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Professor Jakub Gutenbaum
on his 70th birthday

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Fictitious domain approach for numerical solution of elliptic problems

by

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Abstract: In the paper the numerical aspects of the fictitious domain method for elliptic problems are considered. Theoretical results concerning the equivalence of original and embedded domain elliptic problems as well as the convergence of discretization are recalled. The dependence of accuracy of numerical solutions to elliptic problems on the approximation of boundary conditions as well as on the order of shape functions are disscussed. The numerical examples are provided.

Keywords: fictitious domain method, elliptic problems, numerical methods, finite element method.

1. Introduction

The paper deals with the numerical solution of elliptic boundary value problems with mixed boundary conditions by the fictitious domain method. The elliptic boundary value problem is formulated in a bounded domain with Lipschitz continuous boundary. The aim of the paper is to investigate the accuracy of numerical solutions to elliptic boundary value problems obtained by employing fictitious domain method with respect to approximation functions order.

The fictitious domain method for solving the systems described by partial differential equations consists, see Ernst (1996), Glowinski, Pan, Periaux (1994a, b), Glowinski, Pan (1996), Nečas (1967), Peichl, Kunisch (1995), in transforming the original system defined in the complicated geometry domain into a new system defined in a given fixed simple geometry domain containing the original domain with the same differential operator. This method allows to use fairly structured regular meshes on a simple geometry domain containing the actual one. Fast elliptic solvers can be used to solve the transformed problem. The solution of the state equation in the fictitious domain is enforced to satisfy the original boundary conditions. Embedding domain methods for solving elliptic equations with Dirichlet boundary conditions were investigated in Glowinski, Pan, Periaux (1994a, b). The Neumann problem was investigated in Glowinski, Pan (1996), where a penalty approach was employed to impose original boundary conditions and the numerical results were provided.

Our work is motivated by applications of fictitious domain method in numerical solving of optimal shape design problems, see Chambolle, Doveri (1996), Fancello, Haslinger, Feijoo (1993), Haslinger (1993), Haslinger, Hoffmann, Kocvara (1993), Haslinger, Neittaanmaki (1988), Haslinger, Klabring (1995), Neittanmaki, Tiba (1995), Peichl, Kunisch (1995). In these problems the domain where the elliptic boundary value problem is formulated, is a variable subject to optimization. Another field of applications concerns the numerical solution of topological optimization problems, Sokołowski, Żochowski (1999), where the sensitivity of solutions to the elliptic state problem with respect to the variations of small holes or inclusions inside the domain has to be calculated.

In classical approaches to solving these shape optimization problems, the state problem, described by the elliptic boundary value problem, is solved many times on the domain which changes during the computation. The boundary or domain variation methods require calculation of a new discretization of the optimized domain, updating the stiffness matrix and the load vector at each iteration of the numerical algorithm. Since the optimized domain has usually a complicated geometrical structure, the whole computational process is tedious, time consuming, and expensive. To overcome this difficulty, in response to the growing number of industrial applications of the optimal shape design problems, fixed domain methods for solving these problems are being developed. Fixed domain methods are based on using the fictitious or embedding domain method.

The application of the fictitious domain method in solving the optimal shape design problems leads to nonsmooth problems, Dankova, Haslinger (1996), Grisvard (1992), and low accuracy of optimal solutions. In order to improve the accuracy of obtained optimal solution one has to improve the accuracy of solution to the state problem by employing higher order elements or wavelets, see Glowinski, Pan (1996), Peichl, Kunisch (1995).

The aim of this work is to investigate the accuracy of numerical solutions to the elliptic problem for finite element approximations. In this work, by using fictitious domain approach, we shall numerically solve the model Laplace equation with Dirichlet and Neumann boundary conditions. We shall formulate the problem in the fictitious domain and we shall show that the solution to the fictitious domain problem is also the solution to the original problem. The finite-dimensional model is introduced and the results concerning the convergence of the finite-dimensional approximation are recalled. Numerical procedures for solving Dirichlet and Neumann problems are proposed. Numerical solutions to both problems, using bilinear and bicubic test functions in the finite - dimensional formulation, are calculated and their accuracy is discussed.

1.1. Model problem formulation

Let Ω be a bounded domain in R^2 with piecewise C^2 continuous boundary Γ. The boundary Γ consists of two parts, Γ_D and Γ_N. Consider the model Poisson problem:

$$\triangle u = -f \quad \text{in } \Omega, \tag{1}$$

with boundary conditions

$$u = g \quad \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = \phi \quad \text{on } \Gamma_N. \tag{2}$$

Let us introduce a space and a set :

$$V_0 = \{z \in H^1(\Omega) \ : \ z = 0 \ \text{ on } \Gamma_D\}, \quad V_1 = \{z \in H^1(\Omega) \ : \ z = g \ \text{ on } \Gamma_D\}. \tag{3}$$

Let $f \in L^2(\Omega)$, $g \in H^{3/2}(\Gamma_D)$, $\phi \in H^{1/2}(\Gamma_N)$, be given. The problem (1)–(2) has the following variational formulation Nečas (1967): find $u \in V_1$ such that,

$$\int_\Omega \nabla u \nabla \varphi dx = \int_\Omega f \varphi dx + \int_{\Gamma_N} \phi \varphi ds, \quad \forall \varphi \in V_0. \tag{4}$$

For $\Gamma_N = \emptyset$ we have a pure Dirichlet problem, for $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$ it is the mixed problem Aubin (1979), Grisvard (1985, 1992), Nečas (1967). Together with conditions concerning regularity of boundary data, we shall make some additional assumptions about the domain:

1. Each of the parts Γ_N and Γ_D consists of finite number of C^2 arcs,

$$\Gamma_N = \bigcup_{j \in \mathcal{N}} \Gamma_j, \qquad \Gamma_D = \bigcup_{j \in \mathcal{D}} \Gamma_j,$$

   where $\mathcal{N}$ (respectively $\mathcal{D}$) denotes the set of indices j for which the Neumann (respectively Dirichlet) boundary condition is prescribed on arch Γ_j of the boundary Γ.

