Strong decay for one-dimensional wave equations with nonmonotone boundary damping

by

Michel Pierre* and Judith Vancostenoble**

*E.N.S. Cachan, Antenne de Bretagne
and I.R.M.A.R. Université Rennes I
Campus de Ker Lann, 35 170 Bruz, France
E-mail: pierre@bretagne.ens-cachan.fr

**M.I.P. Université Paul Sabatier Toulouse III
118 route de Narbonne, 31 062 Toulouse Cedex 4, France
E-mail: vancoste@mip.ups-tlse.fr

Abstract: This paper is a contribution to the following question: consider the classical wave equation damped by a nonlinear feedback control which is only assumed to decrease the energy. Then, do solutions to the perturbed system still exist for all time? Does strong stability occur in the sense that the energy tends to zero as time tends to infinity? We prove here that the answer to both questions is positive in the specific case of the one-dimensional wave equation damped by boundary controls which are functions of the observed velocity. The main point is that no monotonicity assumption is made on the damping term.

Keywords: wave equation, global existence, asymptotic behavior, stabilization, boundary control, nonmonotone feedback, D'Alembert formula.

1. Introduction

This paper is a contribution to the following question: consider the classical wave equation damped by a nonlinear feedback control which is *only assumed to decrease the energy*. In particular, no monotonicity assumption is made. Then, do solutions to the perturbed system still exist for all time? Does strong stability occur in the sense that the energy tends to zero as time tends to infinity?

We give here a complete and positive answer to this question in the simple case of the one-dimensional wave equation damped by boundary feedback controls. More precisely, let us consider the following system:

$$u_{tt} - u_{xx} = 0, \ x \in (0,1), \ t \geq 0, \tag{1.1}$$

$$u(0,t) = 0, \ t \geq 0, \tag{1.2}$$

$$u_x(1,t) = -q(u_t(1,t)), \ t \geq 0, \tag{1.3}$$

$$(u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), \ x \in (0,1), \tag{1.4}$$

with initial conditions (u_0, u_1) given in $V \times L^2(0,1)$ (where $V = \{v \in H^1(0,1) \mid v(0) = 0\}$), under the basic assumption that

$$q : \mathrm{R} \to \mathrm{R} \text{ is continuous and satisfies } \forall \lambda \in \mathrm{R}, \ \lambda q(\lambda) \geq 0. \tag{1.5}$$

Then the questions are: does the system (1.1)–(1.4) have a global solution in time? Does the energy tend to zero as $t \to +\infty$ if, for instance, $\lambda q(\lambda) > 0$ for $\lambda \neq 0$ (which corresponds to a "strict" decrease of energy)?

Indeed, solutions of (1.1)-(1.3) satisfy the well-known energy equality

$$\forall t_1, t_2 \in \mathrm{R}_+, \ E_u(t_2) - E_u(t_1) = -\int_{t_1}^{t_2} u_t(1,t)q(u_t(1,t)) \, dt,$$

where the energy of u is given by

$$\forall t \in \mathrm{R}_+, \ E_u(t) = \frac{1}{2}(\|u_x(\cdot,t)\|_{L^2(0,1)}^2 + \|u_t(\cdot,t)\|_{L^2(0,1)}^2).$$

Thus, under assumption (1.5), energy is nonincreasing and the trajectories $(u(\cdot), u_t(\cdot))$ are bounded in the energy space. The term q represents a damping force which is a nonlinear function of the observed velocity.

The question is to decide whether this only assumption on the control q provides

— global existence of solutions for (1.1)–(1.4),

— convergence to zero of $E_u(t)$ as $t \to +\infty$, when inequality in (1.5) is strict (see (1.6) below).

Let us recall the state of the art on this question.

