*Dedicated to*
*Professor Jakub Gutenbaum*
*on his 70th birthday*

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# On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions

by

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**Abstract:** The paper introduces the notion of strongly $\alpha(\cdot)$-paraconvex functions. Relations between strong $\alpha(\cdot)$-paraconvexity and $\alpha(\cdot)$-paraconvexity are investigated.

**Keywords:** paraconvex functions, strongly paraconvex functions

Let $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that $\alpha(0) = 0$ and

$$\limsup_{t \to 0+0} \frac{\alpha(t)}{t} < +\infty. \tag{1}$$

Let $(X, \|\cdot\|)$ be a normed space. Let $\Omega$ be a convex subset of $X$. Let $f(\cdot)$ be a real valued function defined on $\Omega$. We say that the function $f(\cdot)$ is *$\alpha(\cdot)$-paraconvex with a constant* $C > 0$ if for all $x, y \in \Omega$ and $0 \le t \le 1$

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) + C\alpha(\|x - y\|). \tag{2}$$

We say that the function $f(\cdot)$ is *$\alpha(\cdot)$-paraconvex*, if there is a constant $C > 0$ such that the function $f(\cdot)$ is $\alpha(\cdot)$-paraconvex with the constant $C > 0$. For $\alpha(t) = t^2$ this definition was introduced in Rolewicz (1979a) and the $t^2$-paraconvex functions were called simply paraconvex functions. In Rolewicz (1979b) the notion was extended of the case $\alpha(t) = t^\gamma, 1 \le \gamma$, and the $t^\gamma$-paraconvex functions were called $\gamma$-paraconvex functions.

We say that the function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex with a constant* $C_1 > 0$ if for all $x, y \in \Omega$ and $0 \le t \le 1$

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) + C_1 \min[t, (1-t)]\alpha(\|x - y\|). \tag{3}$$

If there is a constant $C_1 > 0$ such that the function $f$ is strongly $\alpha(\cdot)$-paraconvex with the constant $C_1 > 0$, we say that the function $f$ is *strongly $\alpha(\cdot)$-paraconvex* .

Of course every function $f(\cdot)$ strongly $\alpha(\cdot)$-paraconvex with a constant $C_1 > 0$ is also $\alpha(\cdot)$-paraconvex with the constant $C_1 > 0$. It was shown in Rolewicz (1979a,b) that for $\alpha(t) = t^\gamma, 1 < \gamma \leq 2$, any $\alpha(\cdot)$-paraconvex function is simultaneously strongly $\alpha(\cdot)$-paraconvex.

In this short paper properties of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions are investigated.

Similarly as in Rolewicz (1979b) we can demonstrate

PROPOSITION 1 *Let $f(\cdot)$ be an $\alpha(\cdot)$-paraconvex function defined on a convex set $\Omega$ of a normed space $X$. Suppose that for each straight line $L$ the function $f(\cdot)$ restricted to the intersection $L \cap \Omega$, $f|_{L\cap\Omega}$ is absolutely continuous. If*

$$\liminf_{t\to 0+0} \frac{\alpha(t)}{t^2} = 0, \tag{4}$$

*then $f(\cdot)$ is convex.*

*Proof.* Observe that the consideration can be restricted to the set $L\cap\Omega$, since convexity of a function on all such sets implies the convexity of the function on the whole set $\Omega$.

Thus, without loss of generality we may assume that we consider an $\alpha(\cdot)$-paraconvex absolutely continuous function $f(\cdot)$ defined on the interval $[a, b] \subset \mathbb{R}$.

