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Second derivatives and sufficient optimality conditions for shape functionals

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Abstract: For some heuristic approaches to boundary variation in shape optimization the computation of second derivatives of domain and boundary integral functionals, their symmetry and a comparison to the velocity field or material derivative method are discussed. Moreover, for these approaches the functionals are Fréchet-differentiable in some sense, because at least a local embedding into a Banach space problem is possible. This allows the discussion of sufficient condition in terms of a coercivity assumption on the second Fréchet-derivative. The theory is illustrated by a discussion of the famous Dido problem.

Keywords: optimal shape design, second directional derivatives, sufficient optimality conditions.

1. Introduction

Shape optimization problems have been intensively studied in the literature throughout the last 25–30 years with respect to various directions of investigation. A lot of methods for description of domain variation have been developed and derivatives of functionals and solutions of state equations with respect to these domain or boundary variations can be computed. Moreover, the necessary optimality conditions are given, and numerical algorithms for a wide variety of problems are applied (see the surveys in Pironneau, 1983, and Sokolowski and Zolesio, 1992). Nevertheless, due to some difficulties arising from the theoretical as well as technical point of view, the study of sufficient conditions seems to be not very well developed at the moment. Only a few number of papers are concerned with related investigations (Fujii, 1994, Belov and Fujii, 1997). Therefore, it seems to make sense to discuss the easiest case of shape functionals only, in order to apply some of the ideas for the more interesting shape

In Eppler (1998a, b, 1999) the author discussed an easy approach to the description of the boundary variation for starshaped domains by the use of polar coordinates. This allows the description of the boundary **and** the boundary perturbation in the same way by functions of the polar angle ϕ. Consequently, a (global) Banach space embedding of the shape problem is possible, which allows the investigation of Fréchet-differentiability using the standard differential calculus for Banach spaces. In this way the existence of first Fréchet-derivatives for domain and boundary integrals of the type

$$J_1(\Omega) = \int_\Omega h\,dx \text{ and } J_2(\Omega) = \int_\Gamma g\,dS_\Gamma,\ \Gamma = \partial\Omega,\ (g, h \text{ are given data}),$$

is shown, which are equivalent to formulas for first (directional) derivatives for other approaches.

As a starting point for this paper we have the following in the case of starshaped domains:

Similar to first derivatives $dJ_i(\Omega)[r_1]$, $(i = 1, 2)$, second derivatives can be directly obtained in the sense of

$$d^2 J_i(\Omega_0)[r_1; r_2] = \lim_{\delta \to 0} \frac{dJ_i(\Omega_{\delta r_2})[r_1] - dJ_i(\Omega_0)[r_1]}{\delta},\ i = 1, 2,$$

because the first derivatives can be expressed as integrals over the interval $[0, 2\pi]$, where only the integrand contains the perturbation parameter δ. These derivatives are of Fréchet-type and therefore they have to be symmetric.

Following the ideas of Kirsch, Kress and Potthast, this is investigated for boundary perturbations by smooth fields for the case of two-dimensionional domains, too. Although this approach allows at least a "local" Banach space embedding, the computation of second derivatives is not straightforward and needs a special definition of the direction of boundary perturbation on perturbed domains (in a neighbourhood of the reference surface). Furthermore, the normal boundary variation method is investigated for the sake of completeness. The derivatives of the area and boundary arc length are discussed as examples.

Based on this, second order sufficient optimality conditions are obtained, at first for the case of starshaped domains. After them, a comparison to other approaches is also discussed. An extension of the results involving equality constraints is given and finally these conditions were applied to the Dido problem.

2. Domain perturbations and first derivatives

In this paper we shall study shape optimization problems for 2-dimensional simply connected bounded domains $\Omega \subset D$, where D is given. In the first part we assume the domains satisfying a condition of starshapeness with respect to a neighbourhood $U_\delta(x_0) = \{y \in \mathbb{R}^2 \mid |y - x_0| < \delta\}$, with some fixed $\delta > 0$.

advantage of this assumption is that the boundary $\Gamma = \partial\Omega$ of such domains can be described by a Lipschitz continuous function $r = r(\phi)$ of the polar angle ϕ (i.e., $\Gamma := \left\{\gamma(\phi) = \begin{pmatrix} r(\phi)\cos\phi \\ r(\phi)\sin\phi \end{pmatrix} \middle| \phi \in [0, 2\pi]\right\}$). Moreover, vice versa, each domain (boundary) can be identified with this describing function.

