let $y(t) = x_1(t) + x_2(t)$. The system shown in Fig. 2 is described by equations (5), where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, and for $R_1 = 1,\ R_2 = 1,\ C_1 = 1,\ C_2 = 1$, we get

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 0.5 & -1 \end{bmatrix},\ B = \frac{1}{3}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{6}$$

It is obvious that the system (5), (6) is positive because A is the Metzler matrix and $B \geq 0, C \geq 0$.

Let $t = kh,\ h > 0,\ k = 0, 1, 2, 3, ...$ and $y(t)$ in (5) be approximated by $y(k)$, and $u(t)$ in (5) for $t \in [kh, (k+1)h)$ be approximated by $u(k)$. Then, from (5) we obtain (3), where (see for example Mitkowski, 1991, p. 141)

$$A := e^{Ah},\quad B := \int_0^h e^{At} B dt,\quad C := C. \tag{7}$$

For matrices given by (6) we have

$$e^{Ah} = \frac{1}{2\sqrt{3}}\begin{bmatrix} e_{11}(h) & e_{12}(h) \\ e_{21}(h) & e_{22}(h) \end{bmatrix}, \tag{8}$$

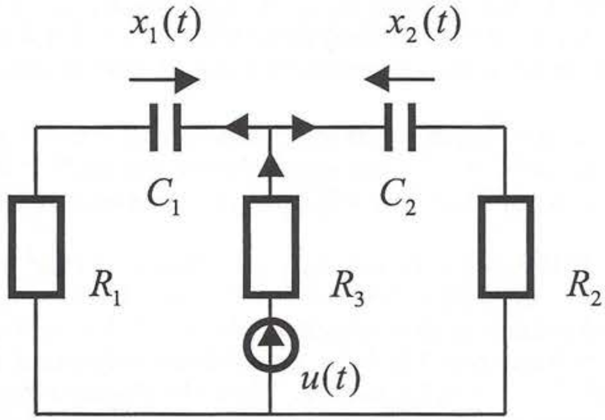

Figure 2. Positive RC-network

where $e_{11}(h) = (\sqrt{3}-1)e^{\lambda_1 h} + (\sqrt{3}+1)e^{\lambda_2 h}$, $e_{12}(h) = 2(e^{\lambda_1 h} - e^{\lambda_2 h})$,
$e_{21}(h) = e^{\lambda_1 h} - e^{\lambda_2 h}$, $e_{22}(h) = (\sqrt{3}+1)e^{\lambda_1 h} + (\sqrt{3}-1)e^{\lambda_2 h}$,
$\lambda_1 = (-3+\sqrt{3})/6$, $\lambda_2 = (-3-\sqrt{3})/6$ and

$$\int_0^h e^{At} B dt = \frac{1}{6}\begin{bmatrix} (3+\sqrt{3})(1-e^{\lambda_1 h}) - (3-\sqrt{3})(1-e^{\lambda_2 h}) \\ (2\sqrt{3}+3)(1-e^{\lambda_1 h}) - (2\sqrt{3}-3)(1-e^{\lambda_2 h}) \end{bmatrix}. \tag{9}$$

Example 5. For $h = 1$ from (8) and (9) with (6) we obtain positive system (3), where

$$A = \begin{bmatrix} 0.5295 & 0.2050 \\ 0.1025 & 0.7345 \end{bmatrix}, \; B = \begin{bmatrix} 0.0349 \\ 0.1630 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{10}$$

and $\lambda(A) = \{0.4544, 0.8095\}$. Let us consider also the scalar linear feedback

$$u(k) = Ky(k), \quad K \in R. \tag{11}$$

The matrix of the closed-loop system (3), (10), (11) is given by

$$A + BKC = \begin{bmatrix} 0.5295 & 0.2050 \\ 0.1025 & 0.7345 \end{bmatrix} + K \begin{bmatrix} 0.0349 & 0.0349 \\ 0.1630 & 0.1630 \end{bmatrix}. \tag{12}$$

The matrix (12) is positive if and only if $K \geq -0.6288$. $M_1[I - A - BKC] > 0 \Leftrightarrow K < 13.4813$ and $M_2[I - A - BKC] > 0 \Leftrightarrow K < 0.8451$. Consequently, from Remark 5 we obtain the following condition of asymptotic stability

$$A + BKC \geq 0 \text{ and } \rho(A + BKC) < 1 \Leftrightarrow K \in [-0.6288, 0.8451). \tag{13}$$