By (4) there is a sequence $\{t_n\}$ tending to 0 such that

$$\lim_{n\to+\infty} \frac{\alpha(t_n)}{t_n^2} = 0. \tag{5}$$

Since the function $f(\cdot)$ is $\alpha(\cdot)$-paraconvex there is a constant $C > 0$ such that for every $k > 0$ such that $(k+1)t_n < b - a$ we have

$$f(a + kt_n) \leq \frac{1}{2}[f(a + (k+1)t_n) + f(a + (k-1)t_n)] + C\alpha(2t_n)$$

and for every $k > 0$ such that $(k+2)t_n < b - a$ we have

$$f(a + (k+1)t_n) \leq \frac{1}{2}[f(a + (k+2)t_n) + f(a + kt_n)] + C\alpha(2t_n).$$

Adding this two inequalities and multiplying by 2 we get for every $k > 0$ such that $(k+1)t_n < b - a$

$$2f(a + kt_n) + 2f(a + (k+1)t_n)$$

$$\leq [f(a + (k+1)t_n) + f(a + (k-1)t_n)]+$$

$$[f(a+(k+2)t_n)+f(a+kt_n)]+2C\alpha(2t_n).$$

Hence,

$$[f(a+kt_n)-f(a+(k-1)t_n)]-[f(a+(k+2)t_n)-f(a+(k+1)t_n)] \leq 2C\alpha(2t_n).(6)$$

Thus, by adding inequalities $(6_k), (6_{k+1}), ..., (6_{k+m-1})$ for $m$ such that $(k+m+1)t_n < b-a$ we obtain

$$[f(a+kt_n)-f(a+(k-1)t_n)]-[f(a+(k+m+1)t_n)-f(a+(k+m)t_n)]$$
$$\leq 2C(m-1)\alpha(2t_n). \tag{7}$$

By our assumptions the function $f(\cdot)$ is absolutely continuous. Thus it is differentiable almost everywhere. Let $\tau_1$ and $\tau_2$ be two arbitrary points in which the function $f(\cdot)$ is differentiable. Let $a < \tau_1 < \tau_2 < b$. Let $k_n$ and $m_n$ be two sequences chosen in such a way that $k_n t_n$ tends to $\tau_1$ and $(k_n+m_n)t_n$ tends to $\hat{t}$.

Dividing both sides of (7) by $t_n$ and recalling that $m_n < \frac{(b-a)}{t_n}$ yields

$$\frac{f(a+kt_n)-f(a+(k-1)t_n)}{t_n}-\frac{f(a+(k+m+1)t_n)-f(a+(k+m)t_n)}{t_n}$$
$$\leq 2C(b-a)\frac{\alpha(2t_n)}{{t_n}^2}. \tag{8}$$

Taking (5) into account and passing to the limit as $n \to \infty$ we get

$$f'(\tau_1) \leq f'(\tau_2)$$

which shows that the function $f(\cdot)$ is convex. ■

It is not clear whether the assumption that for each straight line $L$ the function $f(\cdot)$ restricted to the intersection $L \cap \Omega$, $f|_{L\cap\Omega}$ is absolutely continuous is essential.

Proposition 1 can be reversed in the following way

Proposition 2 *If*

$$\liminf_{t\to 0+0} \frac{\alpha(t)}{t^2} > 0, \tag{9}$$

*then there is an $\alpha(\cdot)$-paraconvex function which is not convex.*

*Proof.* Suppose that $X = \mathbb{R}$. By (9) there are $C, r > 0$ such that

$$\frac{\alpha(t)}{t^2} > C$$

for $0 < t \leq r$. We put

$$f(x) = \begin{cases} -Cx^2 \; for \; |x| \leq r, \\ -Cr^2 \; for \; |x| > r. \end{cases}$$

It is easy to see that $f(\cdot)$ is an $\alpha(\cdot)$-paraconvex function with the constant $C$.

This finishes the proof for the case of $X = \mathbb{R}$. In the general case we simply extend the function $f(\cdot)$ from one-dimensional subspace on the whole space using the fact that any normed space can be decomposed into a direct sum of a one-dimensional subspace and a subspace of codimension 1. ■

The function constructed in the Proposition 2 is also strongly $\alpha(\cdot)$-paraconvex. Under an additional assumption we can construct a function, which is $\alpha(\cdot)$-paraconvex and which is not strongly $\alpha(\cdot)$-paraconvex.