When $q : \mathrm{R} \to \mathrm{R}$ is continuous **increasing** such that $q(0) = 0$, global existence of solutions of (1.1)-(1.4) is known for all initial conditions (u_0, u_1) given in $V \times L^2(0,1)$ (where $V = \{v \in H^1(0,1) \mid v(0) = 0\}$). This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis, 1973).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, \ q(\lambda) \neq 0$, or even the "unilateral" condition

$$\forall \lambda > 0, \ q(\lambda) > 0 \ (\text{or } \forall \lambda < 0, \ q(\lambda) < 0), \tag{1.6}$$

then strong asymptotic stability of solutions occurs in $V \times L^2(0,1)$, i.e.,

This result follows, for instance, from the invariance principle of LaSalle (see for example Haraux, 1979, Lasiecka, 1989, Conrad, Pierre, 1994). This is a very specific situation of a general setting for evolution equations of second order (wave, beam or plate equations...) in a bounded open subset Ω of $\mathbb{R}^N$ with a nonlinear damping $q(u_t)$ applied to a part of Ω or of its boundary: a monotonicity assumption on q and a growth condition at infinity ensure strong compactness of trajectories.

But, if we remove these hypotheses, few results seem to be known. If we assume $q : \mathbb{R} \to \mathbb{R}$ continuous satisfying (1.5) and such that

$$\forall \alpha > 0, \ \inf\{q(\lambda) \mid \lambda \geq \alpha\} > 0,$$

then at least **weak** asymptotic stability of all global solutions holds, i.e.,

$$(u(\cdot, t), u_t(\cdot, t)) \xrightarrow[t \to +\infty]{} (0, 0) \text{ weakly in the energy space.}$$

This is a particular case of a general result of weak stability in Vancostenoble, 1998a, 1998b; see also Slemrod, 1989, for other results in this spirit.

However, strong stability under assumptions (1.5)-(1.6) as well as even global existence under (1.5) seemed to be an open problem. We solve it here completely in the particular case of equation (1.1)-(1.4).

Note also that results of global existence and strong stability may also be found in the literature for other one-dimensional problems with non monotone distributed feedback controls, but with some restrictions on the initial data or on the control (see e.g. Feireisl, 1993a and also Feireisl, 1993b, Feireisl, O'Dowd, 1998, and Vancostenoble, 1998b, c).

We prove here existence, uniqueness and strong stability for the boundary problem (1.1)-(1.4). The proof is elementary and essentially based on the particular structure of solutions of (1.1) given by D'Alembert formula.

2. Results

2.1. Global existence of solutions

The main existence result is

THEOREM 1 *Suppose that* $q : \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies (1.5). For all* $(u_0, u_1) \in V \times L^2(0, 1)$, *there exists* $u(x, t)$ *solution of (1.1)-(1.4) such that*

$$u(x, t) = f(t + x) + g(t - x), \ (x, t) \in (0, 1) \times (0, +\infty), \tag{2.1}$$

where $f : (0, +\infty) \to \mathbb{R}$ *and* $g : (-1, +\infty) \to \mathbb{R}$ *are absolutely continuous functions such that*

REMARKS. 1. For such a solution, the boundary condition (1.3) makes sense for almost every $t \in R_+$, indeed $u_x(1, \cdot) = f'(\cdot + 1) - g'(\cdot - 1)$ and $u_t(1, \cdot) = f'(\cdot + 1) + g'(\cdot - 1)$ exist and belong to $L^2_{\text{loc}}([0, +\infty))$.

2. We can verify that u belongs to the class

$$(u, u_t) \in C([0, +\infty[; V \times L^2(0, 1)). \tag{2.3}$$

3. Condition (1.5) is essential. Global existence may fail even if q is a Lipschitz continuous function. Indeed, we can verify that for $q = -Id$ and $(u_0, u_1) \neq (0, 0)$ the system (1.1)–(1.4) has no solution (see Vancostenoble, 1998b).

4. If we assume $(u_0, u_1) \in (V \cap W^{1,\infty}(0, 1)) \times L^\infty(0, 1)$, then we can easily prove that

$$\|f'\|_{L^\infty(0, +\infty)}, \|g'\|_{L^\infty(-1, +\infty)} \leq \frac{1}{2}(\|u'_0\|_{L^\infty(0,1)} + \|u_1\|_{L^\infty(0,1)}).$$

This was the case considered in Feireisl (1993a) for distributed control.