REMARK 1 Due to a result of Mazja (1979), the boundary function of a domain Ω, starshaped with respect to an open subset U_δ, is Lipschitz continuous with a constant, depending only on δ and on $d_\Omega := \sup\{|x| \mid x \in \Omega\}$. Consequently, if we assume that all domains under consideration are **uniformly** starshaped and bounded (i.e., there exists a bounded outer "security set" D), then they have **uniform** Lipschitz continuous boundaries.

REMARK 2 The assumption $\Gamma \in C^k$, $(k \in \mathbb{N})$ is equivalent to

$$r(\cdot) \in C_p^k[0, 2\pi] := \{r(\cdot) \in C^k[0, 2\pi] \mid r^{(i)}(0) = r^{(i)}(2\pi),\ i = 0, \ldots, k\}. \quad (1)$$

For transformations into polar coordinates we recall the well known formulae for the (local) curvature $\kappa(\cdot)$ (and related curvature radius $R(\cdot) = \kappa^{-1}(\cdot)$ — for $\Gamma \in C^2$), arclength $l(\cdot)$, and unscaled and scaled outer normal of the boundary, given by

$$R^{-1}(\phi) = \kappa(\phi) = \frac{2r'^2(\phi) + r^2(\phi) - r(\phi)r''(\phi)}{\sqrt{r^2(\phi) + r'^2(\phi)}^{\,3}},$$

$$\text{and } l(\phi) = \sqrt{r^2(\phi) + r'^2(\phi)},$$

and

$$\vec{a}(\phi) = \begin{pmatrix} r(\phi)\cos\phi + r'(\phi)\sin\phi \\ r(\phi)\sin\phi - r'(\phi)\cos\phi \end{pmatrix} \text{ (unscaled)}$$

$$\Rightarrow \vec{n}(\phi) = \frac{1}{\sqrt{r^2(\phi) + r'^2(\phi)}}\vec{a}(\phi).$$

In the following a reference domain $\Omega \in C^1$ is given, where the boundary Γ is associated with the describing function $r \in C_p^1[0, 2\pi]$. In this way, the "variables" (the admissible domains) are identified with elements of an open subset of the Banach space $C_p^1[0, 2\pi]$, and differential calculus in Banach spaces can be applied to the study of the problem.

LEMMA 1 *Let $h \in C(D)$ and $g \in C^1(D)$ be given. Then the functionals $J_1 = \int_\Omega h\,dx$ and $J_2 = \int_\Gamma g\,dS_\Gamma$ are Fréchet-differentiable with respect to $C_p^1[0, 2\pi]$ at every admissible Ω with the derivatives*

$$\nabla J_1(r)[r_1] = \int_0^{2\pi} r(\phi)r_1(\phi)h(r(\phi), \phi)\,d\phi, \quad (2)$$

and

$$\nabla J_2(r)[r_1] = \int_0^{2\pi} r_1\sqrt{r^2+r'^2}\,\frac{\partial g}{\partial \vec{r}}(r(\phi),\phi) + g(r(\phi),\phi)\frac{rr_1 + r'r_1'}{\sqrt{r^2+r'^2}}\,d\phi. \quad (3)$$

REMARK 3 For the proof see Eppler (1998a or 1999). Admissible perturbed domains (or boundaries) Ω_ε are now defined by the connection $\Gamma_\varepsilon \Leftrightarrow r_\varepsilon(\phi) = r(\phi) + \varepsilon r_1(\phi)$ with $r_1 \in C_p^1[0,2\pi]$ and $\varepsilon > 0$ sufficiently small, provided that $r_\varepsilon(\phi) > \delta$, $\phi \in [0,2\pi]$ is satisfied. Obviously, we have directional derivatives given by (2) and (3), respectively, which are linear and continuous w.r.t. r_1. Moreover, the related operator-norm of the Gateaux-derivative depends continuously on the $C_p^1[0,2\pi]$-norm of r. This ensures the **continuous Fréchet-**differentiability of the functionals by standard arguments from functional analysis (see Bögel and Tasche, 1974, Ioffe and Tichomirow, 1979).