2. The neighbouring arcs Γ_i, Γ_j make an internal angle ω satisfying

$$0 < \omega < 2\Pi \quad \text{if} \quad i, j \in \mathcal{N} \quad \text{or} \quad \rangle, | \in \mathcal{D},$$
$$0 < \omega < \Pi \quad \text{if} \quad i \in \mathcal{N} \quad \text{and} \quad | \in \mathcal{D}.$$

Then there exists a unique variational solution to (4) in V_1. Moreover, this solution has higher regularity, Grisvard (1985), namely $u \in H^{3/2+\varepsilon}(\Omega)$, for some, possibly small, $\varepsilon > 0$.

2. Fictitious domain formulation

Let us denote by $\hat{\Omega}$ a bounded domain containing Ω, i.e., $\Omega \subset \hat{\Omega}$. By $\hat{\Gamma}$ we denote the Lipschitz continuous boundary of domain $\hat{\Omega}$, $\tilde{f}$ denotes the extension by zero of function f, i.e., $\tilde{f} = f$ in the domain Ω, and $\tilde{f} = 0$ in the domain $\Omega^c = \hat{\Omega} \setminus \Omega$. Finally, $\hat{V}_0 = H_0^1(\hat{\Omega})$.

2.1. Dirichlet boundary condition

Assume $\Gamma_N = \emptyset$. We shall consider the following problem in the domain $\hat{\Omega}$: find $\hat{u}$ satisfying

$$\triangle \hat{u} = -\tilde{f} \quad \text{in } \hat{\Omega}, \tag{5}$$

with boundary conditions:

$$\hat{u} = 0 \quad \text{on } \hat{\Gamma}, \tag{6}$$

$$\hat{u}_{|\Gamma_D} = g \quad \text{on } \Gamma_D . \tag{7}$$

This can be written in the variational form : find $(\hat{u}, \lambda_D) \in \hat{V}_0 \times H^{-1/2}(\Gamma_D)$ such that,

$$\int_{\hat{\Omega}} \nabla \hat{u} \nabla \hat{\varphi} dx - \int_{\hat{\Omega}} \tilde{f} \hat{\varphi} dx + \int_{\Gamma_D} \lambda_D \hat{\varphi}_{|\Gamma_D} ds = 0 \quad \forall \hat{\varphi} \in \hat{V}_0, \tag{8}$$

$$\int_{\Gamma_D} \mu(\hat{u}_{|\Gamma_D} - g) ds = 0 \quad \forall \mu \in H^{-1/2}(\Gamma_D). \tag{9}$$

LEMMA 1 *There exists a unique solution $(\hat{u}, \lambda_D) \in \hat{V}_0 \times H^{-1/2}(\Gamma_D)$ to the system (8)-(9).*

Proof. Define the functional $J : \hat{V}_0 \to R$,

$$J(z) = \frac{1}{2} \int_{\hat{\Omega}} \nabla z \nabla z dx - \int_{\hat{\Omega}} \tilde{f} z dx, \tag{10}$$

and a set

$$K = \{ z \in \hat{V}_0 \mid z = g \quad \text{on } \Gamma_D \}. \tag{11}$$

Since the functional (10) is strictly convex and the set K is a closed and convex subset of $\hat{V}_0$, there exists a unique element $\hat{u} \in \hat{V}_0$ satisfying

$$J(\hat{u}) \le J(z) \quad \forall z \in K.$$

From surjectivity of the trace operator on boundary Γ, Aubin (1979), Nečas (1967), follows the existence of a Lagrange multiplier λ_D ensuring that the Lagrangian L,

$$L(\hat{u}, \lambda_D) = J(\hat{u}) + \int_{\Gamma_D} \lambda_D(\hat{u} - g)ds. \tag{12}$$

is stationary at $\hat{u}$. Hence $(\hat{u}, \lambda_D)$ solves (8)–(9). From uniqueness of $\hat{u}$, surjectivity of the trace operator and the equation (8) follows the uniqueness of λ_D. ∎

LEMMA 2 *Let $(\hat{u}, \lambda_D) \in \hat{V}_0 \times H^{-1/2}(\Gamma_D)$ be a solution to problem (8)–(9). Then $u = \hat{u}_{|\Omega}$ is a solution to the Dirichlet problem.*

Proof. Let $u_1 \in V_1$ be a solution to a problem:

$$\int_\Omega \nabla u_1 \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in V_0. \tag{13}$$

Since $f \in L^2(\Omega)$, the normal derivative $\partial u_1/\partial n \in H^{-1/2}(\Gamma_D)$, and the following Green formula holds:

$$\int_\Omega \nabla u_1 \nabla \varphi dx = \int_\Omega f \varphi dx + \int_{\Gamma_D} \frac{\partial u_1}{\partial n} \varphi ds \quad \forall \varphi \in H^1(\Omega). \tag{14}$$

Hence,

$$-\int_{\Gamma_D} \frac{\partial u_1}{\partial n} \varphi ds = \int_\Omega f \varphi dx - \int_\Omega \nabla u_1 \nabla \varphi dx. \tag{15}$$

Let us introduce a space and a set:

$$\bar{V}_0 = \{\varphi \in H^1(\Omega^c) \mid \varphi = 0 \text{ on } \partial\hat{\Omega}\}, \quad \bar{V}_1 = \{\varphi \in \bar{V}_0 \mid \varphi = g \text{ on } \Gamma_D \}. \tag{16}$$