About uniqueness: it is known that any weak solution of (1.1) has the structure given by (2.1). If we impose that $u_x(1, \cdot)$ and $u_t(1, \cdot)$ belong to $L^2_{\text{loc}}([0, +\infty))$, then f and g verify (2.2). The "natural" space for the solutions of (1.1) is therefore given by (2.1)–(2.2). In this class, we have the following uniqueness result:

PROPOSITION 1 *Under hypotheses of Theorem 1, and if q verifies*

$$\forall \lambda_1, \lambda_2 \in R \text{ such that } \lambda_1 \neq \lambda_2, \ \frac{q(\lambda_1) - q(\lambda_2)}{\lambda_1 - \lambda_2} > -1, \tag{2.4}$$

then u is the unique solution of (1.1)–(1.4) in the class (2.1)–(2.2).

2.2. Asymptotic stability

We give the following result of strong asymptotic stability:

THEOREM 2 *Assume that $q : R \to R$ is continuous and satisfies (1.5) and (1.6). Let (u_0, u_1) be given in $V \times L^2(0, 1)$. Then, for all solution u of (1.1)–(1.4) in the class (2.1)-(2.2),*

$$(u(t), u_t(t)) \xrightarrow[t \to +\infty]{} (0, 0) \text{ strongly in } V \times L^2(0, 1).$$

REMARK. Note that strong asymptotic stability occurs for *all global solutions* even in the case of non-uniqueness of solutions of (1.1)-(1.4).

REMARK. If condition (1.6) is not satisfied, the conclusion is false. Indeed, if there exists $\lambda_0 \neq 0$ such that $q(\lambda_0) = q(-\lambda_0) = 0$, then even weak asymptotic

PROPOSITION 2 *Let* $q : \mathrm{R} \to \mathrm{R}$ *be a continuous function satisfying (1.5) and such that* $q(\lambda_0) = q(-\lambda_0) = 0$ *for some* $\lambda_0 \in \mathrm{R}$. *Then there exists* u *solution of (1.1)–(1.4) such that* $u(t) \xrightarrow[t\to+\infty]{\hspace{0.8cm}/\hspace{0.8cm}} 0$ *weakly in* V.

REMARK. The existence of (at least) one solution of (1.1)–(1.4) is given by Theorem 1. In this counter-example, we explicitly give a solution: its originality is that it satisfies $u_t(1,t) = \pm\lambda_0$ a.e. t and $u_x(1,t) = 0$ a.e. t, so that (1.3) is satisfied a.e. t.

2.3. Comments

We completely solved the problems of existence and of strong stability for equation (1.1)–(1.4) assuming only that the initial conditions belong to the energy space and that q satisfies (1.5)–(1.6). Our proof is elementary and essentially based on the particular structure of solutions of (1.1) given by D'Alembert formula.

It would be interesting to study the same questions when this formula does not apply : for instance, we could replace (1.1) by $u_{tt} - (au_x)_x = 0$ where $a : [0,1] \to \mathrm{R}$ is a regular positive function. In the same spirit, it would also be interesting to study other one-dimensional equations for which some results of existence and strong stability exist in the literature, but only with additional assumptions. For example, E. Feireisl (1993a) obtained similar results for a wave equation with distributed damping for $(u_0, u_1) \in W^{1,\infty}(0,1) \times L^\infty(0,1)$ and q of class $\mathcal{C}^1$. Is it possible to remove the regularity assumptions, especially on the initial data? (See also Feireisl, 1993b, for similar results for a beam equation assuming that q is Lipschitz continuous and see Feireisl, O'Dowd, 1998, Vancostenoble, 1998b, c, for similar results for hybrid systems with the restriction that q is locally increasing at 0).