REMARK 4 Shape derivatives are usually denoted by $d\cdot[r_1]$ or $\nabla\cdot[r_1]$ in the sequel. Spatial gradients ∇_x and partial derivatives with respect to polar coordinates (especially $\frac{\partial\cdot}{\partial\vec{r}} = \langle\nabla_x\cdot,\vec{e}_r\rangle$) or boundary normals often occur in the formulae and should not be confused with shape derivatives. Furthermore, because of $\vec{e}_r\cdot\vec{n} = \frac{r}{\sqrt{r^2+r'^2}} > 0$, ($\vec{e}_r = \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$ — the radial unit vector), the perturbations are always regular, i.e., the perturbation field is a tangential field if and only if $r_1(\cdot)\equiv 0$.

The description of boundary perturbations by smooth fields can be used for more general domains. Especially for 2D-problems boundaries and perturbations can be described by vector parameter functions, based on the usual Cartesian coordinates, more precisely, we have for some $T > 0$

$$\Gamma := \left\{\gamma(t) = \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\,\middle|\, t\in[0,T]\right\},$$
$$\text{with } \gamma(t) = \gamma(t+T), \text{ and } \gamma(\cdot)\in C^2(\mathbb{R}).$$

Moreover, we assume $\gamma(t_1) = \gamma(t_2) \Leftrightarrow t_1 = t_2$, $t_1, t_2 \in (0,T)$, i.e., the curve is free of double points. The curvature (-radius), arclenght and the normal direction are given by $\frac{1}{R(t)} = \kappa(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}^3}$,

$$l(t) = \sqrt{\dot{x}^2+\dot{y}^2},\ \vec{a}(t) = (\pm)\begin{pmatrix}\dot{y}(t)\\ -\dot{x}(t)\end{pmatrix} \Rightarrow \vec{n}(t) = \frac{(\pm)}{l(t)}\vec{a}(t),$$

where the sign for outward normal is "+", if Γ is positive oriented for increasing t. Furthermore, differentiation with respect to arclength is connected with $\frac{d}{\ldots}$ by $\frac{df}{\ldots} = \frac{1}{\ldots}\frac{df}{\ldots} = \frac{\dot{f}}{\ldots}$.

The description of perturbed boundaries γ_ε is similar to γ:

$$\Gamma_\varepsilon := \left\{ \gamma_\varepsilon(t) = \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} + \varepsilon \begin{pmatrix} d_x(t) \\ d_y(t) \end{pmatrix} \middle| \, t \in [0,T] \right\},$$
$$(\vec{d} = \begin{pmatrix} d_x \\ d_y \end{pmatrix} \text{ suff. smooth}),$$

because at least for sufficiently small ε, the same parameter interval for Ω_ε as for Ω can be taken. In order to have a nontrivial perturbation we additionally assume $\binom{d_x(\cdot)}{d_y(\cdot)} \cdot \vec{n}(\cdot) \not\equiv 0$. Although, there are some problems with nonuniqueness, an additional degree of freedom and the existence of smooth tangential fields, the approach is useful and allows at least a "local" Banach space embedding in a neighbourhood of Ω. Formulae for first derivatives are obtained similar to Lemma 1 in terms of integrals on $[0,T]$.

LEMMA 2 *Let $h \in C(D)$ and $g \in C^1(D)$ be given. Then the functionals $J_1 = \int_\Omega h\,dx$ and $J_2 = \int_\Gamma g\,dS_\Gamma$ are Fréchet-differentiable with respect to $\{C_p^1[0,T]\}^2$ at Ω with the derivatives*

$$\nabla J_1(\gamma)[\vec{d}] = \int_\Gamma (\vec{d}\cdot\vec{n})\, h\, dS_\Gamma = \int_0^T h(x(t),y(t))(d_x\dot{y} - d_y\dot{x})(t)\,dt, \tag{4}$$

and

$$\nabla J_2(\gamma)[\vec{d}] = \int_0^T g(t)\frac{\dot{x}\dot{d}_x + \dot{y}\dot{d}_y}{\sqrt{\dot{x}^2+\dot{y}^2}}(t)$$
$$+ \left(\nabla_x g \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix}\right)(t)\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}\,dt. \tag{5}$$