Let $u_2 \in \hat{V}_1$ be given. Since there exists a continuous extension mapping T from Ω^c on Ω the formula

$$\varphi \to \int_{\Omega^c} \nabla u_2 \nabla \varphi dx + \int_{\Gamma_D} \frac{\partial u_1}{\partial n} \varphi ds, \quad \forall \varphi \in \bar{V}_0, \tag{17}$$

defines the linear continuous functional on $\bar{V}_0$, i.e., there exists $\lambda_D \in H^{-1/2}(\Gamma_D)$ such that,

$$\int_{\Gamma_D} \lambda_D \varphi ds = \int_{\Omega^c} \nabla u_2 \nabla \varphi dx + \int_{\Gamma_D} \frac{\partial u_1}{\partial n} \varphi ds \quad \forall \varphi \in \bar{V}_0, \tag{18}$$

and λ_D is bounded in $H^{-1/2}(\Gamma_D)$ norm. From (15) and (18) we have:

$$\int_\Omega \nabla u_1 \nabla \varphi dx + \int_{\Omega^c} \nabla u_2 \nabla \varphi dx = \int_\Omega f \varphi dx + \int_{\Gamma_D} \lambda_D \varphi ds \quad \forall \varphi \in \bar{V}_0. \tag{19}$$

Since $\int_{\Gamma_D} \lambda_D \varphi ds = 0$ for all $\varphi \in V_0$, (19) holds for all $\varphi \in \hat{V}$. Let $\hat{u} = u_1$ on Ω and $\hat{u} = u_2$ on Ω^c. Then (19) is equivalent to

$$\int_{\hat{\Omega}} \nabla \hat{u} \nabla \varphi dx = \int_{\hat{\Omega}} f \varphi dx + \int_{\Gamma_D} \lambda_D \varphi ds \quad \forall \varphi \in \hat{V}. \tag{20}$$

■

2.2. Mixed boundary condition

We shall consider the following problem in the domain $\hat{\Omega}$: find $\hat{u}$ satisfying

$$\triangle \hat{u} = -\tilde{f} \quad \text{in } \hat{\Omega}, \tag{21}$$

with boundary conditions

$$\hat{u} = 0 \quad \text{on } \hat{\Gamma}, \tag{22}$$

$$\hat{u}_{|\Gamma_D} = g \quad \text{on } \Gamma_D, \quad \frac{\partial \hat{u}}{\partial n}\,|_{\Gamma_N} = \phi \quad \text{on } \Gamma_N. \tag{23}$$

Let

$$U = \{h \in H^1(\Gamma_N) \mid h = g \text{ on } \bar{\Gamma}_N \cap \bar{\Gamma}_D \text{ if } \bar{\Gamma}_N \cap \bar{\Gamma}_D \neq \emptyset\}.$$

Instead of solving the mixed problem (21)–(23) we shall solve the following optimization problem: find $h \in U$ minimizing the cost functional

$$J(h) = \int_{\Gamma_N} [\frac{\partial}{\partial n} u(h) - \phi]^2 \, ds, \tag{24}$$

where $u(h)$ is a restriction of the solution of the pure Dirichlet problem,

$$\triangle \hat{u} = -\tilde{f} \quad \text{in } \hat{\Omega}, \quad \hat{u} = g \quad \text{on } \Gamma_D, \quad \hat{u} = h \quad \text{on } \Gamma_N, \tag{25}$$

to the domain Ω, namely $u(h) = \hat{u}|_\Omega$.

LEMMA 3 *There exists a unique solution h^* to the problem (24)–(25). Moreover, $u^* = u(h^*)$ is a solution to (4).*

Proof. Let u be a unique solution to (4). Take $h^\star = u|_{\Gamma_N}$. From the $H^{3/2+\epsilon}(\Omega)$ regularity of u it follows that $h^\star \in U$, and obviously $J(h^\star) = 0$.
Now any h may be expressed as $h = h^\star + v$, where $v \in U_0$ and

$$U_0 = \{v \in H^1(\Gamma_N) \mid v = 0 \text{ on } \Gamma_N \cap \Gamma_D \text{ if } \bar{\Gamma}_N \cap \bar{\Gamma}_D \neq \emptyset \}.$$

The solution $u(h)$ decomposes accordingly into $u(h) = u(h^\star) + w(v)$ for $w(v)$ satisfying the equation

$$\triangle w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \Gamma_D, \quad w = v \text{ on } \Gamma_N. \tag{26}$$

Then, (24)–(25) is equivalent to finding the minimum in U_0 of

$$J_0(v) = \int_{\Gamma_N} [\frac{\partial}{\partial n} w(v)]^2 \, ds. \tag{27}$$

We have $J_0(0) = 0$, U_0 is convex and obviously

$$w(\, tv_1 + (1-t)v_2\,) = tw(v_1) + (1-t)w(v_2)$$

for any $v_1, v_2 \in U_0$ and $t \in (0,1)$. The element $v = 0$ in (27)–(26) corresponds to $h = h^\star$ in (24)–(25).