In higher-dimensional spaces, few results seem to be known. We proved in a very general setting (see Vancostenoble, 1998a, 1998b), that (1.5)–(1.6) imply at least weak stability of all global solutions. Adding some restrictions on the initial data and on the control, we managed to prove strong stability in the case of the wave equation with a distributed control (see Martinez, Vancostenoble). However, even existence (assuming only (1.5)) seems to be open and no result of strong stability (assuming only (1.5)–(1.6)) nor a counter-example to strong stability seem to be known.

3. Proofs

3.1. Proof of global existence of solutions

First of all, let us analyse the structure of solutions of (1.1)–(1.4): a general solution of (1.1) is

where f and g are two absolutely continuous functions respectively defined on $(0, +\infty) \to \mathrm{R}$ and $(-1, +\infty) \to \mathrm{R}$. Initial conditions (1.4) may be written

$$f(x) + g(-x) = u_0(x), \ x \in (0, 1), \tag{3.2}$$

and

$$f'(x) + g'(-x) = u_1(x), \ x \in (0, 1). \tag{3.3}$$

Consequently,

$$\begin{cases} f'(\lambda) = \frac{1}{2}(u_1(\lambda) + u'_0(\lambda)), & \lambda \in (0, 1), \\ g'(\lambda) = \frac{1}{2}(u_1(-\lambda) - u'_0(-\lambda)), & \lambda \in (-1, 0). \end{cases}$$

Condition (1.2) imposes

$$f(\lambda) = -g(\lambda), \ \lambda \in \mathrm{R}_+, \tag{3.4}$$

and condition (1.3) becomes

$$f'(t + 1) - g'(t - 1) = -q(f'(t + 1) + g'(t - 1)), \ t \in \mathrm{R}_+,$$

or, with $\lambda = t + 1$,

$$f'(\lambda) - g'(\lambda - 2) = -q(f'(\lambda) + g'(\lambda - 2)), \ \lambda \geq 1. \tag{3.5}$$

In particular,

$$f'(\lambda) + f'(\lambda - 2) = -q(f'(\lambda) - f'(\lambda - 2)), \ \lambda \geq 2.$$

For the proofs of Theorem 1 and Proposition 1, we will need the following technical result:

LEMMA 1 *Let $q : \mathrm{R} \to \mathrm{R}$ be continuous satisfying (1.5).*
  (i) *For all $A \in \mathrm{R}$, the equation*
$$X \in \mathrm{R}, \ X + A + q(X - A) = 0, \tag{3.6}$$
    has a solution $S(A)$ of smallest absolute value and the application $S : \mathrm{R} \to \mathrm{R}$ is lower-semi-continuous (l.s.c.) on R.
  (ii) *Moreover, any solution X of (3.6) verifies $|X| \leq |A|$.*
  (iii) *Furthermore, if q also verifies (2.4), then equation (3.6) has a unique solution.*

Proof of Lemma 1. (i) Let A be given in R. We denote by $f_A : \mathrm{R} \to \mathrm{R}$ the continuous function defined by

$$\forall X \in \mathrm{R}, \ f_A(X) = X + A + q(X - A).$$

We deduce from hypothesis (1.5) that

Consequently, there exists $X \in \mathrm{R}$ such that $f_A(X) = 0$, i.e. such that X is solution of (3.6). So we can define an application S by

$$S : \begin{cases} \mathrm{R} \to \mathrm{R} \\ A \mapsto X_A = \inf\{X \in \mathrm{R} \mid X \text{ solution of (3.6)}\}. \end{cases}$$

By definition of S, there exists $(X_n)_n$ a sequence of R such that

$$\begin{cases} X_n \to S(A) \text{ as } n \to +\infty, \\ \forall n \in \mathrm{N}, \ X_n + A + q(X_n - A) = 0. \end{cases}$$

Passing to the limit as $n \to +\infty$, we deduce that $S(A)$ is solution of (3.6). Moreover, S is also l.s.c. on R.