PROPOSITION 3 *If (9) holds and*

$$\liminf_{t \to +\infty} \frac{\alpha(t)}{t} = 0, \tag{10}$$

*then there is a function $f(\cdot) : X \to \mathbb{R}$ such that $f(\cdot)$ is $\alpha(\cdot)$-paraconvex and it is not strongly $\alpha(\cdot)$-paraconvex.*

*Proof.* At the beginning we shall prove the Proposition 3 for the case of $X = \mathbb{R}$. By (9) there is $C > 0$ such that

$$\frac{\alpha(t)}{t^2} > C \tag{11}$$

for $0 < t \leq 1$.

We put

$$f(x) = \max_{-\infty < n < +\infty} \left(1 - \left(x - (2n+1)\right)^2\right). \tag{12}$$

The function $f$ is non-negative and not greater than 1, $0 \leq f(x) \leq 1$. Moreover, $f(2n) = 0$, $n = 0, \pm 1, \pm 2, \ldots$ . It is $\alpha(\cdot)$-paraconvex with the constant $\frac{1}{C}$. Indeed, if $|x - y| \geq 1$, then by (11) and the fact that $\alpha(\cdot)$ is nondecreasing $\alpha(|x-y|) > C$. Thus

$$f\big(tx + (1-t)y\big) \leq 1 \leq \frac{1}{C}\alpha(|x-y|) \leq tf(x) + (1-t)f(y) + \frac{1}{C}\alpha(|x-y|). \tag{13}$$

If $|x - y| \leq 1$, then on the interval $[\min(x,y), \max(x,y)]$ the function $f(\cdot)$ is a difference of a convex function and the quadratic function $x^2$. Thus it is $t^2$-paraconvex with constant 1. Therefore by (11) it is $\alpha(\cdot)$-paraconvex with the constant $\frac{1}{C}$.

By (10) there is a sequence $\{t_n\}$ tending to infinity such that

$$\lim_{n\to+\infty} \frac{\alpha(t_n)}{t_n} = 0. \tag{14}$$

Now we consider our function on the interval $[0, t_n]$. Let $t$ be an arbitrary positive number. Let $\lambda_n = \frac{t}{t_n}$. Suppose that the function $f(\cdot)$ considered above is strongly $\alpha(\cdot)$-paraconvex with a constant $C$. It means that

$$f(t) = f\big(\lambda_n t_n + (1-\lambda_n)0\big) \leq \lambda_n f(t_n) + (1-\lambda_n) f(0) + \lambda_n C\alpha(t_n)$$

$$\leq \lambda_n + \lambda_n C\alpha(t_n) = \frac{t}{t_n} + C\frac{\alpha(t_n)}{t_n} \to 0,$$

by (14). Since $f(t)$ is non-negative, we get that $f(t) = 0$, a contradiction.

This finishes the proof for the case of $X = \mathbb{R}$. In the general case we simply extend the function $f(\cdot)$ from the one-dimensional subspace on the whole space using the fact that any normed space can be decomposed into a direct sum of a one-dimensional subspace and a subspace of codimension 1. ■

Proposition 3 can be reversed under an additional assumption in the following way.

PROPOSITION 4 *Suppose that*

$$\int_1^{+\infty} \alpha(\frac{1}{t})dt < +\infty. \tag{15}$$

*If*

$$\liminf_{t\to+\infty} \frac{\alpha(t)}{t} > 0, \tag{16}$$

*then each $\alpha(\cdot)$-paraconvex function $f(\cdot) : X \to \mathbb{R}$ is also strongly $\alpha(\cdot)$-paraconvex.*

The proof is based on the following lemma:

LEMMA 1

$$\int_1^{+\infty} \alpha(\frac{1}{t})dt < +\infty,$$

*if and only if*

$$\sum_{n=1}^{+\infty} 2^n \alpha(\frac{1}{2^n}) < +\infty. \tag{17}$$

*Proof.* Indeed, since $\alpha(\cdot)$ is non-decreasing we have

$$\frac{1}{2}\sum_{n=1}^{+\infty} 2^n\alpha(\frac{1}{2^n}) = \sum_{n=1}^{+\infty} 2^{n-1}\alpha(\frac{1}{2^n}) \leq \int_1^{+\infty} \alpha(\frac{1}{t})dt \leq \sum_{n=0}^{+\infty} 2^n\alpha(\frac{1}{2^n}). \tag{18}$$

■

*Proof of Proposition 4.* By (16) there exists $K > 0$ such that

$$\frac{\alpha(t)}{t} > K$$

for $t > 1$.

Let $0 < \lambda < 1$. Suppose that $\lambda\|x-y\| > 1$. This implies that $\|x-y\| > 1$. Since $f(\cdot)$ is $\alpha(\cdot)$-paraconvex there is a $C > 0$ such that

$$\begin{aligned}
&f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) + C\alpha(\|x-y\|) \leq \\
&\lambda f(x) + (1-\lambda)f(y) + C\alpha(1) \\
&\leq \lambda f(x) + (1-\lambda)f(y) + C\alpha(1)\frac{1}{K}\frac{\alpha(\|x-y\|)}{\|x-y\|} \leq \\
&\lambda f(x) + (1-\lambda)f(y) + \alpha(1)\frac{C}{K}\lambda\alpha(\|x-y\|).
\end{aligned} \tag{19}$$

Suppose that $\lambda\|x-y\| \leq 1$. Let $x \neq y$ and let $F_{x,y}(t) = f(x + t\frac{y-x}{\|x-y\|})$, $0 \leq t \leq 1$. Of course $F_{x,y}(0) = f(x)$ and $F_{x,y}(\|x-y\|) = f(y)$. Observe that the function $f(\cdot)$ is $\alpha(\cdot)$-paraconvex (strongly $\alpha(\cdot)$-paraconvex) if and only if $F_{x,y}(t)$ are $\alpha(\cdot)$-paraconvex (strongly $\alpha(\cdot)$-paraconvex) for all $x, y$.

Since $F_{x,y}(\cdot)$ is a function of a real variable it is enough to restrict ourselves to the case when $X = \mathbb{R}$.

We shall start our proof with the case of $x = 0$ and $y = 1$. Let $g(t)$ be an arbitrary $\alpha(\cdot)$-paraconvex function defined on $\mathbb{R}$. Let $\hat{g}(t) = g(t) - g(0) - t[g(1) - g(0)]$. It is easy to see that $\hat{g}(0) = \hat{g}(1) = 0$ and that $g(t)$ is $\alpha(\cdot)$-paraconvex (strongly $\alpha(\cdot)$-paraconvex) with a constant $\hat{C}$ if and only if $\hat{g}(t)$ is $\alpha(\cdot)$-paraconvex (resp. strongly $\alpha(\cdot)$-paraconvex) with the constant $\hat{C}$.

Recall that we have assumed that (15) (hence (17)) holds. Now we shall show by induction that

$$\hat{g}(\frac{1}{2^n}) \leq \hat{C}\sum_{i=1}^{n} 2^{i-n}\alpha(\frac{1}{2^{i-1}}). \tag{20}$$