By its construction $J_0(v)$ is convex. We shall prove that it is strictly convex. Assume to the contrary,

$$J_0(\, tv_1 + (1-t)v_2\,) = tJ_0(v_1) + (1-t)J_0(v_2).$$

Then, after easy transformation,

$$\frac{\partial}{\partial n}[w(v_1) - w(v_2)] = 0 \quad \text{on} \quad \Gamma_N.$$

Hence $w(v_1) - w(v_2)$ satisfies homogeneous equation with zero Neumann boundary conditions on Γ_N and zero Dirichlet conditions on Γ_D. Therefore $w(v_1) = w(v_2)$ and the same concerns their traces on Γ_N, $v_1 = v_2$, which proves the thesis. ■

3. Finite element approximation

In order to solve numerically the problem (8)–(9) we discretize it by employing a conforming finite element method, Ciarlet (1978).
For the sake of computational simplicity we assume, that $\hat{\Omega}$ is a rectangle, and Ω is a polygonal domain. By $\mathcal{T}_h$ we denote a regular family of partitions of domain $\hat{\Omega}$, Ciarlet (1978), depending on the discretization parameter h, such that $h \to 0^+$. The size of h is given by division of the domain $\hat{\Omega}$ into quadrilateral elements O_i, $i = 1, \ldots, I$:

$$\hat{\Omega} = \bigcup_{i=1}^{I} O_i.$$

Similarly we cover Γ with straight segments S_i, in such a way that the ends of arcs creating Γ coincide with points of division, and

$$\Gamma = \bigcup_{i=1}^{K} S_i, \qquad diam(S_i) \leq h.$$

Observe that this subdivision is described by the same parameter h. We shall approximate function $\hat{u} \in \hat{V}$ using C^0 and C^1 finite elements. As C^0 finite element we shall employ bilinear functions on each polygon O_i. As C^1 finite element we shall employ Bogner - Fox - Schmidt finite element where the function is approximated by bicubic functions on each element O_i, Ciarlet (1978).
Let us introduce the finite dimensional space $\hat{V}_h$ approximating the space $\hat{V}_0$:

$$\hat{V}_h = \{z \in \hat{V}_0 \mid z_{|O_i} \in [P_k(O_i)], \ \ \forall O_i \in \mathcal{T}_h\} \tag{28}$$

where $P_k(O_i)$ denotes the set of polynomials containing all full polynomials of degree less then or equal to k on the element $O_i \in \mathcal{T}_h$. For C^1 approximation it is P_3, for C^0 it is P_1. The space $H^{-1/2}(\Gamma)$ is approximated by

$$\Lambda_h^0(\Gamma) = \{z \in L_2(\Gamma) \mid z_{|O_i} \in [P_0(S_i)], \ \ \forall S_i \subset \Gamma\}, \tag{29}$$

and for $H^1(\Gamma)$ we use

$$\Lambda_h^1(\Gamma) = \{z \in H^1(\Gamma) \mid z_{|O_i} \in [P_1(S_i)], \ \ \forall S_i \subset \Gamma\}. \tag{30}$$

The functions f and ϕ are approximated, respectively, by piecewise constant functions f_h and $\phi_h \in \Lambda_h^0(\Gamma_N)$. The function g is approximated by piecewise linear function $g_h \in \Lambda_h^1(\Gamma_D)$. Thus the discrete model can be characterized by one parameter h.

3.1. Approximation of the Dirichlet problem

The state system (8)–(9) is approximated by the following discrete variational equations: find $(\hat{u}_h, \lambda_{Dh}) \in \hat{V}_h \times \Lambda_h^0(\Gamma_D)$,

$$\int_{\hat{\Omega}_h} \nabla \hat{u}_h \nabla \hat{\varphi} dx - \int_{\hat{\Omega}_h} \tilde{f}_h \hat{\varphi} dx + \int_{\Gamma_{Dh}} \lambda_{Dh} \hat{\varphi}_{|\Gamma_{Dh}} ds = 0 \quad \forall \hat{\varphi} \in \hat{V}_h, \tag{31}$$

$$\int_{\Gamma_{Dh}} \mu(\hat{u}_h|_{\Gamma_{Dh}} - g_h) ds = 0 \quad \forall \mu \in \Lambda_h^0(\Gamma_D). \tag{32}$$

LEMMA 4 *There exists a unique solution $(\hat{u}_h, \lambda_{Dh}) \in \hat{V}_h \times \Lambda_h^0(\Gamma_D)$ to the system (31)–(32).*

Proof. The proof is parallel to the proof of Lemma 1. For details see Haslinger (1993), Peichl, Kunisch (1995). ■

Note that the system (31) - (32) satisfies the LBB condition ensuring the existence and uniqueness of solutions to (31) - (32) for finite dimensional spaces (28), (29), (30), Brezzi, Fortin (1991), Ciarlet (1978). Using the finite dimensional space (29) we assume that the solution to the system (8) - (9) is more regular and the LBB condition is satisfied in $L_2(\Gamma_D)$ space, Haslinger, Klabring (1995). LBB condition is satisfied in $H^{-1/2}(\Gamma_D)$ space if the solution to the system (8) - (9) is assumed to be in $H^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$.

LEMMA 5 *If* $(\hat{u}_h, \lambda_{Dh}) \in \hat{V}_h \times \Lambda_h^0(\Gamma_D)$ *is a solution to the system (31)-(32), then there exist subsequences* $\{\hat{u}_{h'}\}$, $\{\lambda_{Dh'}\}$, *and elements* $\hat{u} \in \hat{V}_0$, $\lambda_D \in L_2(\Gamma_D)$ *such that,*

$$\begin{aligned} &\hat{u}_{h'} \to \hat{u} \quad in \;\; \hat{V}_0, \\ &\lambda_{Dh'} \to \lambda_D \;\; in \;\; L_2(\Gamma_D), \end{aligned} \tag{33}$$

and $(\hat{u}, \lambda_D) \in \hat{V}_0 \times L_2(\Gamma_D)$ *is a solution to the problem (8)-(9).*

Proof. For details of the proof see Glowinski, Pan (1996), Grisvard (1992), Haslinger (1993), Peichl, Kunisch (1995). ■
The rate of convergence was investigated in Glowinski, Pan, Periaux (1994b). Assuming that Ω is more regular, i.e., Ω is a bounded C^2 domain and the solution $u \in H^2(\Omega)$, u_h converges to u linearly in $H^1(\Omega)$ and quadratically in $L^2(\Omega)$.