(ii) For all $A \in \mathrm{R}$ and for all $X \in \mathrm{R}$ solution of (3.6), we multiply (3.6) by $(X - A)$, which gives

$$(X^2 - A^2) = -q(X - A)(X - A) \leq 0.$$

Consequently, $|X| \leq |A|$.

(iii) Assume that q also verifies (2.4). Let A be given in R and assume that equation (3.6) has two distinct solutions X and X'. Then

$$\frac{q(X - A) - q(X' - A)}{(X - A) - (X' - A)} = -1,$$

which is in contradiction with (2.4). ∎

Proof of Theorem 1. We define f' on $(0, +\infty) \to \mathrm{R}$ by

$$f'(\lambda) = \frac{u_1(\lambda) + u'_0(\lambda)}{2} \text{ a.e. } \lambda \in (0, 1), \tag{3.7}$$

$$f'(\lambda) = S\left(-\frac{u_1(-\lambda + 2) - u'_0(-\lambda + 2)}{2}\right) \text{ a.e. } \lambda \in (1, 2), \tag{3.8}$$

and

$$f'(\lambda) = S(f'(\lambda - 2)) \text{ a.e. } \lambda \in (2, +\infty). \tag{3.9}$$

Then we define g' on $(-1, +\infty) \to \mathrm{R}$ by

$$g'(\lambda) = \frac{u_1(-\lambda) - u'_0(-\lambda)}{2} \text{ a.e. } \lambda \in (-1, 0),$$

and

$$f(\lambda) = -g(\lambda) \text{ a.e. } \lambda \in (0, +\infty).$$

Clearly, f' and g' satisfy conditions (3.2), (3.3) and (3.4). And, from Lemma 1, (3.5) is also verified.

Moreover, we prove

Indeed, from (3.8) and Lemma 1, it follows

$$|f'(\lambda)| \leq \left| -\frac{u_1(-\lambda+2) - u_0'(-\lambda+2)}{2} \right| \quad \text{a.e. } \lambda \in (1,2). \tag{3.10}$$

Consequently, from (3.7) and (3.10), we have

$$\|f'\|_{L^2(0,2)}^2 \leq \|u_0'\|_{L^2(0,1)}^2 + \|u_1\|_{L^2(0,1)}^2.$$

Similarly, from (3.9), we obtain for all $k \in \mathrm{N}$,

$$\|f'\|_{L^2(2k,2k+2)}^2 \leq \|f'\|_{L^2(2k-2,2k)}^2 \leq \cdots \leq \|u_0'\|_{L^2(0,1)}^2 + \|u_1\|_{L^2(0,1)}^2. \tag{3.11}$$

Finally, we define for $(x,t) \in (0,1) \times (0,+\infty)$,

$$u(x,t) = \int_0^{t+x} f'(s)\,ds + \int_0^{t-x} g'(s)\,ds + u_0(0). \tag{3.12}$$

Clearly u belongs to the class (2.1)–(2.2) and is solution of (1.1)–(1.4). ■

REMARK. When (2.4) is not satisfied, we can use, instead of S, any other bounded and measurable section of (3.6). This also leads to a solution of (1.1)–(1.4). On examples, we can construct several distinct solutions of (1.1)–(1.4).

REMARK. Using the properties of f and g, we can prove that u belongs to the class (2.3). Indeed, since $f' \in L^2_{\mathrm{loc}}([0,+\infty))$ and $g' \in L^2_{\mathrm{loc}}([-1,+\infty))$, we deduce from the expression (2.1) of u that $u, u_t, u_x \in \mathcal{C}([0,+\infty[; L^2(0,1))$. Let t_0 be fixed in R. Then

$$\|u_t(\cdot,t) - u_t(\cdot,t_0)\|_{L^2(0,1)}^2$$
$$\leq 2\|f'(t+\cdot) - f'(t_0+\cdot)\|_{L^2(0,1)}^2 + 2\|g'(t-\cdot) - g'(t_0-\cdot)\|_{L^2(0,1)}^2.$$