REMARK 5 Relation (5) is directly clear from

$$J_2 = \int_\Gamma g\,dS_\Gamma = \int_0^T g(x(t),y(t))\cdot\sqrt{\dot{x}^2+\dot{y}^2}\,dt.$$

Moreover, for $\Gamma \in C^2$, (5) is equivalent to (see (11)),

$$dJ_2(\gamma)[\vec{d}] = \int_\Gamma (\vec{d}\cdot\nabla_x g) + g\,\mathrm{div}_\Gamma\,\vec{d}\,dS_\Gamma,$$

where $\mathrm{div}_\Gamma\,\vec{d} := \mathrm{div}_\Gamma\{\vec{d} - (\vec{n}\cdot\vec{d})\vec{n}\} + \kappa(\vec{n}\cdot\vec{d})$ — for the definition of div_Γ (or

From the historical point of view the first approach (see Hadamard, 1910) was the method of normal boundary perturbation by using

$$\Gamma_\varepsilon : \gamma_\varepsilon(t) = \gamma(t) + \varepsilon\rho(t)\vec{n}(t),\ t \in [0, T].$$

However, this approach does not allow a direct embedding of the optimization problem into a Banach space, because at each step of approximation one degree of smoothness is lost. Nevertheless, directional derivatives exist for sufficiently smooth domains.

LEMMA 3 *Let $h \in C(D)$, $g \in C^1(D)$ and $\Omega \in C^2$ be given. Then the functionals $J_1(\cdot)$ and $J_2(\cdot)$ are directional differentiable with respect to $\rho(\cdot) \in C^1$ at Ω with the derivatives*

$$dJ_1(\gamma)[\rho] = \int_\Gamma \rho\, h\, dS_\Gamma = \int_0^T h(x(t), y(t))\rho(t)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt, \tag{6}$$

and

$$dJ_2(\gamma)[\rho] = \int_\Gamma \rho \cdot \left(\frac{\partial g}{\partial n} + \frac{g}{R}\right) dS_\Gamma$$

$$= \int_0^T \rho(t)\left(\kappa(t)g(t) + \frac{\partial g}{\partial n}(t)\right)\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt. \tag{7}$$

REMARK 6 Because of $\vec{d} = r_1\vec{e}_r$ and $(\vec{d} \cdot \vec{n})dS_\Gamma = r(\phi)r_1(\phi)d\phi$ for the polar coordinates, we have the equivalence of (2) and (4), as well as (3) and (5), respectively. Moreover, for $\Gamma \in C^2$, (3) is similar to (7), which can be seen after integration by parts.

REMARK 7 The assumptions on the data fields f and g can be weakened to fields with weak singularities (see Eppler, 1998a). Furthermore, regularity of the boundaries can be reduced, but this will not be studied in the paper.

The next result contains some technical details, useful for the computation and the transformation of higher order derivatives.

LEMMA 4 *Let Ω and the perturbations be sufficiently smooth. Then it holds for the shape derivative $d\vec{n}$ of the normal ($\vec{n}_\varepsilon$ and κ_ε are related to Γ_ε)*

$$d\vec{n}[\vec{d}](t) = \frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} = \frac{\dot{x}\dot{d}_y - \dot{y}\dot{d}_x}{\dot{x}^2 + \dot{y}^2}(t) \cdot \vec{\tau}(t) = -\langle \vec{n}, \frac{d}{ds}\vec{d}\rangle \cdot \vec{\tau} \perp \vec{n}(t),$$

$$d\vec{n}[r_1](\phi) = \frac{d}{d\varepsilon}\vec{n}_\varepsilon(\phi)|_{\varepsilon=0} = \frac{rr_1' - r'r_1}{r^2 + r'^2}(\phi) \cdot \vec{\tau}(\phi) \perp \vec{n}(\phi), \tag{8}$$

$$d\vec{n}[\rho](t) = \frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} = -\frac{\dot{\rho}}{\cdots}(t) \cdot \vec{\tau}(t) = -\frac{d\rho}{\cdots} \cdot \vec{\tau} \perp \vec{n}(t),$$

where $\vec{\tau}(\phi/t)$ denotes the unit tangential vector on Γ directed to increasing $\phi(t)$. Furthermore, the shape derivative $d\kappa$ of the curvature is given by