3.2. Approximation of the mixed problem

The general mixed problem is approximated by finding v_h minimizing the cost functional

$$J(v_h) = \int_{\Gamma_N} [\frac{\partial}{\partial n}\hat{u}_h(v_h) - \phi_h]^2 \, ds, \tag{34}$$

where $\hat{u_h}$ is a solution to the problem

$$\Delta \hat{u}_h = 0 \quad \text{in } \hat{\Omega}_h, \quad \hat{u}_h = g_h \quad \text{on } \Gamma_D, \quad \hat{u}_h = v_h \quad \text{on } \Gamma_N, \tag{35}$$

and $g_h \in \Lambda_h^1(\Gamma_D)$, $v_h \in \Lambda_h^1(\Gamma_N)$. In addition, $v_h \in U_h$, where the admissible set U_h has the form

$$U_h = \{ \, v_h \in \Lambda_h^1(\Gamma_N) \mid v_h = g_h \quad \text{on} \quad \Gamma_N \cap \Gamma_D \, \}.$$

LEMMA 6 *There exists a unique solution* $\hat{u}_h$ *to the problem (34)-(35).*

Proof. The proof is parallel to the proof of Lemma 3. Namely, it is easy to show that the functional J is strictly convex on U_h.

LEMMA 7 *If* $(\hat{u}_h, v_h) \in \hat{V}_h \times \Lambda_h^1(\Gamma_N)$ *is a solution to the system (34)–(35), then there exist subsequences* $\{\hat{u}_{h'}\}$, $\{v_{h'}\}$, *and elements* $\hat{u} \in \hat{V}_0$, $v \in H^1(\Gamma_N)$ *such that,*

$$\begin{aligned} &\hat{u}_{h'} \to \hat{u} \quad in \quad \hat{V}_0, \\ &v_{h'} \to v \quad in \quad H^1(\Gamma_N), \end{aligned} \tag{36}$$

and $(\hat{u}, v) \in \hat{V} \times H^1(\Gamma_N)$ *is a solution to the problem (24)–(25).*

Proof. Let $\bar{v}_h$ be a unique solution to (34)–(35) and $\hat{u}$ a solution to the continuous problem (24)–(25). Define $v_h^\star = (\hat{u}|_{\Gamma_N})_{\Lambda_h^1(\Gamma_N)}$. Then, $v_h^\star \in U_h$, and therefore

$$0 \le J(\bar{v}_h) \le J(v_h^\star).$$

Since $J(v_h^\star) \to 0$, we have $J(\bar{v}_h) \to 0$, and consequently

$$\|\frac{\partial}{\partial n}\hat{u}_h(v_h) - \phi_h\| \longrightarrow 0 \qquad \text{in} \quad L_2(\Gamma_N).$$

Hence $\hat{u}_h(\bar{v}_h) \to \hat{u}$ in $\hat{V}_0$, what implies the thesis. ■

4. Numerical aspects of the fictitious domain method

Let us assume that the reference domain $\hat{\Omega} \subset \mathbb{R}^2$ constitutes a rectangle, and that it has already been discretized into squares of size h. Two types of approximations will be tested: bilinear of C^0 regularity and bicubic of C^1 regularity. The original domain Ω will be contained in $\hat{\Omega}$, so that $dist(\partial\Omega, \partial\hat{\Omega}) > \frac{1}{5} diam(\hat{\Omega})$.

4.1. Imposing Dirichlet boundary conditions

Consider first the Dirichlet problem (5)–(7), where $\Gamma_D = \partial\Omega$ and $\tilde{f} = 0$,

$$\Delta\hat{u} = 0 \quad \text{in} \ \hat{\Omega}, \qquad \hat{u} = g \quad \text{on} \ \Gamma_D. \tag{37}$$

As has been stated before, the Lagrange functional L for the Dirichlet problem (37) has the form (12). Let us introduce the notation:

$$S_h = \{\int_{\hat{\Omega}} \nabla\hat{\varphi}_i \nabla\hat{\varphi}_j dx\}_{i,j=1,\dots,M}, \quad \hat{u}_h = \Sigma_{i=1}^M u_i \hat{\varphi}_i, \tag{38}$$

where S_h denotes $M \times M$ stiffness matrix for the Laplace equation (37) , $\hat{u}_h$ approximates the solution $\hat{u}$ to system (5) - (7), $u_h = [u_1, ..., u_M]$ is a vector of coefficients. We denote by $\Gamma_{Dh} = polyline(p_1, \dots, p_{K+1})$. For simplicity we assume here that Γ_{Dh} has only one component, and $p_{K+1} = p_1$ (closed line). We assume also that $\frac{1}{2}h < dist(p_i, p_{i+1}) < h$, because finer subdivision of the boundary does not improve accuracy. Moreover, let $g_h = [g_1, \dots, g_K]^T$ denote the values in vertices for the piecewise linear approximation of g. The multiplier λ_{Dh} is approximated by the piecewise constant function, so that $\lambda_{Dh} = [\lambda_1, \dots, \lambda_K]^T$. Using this notation, the discretized version of (12) may be written as:

$$L(\hat{u}_h, \lambda_{Dh}) = \frac{1}{2} u_h^T S_h u_h - \sum_{i=1}^{K} \lambda_i \int_{[p_i, p_{i+1}]} (\hat{u}_h - g_h)\, ds. \tag{39}$$