The application "translation" is continuous on $\mathrm{R} \to L^2(0,1)$ (see, for example, Rudin, 1991), i.e. $\forall t_0 \in \mathrm{R}, \forall h \in L^2(0,1)$, $h(\cdot+t) \xrightarrow[t \to t_0]{} h(\cdot+t_0)$ in $L^2(0,1)$. Consequently, $u_t(\cdot,t) \xrightarrow[t \to t_0]{} u_t(\cdot,t_0)$ in $L^2(0,1)$. And similarly, we have $u_x(\cdot,t) \xrightarrow[t \to t_0]{} u_x(\cdot,t_0)$ in $L^2(0,1)$ and $u(\cdot,t) \xrightarrow[t \to t_0]{} u(\cdot,t_0)$ in $L^2(0,1)$. ■

Proof of Proposition 1. Let u be a solution of (1.1)–(1.4) belonging to the class (2.1)–(2.2). The analysis of the structure of solutions imposes that f and g verify (3.2)–(3.5). Since q satisfies (2.4), Lemma 1 implies that f' and g' are uniquely determined respectively in $L^2_{\mathrm{loc}}([0,+\infty))$ and $L^2_{\mathrm{loc}}([-1,+\infty))$. Consequently, u is uniquely determined by (3.12). ■

3.2. Proof of asymptotic stability

LEMMA 2 *For all sequence $(t_n)_n$ of R such that $t_n \xrightarrow[n \to +\infty]{} +\infty$, and for all $T > 0$,*

$$\int_0^T |q(u_t(1, t + t_n))| \, dt \xrightarrow[n \to +\infty]{} 0, \tag{3.13}$$

and

$$\int_0^T (u_t(1, t + t_n))^+ \, dt \xrightarrow[n \to +\infty]{} 0, \tag{3.14}$$

where $(\cdot)^+ = \max(\cdot, 0)$.

We choose $T = 2$, $t_n = 2n$ for all $n \in \mathbb{N}^*$ and we set $F_n(s) = f'(s + 2n)$. With the new variable $s = t - 1$, we obtain

$$\int_{-1}^1 |q(u_t(1, s + 1 + 2n))| \, ds = \int_{-1}^1 |q(F_{n+1}(s) - F_n(s))| \, ds \xrightarrow[n \to +\infty]{} 0,$$

and

$$\int_{-1}^1 (u_t(1, s + 1 + 2n))^+ \, ds = \int_{-1}^1 (F_{n+1}(s) - F_n(s))^+ \, ds \xrightarrow[n \to +\infty]{} 0.$$

Condition (1.3) may be written: $\forall n \in \mathbb{N}^*$, a.e. $s \in (-1, 1)$,

$$F_{n+1}(s) + F_n(s) = -q(F_{n+1}(s) - F_n(s)). \tag{3.15}$$

Thus

$$\int_{-1}^1 |F_{n+1}(s) + F_n(s)| \, ds \xrightarrow[n \to +\infty]{} 0. \tag{3.16}$$

Using

$$2(F_{n+1})^+ \le (F_{n+1} + F_n)^+ + (F_{n+1} - F_n)^+,$$
$$2(F_n)^- \le (F_{n+1} - F_n)^+ + (F_{n+1} + F_n)^-,$$

we deduce from (3.15)–(3.16) that $F_n \xrightarrow[n \to +\infty]{} 0$ in $L^1(-1, 1)$. Since $n \mapsto |F_n(s)|$ is nonincreasing (by (3.9) and Lemma 1), we deduce $F_n(s) \xrightarrow[n \to +\infty]{} 0$ a.e. s and

$$\int_{-1}^1 f'(s + 2n)^2 \, ds = \int_{-1}^1 F_n(s)^2 \, ds \xrightarrow[n \to +\infty]{} 0.$$

Finally, $u_t(\cdot, 2n) = f'(2n + \cdot) - f'(2n - \cdot)$ and $u_x(\cdot, 2n) = f'(2n + \cdot) + f'(2n - \cdot)$ converge strongly to 0 in $L^2(0, 1)$. Since $E_u(t)$ is a nonincreasing, we deduce