$$\begin{aligned}
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= \frac{\dot{x}\ddot{d}_y + \ddot{y}\dot{d}_x - \dot{y}\ddot{d}_x - \ddot{x}\dot{d}_y}{\sqrt{\dot{x}^2+\dot{y}^2}^3} - 3\kappa \cdot \frac{\dot{x}\dot{d}_x + \dot{y}\dot{d}_y}{\dot{x}^2+\dot{y}^2}, \\
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= \frac{2rr_1 + 4r'r_1' - rr"_1 - r"r_1}{\sqrt{r^2+r'^2}^3} - 3\kappa \cdot \frac{rr_1 - r'r_1'}{r^2+r'^2}, \\
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= -\frac{\ddot{\rho}}{\dot{x}^2+\dot{y}^2} + \frac{\dot{\rho}[\dot{y}\ddot{y} + \dot{x}\ddot{x}]}{(\dot{x}^2+\dot{y}^2)^2} - \rho\kappa^2 \\
&= \frac{-\frac{d}{dt}\left[\dot{\rho}\sqrt{\dot{y}^2+\dot{x}^2}\right] + 2\frac{d}{dt}\left[\rho\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})\right]}{\sqrt{\dot{x}^2+\dot{y}^2}^3} - \rho\left\{\kappa^2 + 2\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}^2+\dot{y}^2})}{\sqrt{\dot{x}^2+\dot{y}^2}^3}\right\}.
\end{aligned} \tag{9}$$

REMARK 8 The relation $\frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} \perp \vec{n}$ is also known for more general cases (see Sokolowski and Zolesio, 1992). The last transformation of (9) needs obviously $\Omega \in C^3$. Moreover, a well founded derivation of the derivative formula in the case of normal variation needs formally also $\Omega \in C^3$. However, the result is valid for C^2-boundaries, too.

Similar formulas for the first directional derivatives hold for the velocity field (or material derivative) method, developed by Sokolowski and Zolesio. We present for the sake of completeness the main idea of the approach (for a detailed investigation see Sokolowski and Zolesio, 1992):

Given a so called "velocity field" $V(t,x) : V \in C(0,\varepsilon; C^k(\bar{D}, \mathbb{R}^N))$, one direction of perturbation of a reference domain Ω is described by a family of domains Ω_t, defined by

$$\Omega_t := \left\{ x(t,X) \in \mathbb{R}^N \,\middle|\, \frac{dx(\tau,X)}{d\tau} = V(\tau,X),\ x(0,X) = X \in \Omega \right\}.$$

The main advantage is that the direction of the domain perturbation is well defined on $\bar{D}$, where $V(0)|_\Gamma$ can be viewed as the boundary perturbation in comparison to other approaches. The first directional derivatives are given by

LEMMA 5 *Let $h \in C(D)$ and $g \in C^1(D)$ and $\Omega \in C^2$ be given. Then the functionals $J_1(\cdot)$ and $J_2(\cdot)$ are directional differentiable with respect to $V(\cdot) \in C^1$ at Ω with the derivatives*

$$dJ_1(\Omega)[V(0)] = \lim_{t\to 0} \frac{J_1(\Omega_t) - J_1(\Omega)}{t} = \int_\Gamma \langle V(0), \vec{n}\rangle h\, dS_\Gamma, \tag{10}$$

and

$$dJ_2(\Omega)[V(0)] = \int \langle V(0), \nabla g\rangle + g(\operatorname{div} V(0) - [DV(0)\vec{n}, \vec{n}])\, dS_\Gamma. \tag{11}$$

REMARK 9 $DV(0)$ denotes the Jacobian of the mapping $x \in \mathbb{R}^2 \mapsto V(0,x) \in \mathbb{R}^2$. Furthermore, the following transformation of (10)

$$dJ_1(\Omega)[V(0)] = \int_\Gamma (\vec{n} \cdot V(0))h \, dS_\Gamma$$

$$= \int_\Omega \operatorname{div}[h \cdot V(0)] \, dx, \; h \in C^1, \; (h \in W^{1,1}),$$

shows that the velocity method allows the definition of shape derivatives under essentially weaker assumptions on the domains. Additional degrees of freedom ($V_1(0)|_\Gamma = V_2(0)|_\Gamma \Rightarrow$ both "velocity fields" represent the "same boundary variation") cause no difficulties.