3.2. Electric RC-ladder network

Let us consider an electric RC-ladder network shown in Fig. 3 (for $n = 3$). The parameters of the network, $R_i > 0$ and $C_i > 0$, are known. The system shown in Fig. 3 is described by the equations (Mitkowski, 1997)

$$\dot{x}_i(t) = a_i x_{i-1}(t) + b_i x_i(t) + c_i x_{i+1}(t), \ \ i = 1, 2, 3, ..., n \tag{14}$$

where $x_0(t) = 0, \ \ x_{n+1}(t) = 0$ and

$$a_i = \frac{1}{R_i C_i}, \ \ c_i = \frac{1}{R_{i+1} C_i}, \ \ b_i = -(a_i + c_i), \ \ i = 1, 2, 3, ..., n. \tag{15}$$

The RC-ladder system can be described by the equation

$$\dot{x}(t) = Ax(t), \ \ x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \end{bmatrix}^T, \tag{16}$$

where A is the $n \times n$ tridiagonal real Jacobi matrix with parameters given in (15). For $n = 5$ A is given by

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{bmatrix} \tag{17}$$

or

$$\begin{aligned} & A = diag(\tfrac{1}{C_1}, \tfrac{1}{C_2}, \tfrac{1}{C_3}, \tfrac{1}{C_4}, \tfrac{1}{C_5}) \cdot M, \\ & M = -L \cdot diag(r_1, r_2, r_3, r_4, r_5) \cdot L^T, \end{aligned} \tag{18}$$

where

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \ \ r_i = \frac{1}{R_i}, \ \ i = 1, 2, 3, ..., n.$$

If $a_{i+1} c_i > 0$ for $i = 1, 2, 3, ..., n-1$, then tridiagonal matrix is called Jacobi matrix. The tridiagonal real Jacobi matrix has only single real eigenvalues (Ilin and Kuznyetsow, 1985, p. 83, p. 104). The matrix A is diagonalizable. The Jordan canonical form of A is $J = \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are real eigenvalues of A . From the Gershgorin's criterion (Gantmacher, 1988, p. 390) $\lambda_k \in [-m, 0]$, $k = 1, 2, ..., n$, $m = \max_k(|a_k| + |c_k|)$. From (15) $a_i > 0$ and $c_i > 0$. $Rank L = n$ (rank of matrix L), $R_i > 0$ and $C_i > 0$. Consequently, (see (18)) we have $\det A \neq 0$ ($Rank A = n$). Thus $\lambda_k \in [-m, 0)$, $k = 1, 2, ..., n$.

Example 6. Consider the tridiagonal real Jacobi matrix A with $a_i = a, \ b_i = b, \ c_i = c$. In this case we have analytic formula for eigenvalues of

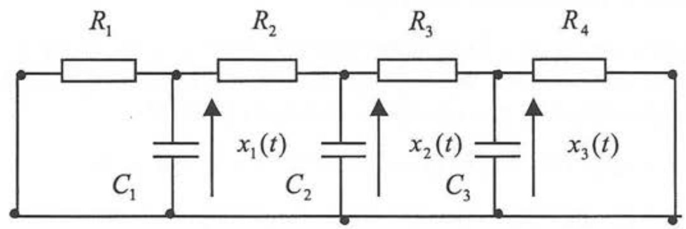

Figure 3. Positive RC-ladder network.

in the following form (Bellman, 1960, p. 215; Ilin and Kuznyetsow, 1985, p. 159; Lancaster, 1969, p. 104):

$$\lambda_k = b + 2\sqrt{ac}\cos\varphi_k, \quad \varphi_k = \frac{k\pi}{n+1}, \tag{19}$$

$k = 1, 2, 3, ..., n$.

Remark 7. The tridiagonal real matrix A (see (17)) with (15) is a Metzler matrix.