Proof of Lemma 2. Let $T > 0$ be fixed and let $(t_n)_n$ be a sequence of R such that $t_n \xrightarrow[t \to +\infty]{} +\infty$. The energy equality gives

$$E_u(t_n + T) - E_u(t_n) = -\int_0^T u_t(1, t + t_n) q(u_t(1, t + t_n)) \, dt.$$

There exists $L = \lim\limits_{t \to +\infty} E_u(t)$. So $\lim\limits_{n \to +\infty} E_u(t_n) = \lim\limits_{n \to +\infty} E_u(t_n + T) = L$. Therefore

$$\lim_{n \to +\infty} \int_0^T u_t(1, t + t_n) q(u_t(1, t + t_n)) \, dt = 0. \tag{3.17}$$

First, in order to prove (3.13), we fix $\varepsilon > 0$. By continuity of q at 0, there exists $\eta(\varepsilon)$ such that

$$\int_0^T_{|u_t(1, t+t_n)| \leq \eta(\varepsilon)} |q(u_t(1, t + t_n))| \, dt \leq \varepsilon.$$

So, we have

$$\int_0^T |q(u_t(1, t + t_n))| \, dt \leq \int_0^T_{|u_t(1, t+t_n)| \geq \eta(\varepsilon)} |q(u_t(1, t + t_n))| \, dt$$
$$+ \int_0^T_{|u_t(1, t+t_n)| \leq \eta(\varepsilon)} |q(u_t(1, t + t_n))| \, dt$$
$$\leq \frac{1}{\eta(\varepsilon)} \int_0^T u_t(1, t + t_n) q(u_t(1, t + t_n)) \, dt + \varepsilon.$$

Finally, from (3.17), we obtain (3.13).

Next, in order to prove (3.14), we fix $\alpha > 0$ and $k > 0$. Then

$$\int_0^T \left(u_t(1, t + t_n) \right)^+ dt = \int_0^T_{0 \leq u_t(1, t+t_n) \leq \alpha} u_t(1, t + t_n) \, dt$$
$$+ \int_0^T_{\alpha \leq u_t(1, t+t_n) \leq k} u_t(1, t + t_n) \, dt + \int_0^T_{k \leq u_t(1, t+t_n)} u_t(1, t + t_n) \, dt$$
$$\leq \alpha T + \frac{1}{C_{\alpha,k}} \int_0^T u_t(1, t + t_n) q(u_t(1, t + t_n)) \, dt + \frac{C_T}{k},$$

where $C_{\alpha,k} = \inf\{q(\lambda) \mid \alpha \leq \lambda \leq k\} > 0$, (since $q(\lambda) > 0$ for all $\lambda > 0$, we have $C_{\alpha,k} > 0$), and where $C_T > 0$ is a constant. Indeed, we have

$$u_t(1, t + t_n) = f'(t + t_n + 1) + g'(t + t_n - 1), \ t \in (0, T).$$

From (3.11), we deduce that there exists a constant $C_T > 0$ such that

Consequently,

$$k \int_0^T {}_{k \le u_t(1,t+t_n)} u_t(1, t + t_n)\, dt \le \int_0^T |u_t(1, t + t_n)|^2\, dt \le C_T,$$

which proves the result. Finally we deduce (3.14) from (3.17).   ∎

3.3. Counter-example to weak stability

Proof of proposition 2. We introduce $f : (-1, +\infty) \to \mathrm{R}$ an absolutely continuous function such that

$$f'(\lambda) = -\frac{\lambda_0}{2} \text{ a.e. } \lambda \in (-1, 0) \text{ and a.e. } \lambda \in (4k+2, 4k+4), \ k \in \mathrm{N},$$

$$f'(\lambda) = \frac{\lambda_0}{2} \text{ a.e. } \lambda \in (4k, 4k+2), \ k \in \mathrm{N}.$$