It is obvious, how to compute

$$d_i = \int_{[p_i, p_{i+1}]} g_h\, ds = \frac{1}{2} dist(p_i, p_{i+1})(g_i + g_{i+1}), \tag{40}$$

and we denote $d_D = [d_1, \dots, d_K]^T$.
The integral of $\hat{u}_h$ is computed using numerical Gauss quadrature. Let (ξ_i, w_i), $i = 1, \dots 8$ be points and weights for the integral over $[0, 1]$. Then

$$\int_{[p_i, p_{i+1}]} \hat{u}_h\, ds = dist(p_i, p_{i+1}) \sum_{k=1}^{8} w_k \hat{u}_h(p_{\xi_k}), \quad p_{\xi_k} = (1 - \xi_k) p_i + \xi_k p_{i+1}. \tag{41}$$

On the other hand, each value $\hat{u}_h(p_{\xi_k})$ may be expressed as a sum of nodal values of u_h, with easily computed coefficients. First, we identify the square in which p_{ξ_k} is located, then its relative position with respect to corners, and finally the coefficients depending on the type of approximating functions. The final result is the representation

$$\int_{[p_i, p_{i+1}]} \hat{u}_h\, ds = u_h^T \cdot c_i, \tag{42}$$

where c_i represents a constant column vector of the same size M as u_h.
Let us denote by $C_D = [c_1, \dots, c_K]$ the matrix of dimensions $M \times K$. Then, the functional (39) takes on the form:

$$L(u_h, \lambda_{Dh}) = \frac{1}{2} u_h^T S_h u_h - \lambda_{Dh} \cdot (u_h^T \cdot C_D - d_D^T). \tag{43}$$

For this purely discrete problem we may write down the necessary optimality conditions,

$$S_h u_h - C_D \lambda_{Dh} = 0, \tag{44}$$
$$u_h^T C_D - d_D^T = 0. \tag{45}$$

Note that since the employed finite elements satisfy discrete LBB condition, Brezzi, Fortin (1991), Ciarlet (1978), it follows that there exists a unique solution to the system (44)–(45).

This system may be solved simultaneously for both u_h and the multiplier λ_{Dh}. From (44) we find

$$u_h = S_h^{-1} C_D \lambda_{Dh}.$$

After substitution into (45) one gets

$$(C^T S_h^{-1} C_D)\lambda_{Dh} = d_D.$$

In practice, the order of operations is as follows:

$$F = S_h^{-1} C_D \quad \Rightarrow \quad V = C_D^T F \quad \Rightarrow \quad \lambda_{Dh} = V^{-1} d_D \quad \Rightarrow \quad u_h = S_h^{-1} F \lambda_{Dh}.$$

4.2. Imposing the Neumann boundary conditions

Consider the mixed boundary value problem (21)–(23). Let $\Gamma_{Nh} = polyline(q_1, \dots, q_L)$, $\Gamma_{Dh} = polyline(p_1, \dots, p_K)$, where $p_K = q_1$ and $q_L = p_1$. Similarly as in previous subsection, we approximate g by $g_h = [g_1, \dots, g_K]^T$, but for ϕ we use a piecewise constant representation $\phi_h = [\phi_1, \dots, \phi_{L-1}]^T$. Then we shall solve the problem (34)–(35).

It is evident that we cannot escape from the problem of approximating $\partial u/\partial n$ on the boundary. In addition, we must be careful: $\partial u/\partial n$ denotes here a limit of the normal derivative, as the current point approaches Γ_N from the inside of Ω. This means that in order to get $\partial u/\partial n$ we must use only values of u inside Ω. In practice, one of the solutions, which we use here, is as follows. In the case of bilinear approximation we estimate $\partial u/\partial n$ using three points located on the internal normal to the boundary at the distances $0.5h, h, 1.5h$ from the boundary point and extrapolation. In case of the C^1 approximation we take as an estimate the value of $\partial u/\partial n$ in the point on the internal normal at the distance $0.5h$ from the boundary.

Leaving out the details of the discretization, the optimization problem (34)–(35) takes the form:

$$S_h u_h = C_D \lambda_{Dh} + C_N \lambda_{Nh}, \tag{46}$$

$$C_D^T u_h = d_D, \tag{47}$$

$$C_N^T u_h = d_N, \tag{48}$$

where S_h, u_h are defined by (37), d_D and C_D are defined in previous subsection, C_N is matrix $L \times L$, and d_N is vector $1 \times L$ dependent on v. The method of computing C_N and d_N is similar to computing C_D and d_D. Since the employed finite elements satisfy discrete LBB condition, Brezzi, Fortin (1991), Ciarlet (1978), it follows that there exists a unique solution to the system (46)–(48).

The goal functional (34) may be expressed as

$$J_h(d_N) = \|L_N^T u_h - l\|^2,$$

with l resulting from integrating ϕ over Γ_N as in (40). The matrix L_N is obtained in a similar way as C_D by numerical integration and approximation of $\partial \hat{u}_h / \partial n$ in the way described above.

The solution algorithm for the system (46)–(48) consists of several steps:

1. We transform (47),(48) into

$$C_D^T S_h^{-1} C_D \cdot \lambda_{Dh} + C_D^T S_h^{-1} C_N \cdot \lambda_{Nh} = d_D, \tag{49}$$

$$C_N^T S_h^{-1} C_D \cdot \lambda_{Dh} + C_N^T S_h^{-1} C_N \cdot \lambda_{Nh} = d_N. \tag{50}$$

Denoting $\lambda = [\lambda_{Dh}, \lambda_{Nh}]^T$, we may write this as

$$H\lambda = \begin{bmatrix} d_D \\ d_N \end{bmatrix}. \tag{51}$$

2. Observe that the solution λ is a sum of two componenets: $\lambda = \lambda_1 + \lambda_2$, where

$$\lambda_1 = H^{-1} \begin{bmatrix} d_D \\ 0 \end{bmatrix}$$

and

$$\lambda_2 = H^{-1} \begin{bmatrix} 0 \\ d_N \end{bmatrix} = (H^{-1}P) \cdot d_N.$$

Here P is a matrix of the form

$$P = \begin{bmatrix} 0 \\ I_N \end{bmatrix},$$

and I_N is an identity matrix of the same size as d_N.