THEOREM 2 *Consider the Jacobi matrix A (see (17)) with $a_i > 0$ and $c_i > 0$. Let $a_{max} = \max_i a_i$, $a_{min} = \min_i a_i$, $i = 2, 3, ..., n$, $b_{max} = \max_i b_i$, $b_{min} = \min_i b_i$, $i = 1, 2, ..., n$, $c_{max} = \max_i c_i$, $c_{min} = \min_i c_i$, $i = 1, 2, ..., n-1$. Then*

$$b_{min} + 2\sqrt{a_{min}c_{min}}\cos\frac{\pi}{n+1} \le \alpha(A) \le b_{max} + 2\sqrt{a_{max}c_{max}}\cos\frac{\pi}{n+1}. \tag{20}$$

Proof. Using Theorem 1 and (19) we obtain (20). ■

Example 7. Let us consider an electric RC-ladder network shown in Fig. 3. Let $R_1 = 1$, $R_2 = 0.9$, $R_3 = 1.1$, $R_4 = 0.8$, $C_1 = 1$, $C_2 = 1$, $C_3 = 1$. In this case we obtain

$$A = \begin{bmatrix} -2.1111 & 1.1111 & 0.0000 \\ 1.1111 & -2.0202 & 0.9091 \\ 0.0000 & 0.9091 & -2.1591 \end{bmatrix}, \tag{21}$$

$$A_{min} = \begin{bmatrix} -2.1591 & 0.9091 & 0.0000 \\ 0.9091 & -2.1591 & 0.9091 \\ 0.0000 & 0.9091 & -2.1591 \end{bmatrix},$$

$$A_{max} = \begin{bmatrix} -2.0202 & 1.1111 & 0.0000 \\ 1.1111 & -2.0202 & 1.1111 \\ 0.0000 & 1.1111 & -2.0202 \end{bmatrix}. \tag{22}$$

From numerical calculations we obtain $\lambda(A) = \{-3.5122, -2.1398, -0.6384\}$ and $\alpha(A) = -0.6384$. Using inequalities (20) for eigenvalues of the matrices (22) we obtain the following estimation: $\alpha(A) \in [-0.8734, -0.4489]$.

4. Positive Frobenius systems

Consider the system (3) or (5) with $A = F$, $B = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T$, $C = 0$, where F is a real Frobenius matrix. For $n = 3$ we have

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \text{ and } \det[\lambda \cdot I - F] = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \quad (23)$$

Let us consider the scalar linear state-feedback $u = Kx$, $K = \begin{bmatrix} k_0 & k_1 & \dots & k_{n-1} \end{bmatrix}$. The matrix of the closed-loop system (for $n = 3$) is given by

$$F + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_0 - a_0 & k_1 - a_1 & k_2 - a_2 \end{bmatrix}. \quad (24)$$

It is clear that

$$\det[\lambda \cdot I - F - BK] = \lambda^3 + (a_2 - k_2)\lambda^2 + (a_1 - k_1)\lambda + a_0 - k_0. \quad (25)$$

Remark 8. For any F there exists a real matrix $K = \begin{bmatrix} k_0 & k_1 & \dots & k_{n-1} \end{bmatrix}$ such that $\rho(F + BK) < 1$ and $F + BK \geq 0$. Particularly, for $k_i = a_i$ we have $\lambda_i(A) = 0$ (see (24)). But a real matrix $K = \begin{bmatrix} k_0 & k_1 & \dots & k_{n-1} \end{bmatrix}$, such that $Re\lambda(F + BK) < 0$ and $F + BK \geq 0$, does not exist .

Remark 9. If $a_i < 0$, then matrix $F \geq 0$ (see (23)). In this case from the Descartes criterion the matrix $F \geq 0$ has only one real eigenvalue $\lambda_{max} > 0$ (Turowicz, 1967, p. 37). It is clear that $\rho(F) = \lambda_{max}$.

Example 8. For $n = 5$ we have

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 10 & 1 & 15 & 1 \end{bmatrix}$$

and $\lambda(F) = \{4.4972, 0.0366 + 0.8015i, 0.0366 - 0.8015i, -0.0995, -3.4709\}$.

5. Concluding remarks

The finite dimensional positive discrete-time and continuous-time linear systems were considered. The effective conditions of asymptotic stability were given. The results obtained in Section 3 can be extended to the nonuniform ladder networks of the RL and GC-type. The positive ladder networks can be applied in approximation of some positive distributed parameter systems (Schanbacher, 1989). A very interesting method for stabilization of positive linear systems by state-feedback with single input was considered in Kaczorek (1999a). The Kaczorek's method is based on Gersgorin's theorem and quadratic programming. The reachability and controllability of positive systems were considered in papers by Klamka (1998) and Kaczorek (1999c).

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