For $n = 1$ it trivially follows from the fact that $\hat{g}(t)$ is $\alpha(\cdot)$-paraconvex. Suppose that (20) is true for certain $n$. Then by the fact that $\hat{g}(t)$ is $\alpha(\cdot)$-paraconvex we have

$$\hat{g}(\frac{1}{2^{n+1}}) \leq \frac{1}{2}\big(\hat{g}(0) + \hat{g}(\frac{1}{2^n})\big) + \hat{C}\alpha(\frac{1}{2^n}) \leq \hat{C}\alpha(\frac{1}{2^n}) + \hat{C}\sum_{i=1}^{n} 2^{i-n-1}\alpha(\frac{1}{2^{i-1}})$$

$$= \hat{C} \sum_{i=1}^{n+1} 2^{i-(n+1)} \alpha(\frac{1}{2^{i-1}}). \tag{21}$$

Now let $\lambda = \frac{1}{2^n}$. By (20)

$$\frac{\hat{g}(\lambda)}{\lambda} \leq \hat{C} \sum_{i=1}^{n} 2^i \alpha(\frac{1}{2^i}) \leq \hat{C} \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}) < \infty. \tag{22}$$

Let $\lambda \leq \frac{1}{2}$. Let $\frac{1}{2^n} \leq \lambda < \frac{1}{2^{n-1}}$, $n = 2, 3, \dots$ . Since $\hat{g}(t)$ is $\alpha(\cdot)$-paraconvex we have

$$\hat{g}(\lambda) \leq \max[\hat{g}(\frac{1}{2^n}), \hat{g}(\frac{1}{2^{n-1}})] + \hat{C}\alpha(\frac{1}{2^n}) \leq \frac{\hat{C}}{2^{n-1}} \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}) + \hat{C}\alpha(\frac{1}{2^n}). \tag{23}$$

Hence

$$\frac{\hat{g}(\lambda)}{\lambda} \leq 2^n \left( \frac{\hat{C}}{2^{n-1}} \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}) + \hat{C}\alpha(\frac{1}{2^n}) \right)$$

$$\leq 3\hat{C} \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}) < +\infty. \tag{24}$$

In a similar way we can show that for $\frac{1}{2} \leq \lambda \leq 1$

$$\frac{\hat{g}(\lambda)}{1-\lambda} \leq 3\hat{C} \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}) < +\infty. \tag{25}$$

It means that

$$\hat{g}(\lambda) \leq L\hat{C} \min[\lambda, (1-\lambda)], \tag{26}$$

where

$$L = 3 \sum_{i=1}^{\infty} 2^i \alpha(\frac{1}{2^i}).$$

Thus we have proved sufficiency in the case of $x = 0$ and $y = 1$. Now let $x, y$ be arbitrary. Let $h(t) = \hat{g}(x + t(y - x))$. Observe that $\hat{g}(\lambda x + (1-\lambda)y) = h(\lambda)$, in particular $h(0) = \hat{g}(x)$ and $h(1) = \hat{g}(y)$.

Suppose that the function $\hat{g}(x)$ is $\alpha(\cdot)$-paraconvex with a constant $\hat{C} > 0$

$$\hat{g}(\lambda x + (1-\lambda)y) \leq \lambda\hat{g}(x) + (1-\lambda)\hat{g}(y) + \hat{C}\alpha(|x-y|).$$

Thus

$$h(\lambda) \leq \lambda h(0) + (1-\lambda)h(1) + \hat{C}\alpha(|x-y|).$$

Applying (26) to $h(\lambda)$ we get

$$h(\lambda) \leq \min[\lambda, (1-\lambda)] L\hat{C}\alpha(|x-y|).$$

Therefore

$$\hat{g}(\lambda x + (1-\lambda)y) \leq \min[\lambda, (1-\lambda)] L\hat{C}\alpha(|x-y|). \tag{27}$$

Now we shall apply (27) to $\hat{g}(t) = F_{x,y}(t)$ and recalling the definition of $F_{x,y}(t)$ in the case of $\lambda\|x-y\| \leq 1$ we get

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) + \min[\lambda, (1-\lambda)] L\hat{C}\alpha(\|x-y\|) \tag{28}$$

and finally combining (28) and (19) we obtain

$$f\big(\lambda x + (1-\lambda)y\big) \leq \min[\lambda, (1-\lambda)] \max(\alpha(1)\frac{C}{K}, L\hat{C})\alpha(\|x-y\|).$$