3. The solution u_h may be split in a similar way: $u_h = u_{h1} + u_{h2}$. Let $F = [C_D, C_N]$. Then

$$u_{h1} = S_h^{-1} F \cdot \lambda_1,$$

$$u_{h2} = S_h^{-1} F \cdot \lambda_2.$$

Finally, let z_1 be an error of u_{h1} in satisfying the Neumann boundary condition,

$$z_1 = l - L_N^T \cdot u_{h1}.$$

Then the functional $J_h(d_N)$ reduces to

$$J_h(d_N) = \|L_N^T S_h^{-1} F (H^{-1}P) \cdot d_N - z_1\|^2,$$

from which we may calculate d_N:

$$G = L_N^T S_h^{-1} F (H^{-1}P) \quad \Rightarrow \quad d_N = G^{-1} z_1.$$

Finally we substitute back:

$$\lambda_2 = H^{-1} P \cdot d_N \quad \Rightarrow \quad u_{h2} = S_h^{-1} F \cdot \lambda_2.$$

Let us comment here that we cannot simply find λ_2 without getting d_N first, because we have too little information.

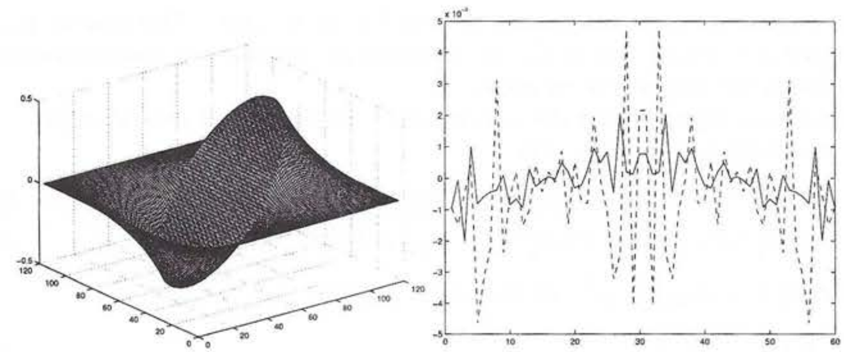

Figure 1. On the left: the solution for the first example using the fine C^1 mesh. On the right: the difference between true and discrete solutions along the boundary for C^0 (dashed line) and C^1 fine approximations with the same number of unknowns.

4.3. Numerical examples

In the numerical examples we will have always $\hat{\Omega} = [-1,-1] \times [-1,1]$. As it was mentioned before, we shall use the C^0 finite element approximation (bilinear) on elementary squares, with one degree of freedom per vertex (function value), and the C^1 bicubic approximation, using well known Hermitian elements, with four degrees of freedom (value, x_1–derivative, x_2–derivative, x_1x_2–derivative). For the C^1 case two discretizations are used: coarse, with 20×20 division of $\hat{\Omega}$, and fine 40×40, with twice smaller elements. The corresponding C^0 discretizations having the same number of unknowns have the sizes 40×40 and 80×80. The approximate solutions are compared with the analytical ones, i.e. the error $e = u_h - u$ is computed.

First example. The equation is defined in the circle,

$$\Delta u = 0 \quad \text{in } \Omega = \{x \mid \|x\| \leq 0.5\},$$
$$u = x_1 \quad \text{on } \partial\Omega,$$

where $\|\cdot\|$ denotes a Euclidean norm. The known smooth solution is obviously $u = x_1$. Both types of approximations work very well. In Fig.1 we see the results for the C^1 fine mesh. The grid has more points than discretization, since the C^1 basis functions allow computing functions values also inside elements. The C^1 approximation gives much more accurate results for both resolutions.

Second example. This is the mixed problem defined in the ring,

$$\Delta u = 0 \quad \text{in } \Omega = \{x \mid 0.3 \leq \|x\| \leq 0.7\},$$
$$u = x_1 \quad \text{on } \Gamma_D = \{x \mid \|x\| = 0.7\},$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N = \{x \mid \|x\| = 0.3\}.$$

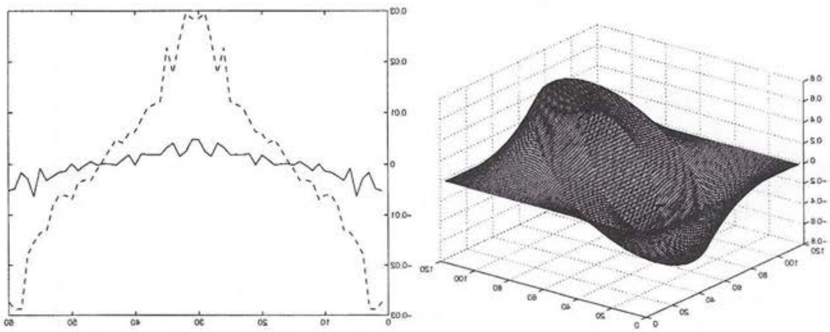

Figure 2. On the left: the solution for the second example using the fine C^1 mesh. On the right: the distribution of error along the Neumann boundary for C^0 (dashed line) and C^1 fine approximations with the same number of unknowns.