Let then u be defined by

$$u(x, t) = f(t + x) - f(t - x), \ (x, t) \in (0, 1) \times \mathrm{R}_+.$$

We easily verify that u is solution of (1.1) with (1.2) and (1.4). And

$$u_x(1, t) = f'(t + 1) + f'(t - 1) = \pm \frac{\lambda_0}{2} + \left(\mp \frac{\lambda_0}{2} \right) = 0,$$

$$u_t(1, t) = f'(t + 1) - f'(t - 1) = \pm \frac{\lambda_0}{2} - \left(\mp \frac{\lambda_0}{2} \right) = \pm \lambda_0.$$

This implies (1.3) since $q(u_t(1, t)) = q(\pm \lambda_0) = 0 = -u_x(1, t)$ a.e. $t \in \mathrm{R}_+$.

On the other hand, with $t_k = 4k + 1$ for all $k \in \mathrm{N}$, we have for all $k \in \mathrm{N}$,

$$u_x(x, t_k) = f'(t_k + x) + f'(t_k - x) = \frac{\lambda_0}{2} + \frac{\lambda_0}{2} = \lambda_0 \text{ a.e. } x \in (0, 1),$$

(since $t_k + x$ and $t_k - x \in (4k, 4k+2)$). We denote by φ the function of V defined by $\varphi(x) = x$ for all $x \in [0, 1]$.

$$\forall k \in \mathrm{N}, \ (u(t_k), \varphi)_V = (u_x(t_k), \varphi_x)_{L^2(0,1)} = \int_0^1 u_x(x, t_k)\, dx = \lambda_0.$$

Thus, $u(t_k) \underset{k \to +\infty}{\not\longrightarrow} 0$ in V.   ∎

References

BREZIS H. (1973) *Opérateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert.* North-Holland, Amsterdam.

CONRAD F., PIERRE M. (1994) Stabilization of second order evolution equations by unbounded nonlinear feedbacks. *Ann. Inst. H. Poincaré Anal.*

FEIREISL E. (1993A) Strong decay for wave equations with nonlinear nonmonotone damping. *Nonlinear Anal.*, **21**, 1, 49-63.

FEIREISL E. (1993B) Strong asymptotic stability for a beam equation with weak damping. *Proc. Roy. Soc. Edinburgh*, **123A**, 365-371.

FEIREISL E., O'DOWD G. (1998) Stabilisation d'un système hybride par un feedback non linéaire, non monotone. *C. R. Acad. Sci. Paris Sér. I*, **326**, 323-327.

HARAUX A. (1979) Comportement à l'infini pour certains systèmes dissipatifs non linéaires. *Proc. Roy. Soc. Edinburgh*, **84A**, 213-234.

LASIECKA I. (1989) Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary. *J. Differential Equations*, **79**, 340-381.

MARTINEZ P., VANCOSTENOBLE J. (1999) Exponential stability for the wave equation with weak nonmonotone damping. *Portugal. Math.*, to appear.

RUDIN W. (1991) *Analyse réelle et complexe.* Masson, Paris.

SLEMROD M. (1989) Weak asymptotic decay via a "relaxed invariance principle" for a wave equation with nonlinear, nonmonotone damping. *Proc. Roy. Soc. Edinburgh*, **113A**, 87-97.

VANCOSTENOBLE J. (1997) Stabilisation faible de l'équation des ondes par un contrôle non linéaire, non monotone. Inst. Elie Cartan Univ. Nancy I, 3.

VANCOSTENOBLE J. (1998A) Weak asymptotic stability of second order evolution equations by nonlinear and nonmonotone feedbacks. *SIAM J. Math. Anal.*, **30**, 1, 140-154.

VANCOSTENOBLE J. (1998B) *Stabilisation non monotone de systèmes vibrants et Contrôlabilité.* Ph.D. Thesis, University of Rennes I.

VANCOSTENOBLE J. (1998C) Strong stabilization (via weak stabilization) of hybrid systems with a nonmonotone feedback. To appear.

ZUAZUA E. (1990) Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Contr. Opt.*, **28**, 2, 466-477.