■

COROLLARY 1 *(Jourani (1996)). Let $(X, \|\cdot\|)$ be a normed space. Let $1 \leq \gamma \leq 2$. Then every $\gamma$-paraconvex (i.e. $t^\gamma$-paraconvex) function is simultaneously strongly $\gamma$-paraconvex if and only if $1 < \gamma$.*

Since $\alpha(\cdot)$ is a non-decreasing function, condition (15) implies

$$\lim_{t\to 0} \frac{\alpha(t)}{t} = 0. \tag{29}$$

There are also non-decreasing functions $\alpha(\cdot)$ different from $t^\gamma$, $1 < \gamma$, such that (15) holds. Indeed let $1 \leq \gamma \leq 2$ and let

$$\alpha(t) = \begin{cases} \frac{t}{|\lg t|^\gamma} \ for \ 0 < t \leq \frac{1}{e}, \\ \frac{1}{e} \ for \ \frac{1}{e} < t. \end{cases}$$

It is easy to see that $\alpha(\cdot)$ is a nondecreasing function and that (11) holds for $1 < \gamma \leq 2$ and does not hold for $\gamma = 1$. Thus, we have

COROLLARY 2 . *Let $(X, \|\cdot\|)$ be a normed space. Let $1 \leq \gamma \leq 2$ and let*

$$\alpha(t) = \begin{cases} \frac{t}{|\lg t|^\gamma} \ for \ 0 < t \leq \frac{1}{e}, \\ \frac{1}{e} \ for \ \frac{1}{e} < t, \end{cases}$$

*Then every $\alpha(\cdot)$-paraconvex function is simultaneously strongly $\alpha(\cdot)$-paraconvex if and only if $1 < \gamma$.*

Repeating the consideration of Jourani (1996) we shall prove

PROPOSITION 5 *Let $(X, \|\cdot\|)$ be a normed space. Let a real-valued function $f$ defined on a convex set $\Omega \subset X$ be strongly $\alpha(\cdot)$-paraconvex with constant $C$. Suppose that it is locally bounded. Then it is locally Lipschitzian.*

*Proof.* Let $x_0 \in \Omega$ be arbitrary. Since $f$ is locally bounded, there are $r, a > 0$ such that for any $z \in \Omega$ such that $\|z - x_0\| < r$ we have

$$|f(z)| < a.$$

Let $x, u$ be two arbitrary elements of $\Omega$ such that $\|x - x_0\| < \frac{r}{2}$, $\|u - x_0\| < \frac{r}{2}$. Let $\varepsilon$ be an arbitrary positive number, let $\beta = \varepsilon + \|x - u\|$ and let

$$z = u + \frac{r}{2\beta}(u - x). \tag{30}$$

Observe that

$$\|z - x_0\| < \|u - x_0\| + \frac{r}{2\beta}\|u - x\| < \frac{r}{2} + \frac{r}{2}\frac{\|x - u\|}{\varepsilon + \|x - u\|} < r$$

and so

$$|f(z)| < a.$$

Let $\lambda = \frac{2\beta}{r + 2\beta}$. Observe that $u = \lambda z + (1 - \lambda)x$.
Since the function $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex with constant $C$,

$$f(u) = f\big(\lambda z + (1 - \lambda)x\big) \leq \lambda f(z) + (1 - \lambda)f(x) + C\lambda\alpha(\|x - z\|). \tag{31}$$

Thus,

$$f(u) - f(x) \leq \lambda(f(z) - f(x)) + C\lambda\alpha(\|x - z\|). \tag{32}$$

Since $\lambda\|z - x\| = \|u - x\|$ we get

$$f(u) - f(x) \leq \lambda(f(z) - f(x)) + C\lambda\alpha(\frac{\|u - z\|}{\lambda}). \tag{33}$$