We see that $\Gamma_D \cap \Gamma_N = \emptyset$, and as a result the solution is smooth and has a form, Sokołowski, Żochowski (1999)

$$u = \frac{R^2}{R^2+\rho^2}(1+\frac{\rho^2}{r^2})x_1,$$

where $R = 0.7$, $\rho = 0.3$, $r^2 = x_1^2 + x_2^2$. Some of the results of computations are shown in Fig.2. In addition, we may compute here the rate of convergence q for both approximations. The result is $q = 1.6$ for C^0 elements, and $q = 2.1$ for C^1. In theory, we should expect $q = 2$. Here C^1 approximation is also much more accurate for both resolutions.

Third example. This is also a mixed problem defined in the circle with one quarter cut out,

$$\begin{aligned} \Delta u = 0 \quad &\text{in } \Omega = \{(r,\theta) \mid 0 \le r \le 0.5,\ 0 \le \theta \le \frac{3}{2}\pi\}, \\ u = \sqrt{0.125}\cos\theta \quad &\text{on } \Gamma_D = \{(r,\theta) \mid r = 0.5,\ 0 \le \theta \le \frac{3}{2}\pi\}, \\ \frac{\partial u}{\partial n} = 0 \quad &\text{on } \Gamma_N = \{(r,\theta) \mid 0 \le r \le 0.5, \theta = 0 \text{ or } \theta = \frac{3}{2}\pi\}. \end{aligned}$$

where r and θ denote polar coordinates. The exact solution, Grisvard (1985), is

$$u = r^{2/3}\cos\frac{2}{3}\theta,$$

and does not have full regularity, namely $u \in H^{5/3-\epsilon}$, for any $\epsilon > 0$, Grisvard (1992). This fact should change the convergence of the approximation in comparison with the second example, lowering it to $q = 1.66$. From the experiment

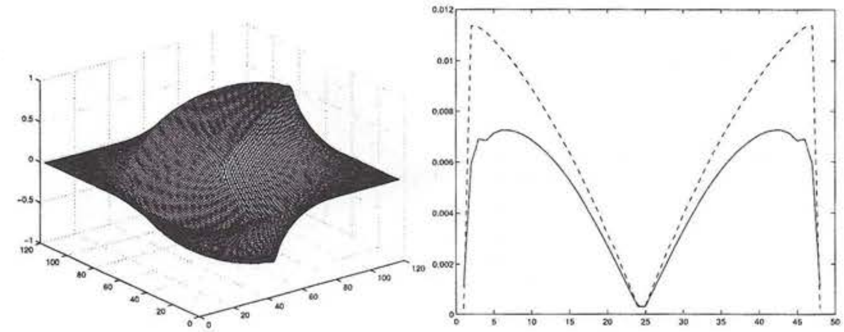

Figure 3. On the left: the solution for the third example using the fine C^1 mesh. On the right: the distribution of error along the line at half the distance from the origin to Dirichlet boundary, for C^0 (dashed line) and C^1 fine approximations with the same number of unknowns.

we obtain $q = 1.56$ and $q = 1.20$, corespondingly. In addition, C^1 is now about two times more accurate in terms of the L_2–norm. However, for such problems with lower regularity of solutions, the pointwise error of smooth elements may be sometimes bigger than for simple linear basis functions. The results of computation in this case are presented in Fig.3.

In Fig.4 we see the accuracy, with which the value of the solution along the Neumann boundary for the second and third examples are reproduced.

5. Concluding remarks

The formulae derived here work well in practice, but it is evident that the theoretical analysis of their stability would be extremely difficult. We do not attempt it here.

It should be added that the seemingly lower convergence rates of C^1 elements for singular problems are probably the result of the less sophisticated estimation of the normal derivative along the boundary. Construction of good formulae is here an open problem. Nethertheless, for the same number of unknowns, they are in each case much more accurate and give good results even in a very straightforward implementation.

Another very visible feature of the fictitious domain approach is its similarity to the boundary element method. In both cases we must solve nonsymmetric, dense systems of equations for certain values given on the boundary. Here the role of fundamental solutions is played by the inverse of the stiffness matrix S. There is also an additional element: a subtle interdependence between the discretization of $\hat{\Omega}$ and the discretization of the boundary of Ω, which is absent in the boundary element method. However, both methods share the inability to cope with domains containing narrow "necks" and lose some of their at-

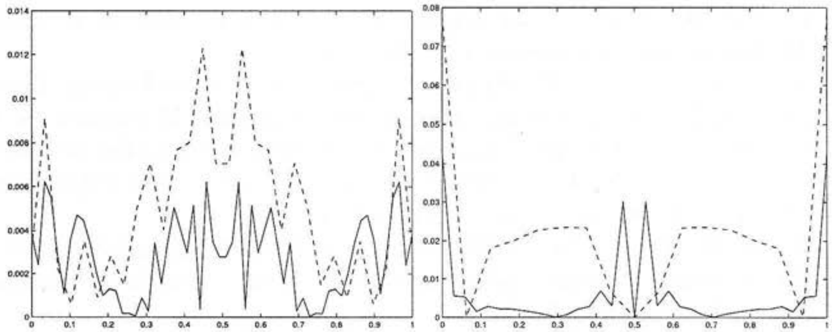

Figure 4. The comparison of absolute errors for the C^1 approximation using coarse (dashed line) and fine mesh. On the left: mixed problem and Neumann boundary, on the right: singular problem and Neumann boundary.

tractivness for the nonhomogeneous case. In one respect the fictitious domain method has an advantage: the computation of matrices needed for solving the intermediate problems is easier.

There are also cases where the fictitious domain method seems indispensable, i.e. for domains with holes (see Chambolle, Doveri, 1996), and indeed works there very well.

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