Recall that $0 < \lambda < 1$ and thus

$$f(u) - f(x) \leq \lambda(f(z) - f(x)) + C\lambda\alpha(\|x - z\|) \leq \lambda\big(2a + C\alpha(2r)\big)$$

$$\leq \frac{2\beta}{r}\big(2a + C\alpha(2r)\big) \leq L(\varepsilon + \|u - x\|),$$

where $L = \frac{2}{r}\big(2a + C\alpha(2r)\big)$.
Exchanging the role of $x$ and $u$ we get

$$|f(u) - f(x)| \leq L(\varepsilon + \|u - x\|).$$

The arbitrariness of $\varepsilon$ implies

$$|f(u) - f(x)| \leq L\|u - x\|. \tag{34}$$

■

As every continuous function is locally bounded we get the following consequence:

COROLLARY 3 *Let $(X, \|\cdot\|)$ be a normed space. Let a continuous real-valued function $f(\cdot)$ defined on a convex set $\Omega \subset X$ be strongly $\alpha(\cdot)$-paraconvex with constant $C$. Then it is locally Lipschitzian.*

PROPOSITION 6 *Let $(X, \|\cdot\|)$ be a Banach space. Let a real-valued function $f(\cdot)$ defined on an open convex set $\Omega \subset X$ be strongly $\alpha(\cdot)$-paraconvex with a constant $C$. Then it is locally Lipschitzian.*

*Proof.* Let $\Omega_0$ be a closed convex set with non-empty interior. Let $\Omega_m = \{x \in \Omega_0 : |f(x)| \leq m\}$. Using the category method we can show that one among those sets contains an open ball, $B(x_0, r) \subset \Omega_{m_0}$. Take an arbitrary point $x \in \Omega$. Since $\Omega$ is open there is $y \in \Omega$ and $t > 0$ such that $x = (1-t)y + tx_0$. Let $z \in \Omega$ be such that $\|x - z\| < rt$. Then we can represent $z$ in the form $z = tu + (1-t)y$, where $u \in B(x_0, r) \subset \Omega_{m_0}$. Thus

$$f(z) \leq tf(y) + (1-t)m_0 \leq \max(f(y), m_0),$$

i.e. the function $f(z)$ is majorized on $B(x, rt)$. Thus by Proposition 5 it is locally Lipschitzian. ■

As a consequence of Propositions 1 and 6 we obtain

PROPOSITION 7 *Let $f(\cdot)$ be a strongly $\alpha(\cdot)$-paraconvex function defined on a convex set $\Omega$ of a normed space $X$. If*

$$\liminf_{t \to 0+0} \frac{\alpha(t)}{t^2} = 0, \tag{35}$$

*then $f(\cdot)$ is convex.*

*Proof.* Take an arbitrary straight line $L$ and let $\Omega_L = \Omega \cap L$. Now let $f_L(\cdot)$ denote the restriction of the function $f(\cdot)$ to the set $\Omega_L$, $f_L(\cdot) = f|_{\Omega_L}(\cdot)$. By $Int_a\Omega_L$ we shall denote the relative interior ( with respect to the line $L$) of the set $\Omega_L$. By Proposition 10 the function the function $f(\cdot)$ is locally Lipschitz on $Int_a\Omega_L$. Then, it is absolutely continuous on every closed interval $[a, b]$ contained in $\Omega_L$. By Proposition 1 it is convex on [a,b]. The arbitrariness of [a,b] implies that $f_L$ is convex on $Int_a\Omega_L$. It finishes the proof in the case of $\Omega_L = Int_a\Omega_L$. Suppose that $c \in \Omega_L$ and $c \notin Int_a\Omega_L$. Since $f_L$ is $\alpha(\cdot)$-paraconvex, we trivially obtain that

$$\lim_{\substack{x \to c, \\ x \in \Omega_L}} f(x) \leq f(c).$$

Therefore the function $f_L$ is convex on $\Omega_L$. The arbitrariness of $L$ implies that $f(\cdot)$ is convex. ■

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