3. Second derivatives

As we had already announced, the second shape derivatives for starshaped domains can be computed "straight forward", if the data fields are smooth enough.

THEOREM 1 *Let $h \in C^1(D)$ and $g \in C^2(D)$ be given. Then the functionals $J_1 = \int_\Omega h \, dx$ and $J_2 = \int_\Gamma g \, dS_\Gamma$ are twice Fréchet-differentiable with respect to $C_p^1[0,2\pi]$ at Ω with the second derivatives*

$$\nabla^2 J_1(r)[r_1; r_2] = \int_0^{2\pi} r_2(\phi) r_1(\phi) h(r,\phi) + r(\phi) r_1(\phi) r_2(\phi) \frac{\partial h}{\partial \vec{r}}(r,\phi) \, d\phi, \quad (12)$$

and

$$\begin{aligned}\nabla^2 J_2(r)[r_1; r_2] = \int_0^{2\pi} d\phi \Bigg\{ & r_2(\phi) r_1(\phi) \sqrt{r^2 + r'^2} \, \frac{\partial^2 g}{\partial \vec{r}^2} \\ & + \frac{\partial g}{\partial \vec{r}} \left[r_1 \frac{r r_2 + r' r_2'}{\sqrt{r^2 + r'^2}} + r_2 \frac{r r_1 + r' r_1'}{\sqrt{r^2 + r'^2}} \right] \\ & + g \frac{(r_1 r_2 + r_1' r_2')(r^2 + r'^2) - (r r_1 + r' r_1')(r r_2 + r' r_2')}{\sqrt{r^2 + r'^2}^3} \Bigg\}. \end{aligned} \quad (13)$$

REMARK 10 Due to the Banach space embedding, the boundary variation r_2 on perturbed boundaries $\Gamma_{\delta r_1}$ and on Γ is defined in the same way without any additional problem. Therefore, differentiation can be carried out and leads obviously to symmetry with respect to r_1 and r_2. Moreover, we need no additional regularity of the boundary for the definition of higher order derivatives of shape

In order to investigate higher order derivatives for the other cases, a definition of the boundary variation on perturbed boundaries is necessary. Following Potthast and Kirsch, in the case of boundary variation by smooth fields we may proceed for $N = 2$ as follows, in some sense similarly to the case of polar coordinates:

We compute the derivative of $dJ_i(\vec{d})$, $(i = 1, 2)$, after the transformation into an integral over the fixed interval $[0, T]$ with

$$\Gamma_\delta := \left\{ \gamma_\delta(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \delta \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix} \,\middle|\, t \in [0, T] \right\},$$

because a smooth parametrization of the perturbed domain exists on the same interval $[0, T]$ for δ sufficiently small. The "transformation" of direction $\vec{d}$ onto Γ_δ is defined by an "unchanged translation", i.e., $\vec{d}(\gamma_\delta(t)) := \vec{d}(\gamma(t)) = \vec{d}(t)$. From

$$dJ_1[\vec{d}]|_\delta = \int_0^T h(x_\delta, y_\delta)(d_x \dot{y}_\delta - d_y \dot{x}_\delta)\, dt = \int_0^T h_\delta(t) \langle \vec{d}, \vec{a}_\delta \rangle (t)\, dt$$

and

$$dJ_2[\vec{d}]|_\delta = \int_0^T g_\delta \frac{\dot{x}_\delta \dot{d}_x + \dot{y}_\delta \dot{d}_y}{\sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2}} + \left\langle \nabla_x g_\delta, \begin{pmatrix} d_x \\ d_y \end{pmatrix} \right\rangle \sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2}\, dt$$

we immediately obtain

COROLLARY 1 *Let $h \in C^1(D)$ and $g \in C^2(D)$ be given. Then the functionals $J_1 = \int_\Omega h\, dx$ and $J_2 = \int_\Gamma g\, dS_\Gamma$ are twice Fréchet-differentiable with respect to $\{C_p^1[0, T]\}^2$ at Ω with the second derivatives*

$$\nabla^2 J_1(\gamma)[\vec{d}; \vec{f}] = \int_0^T h(d_x \dot{f}_y - d_y \dot{f}_x)$$
$$+ \left\langle \nabla_x h, \begin{pmatrix} f_x \\ f_y \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} d_x \\ d_y \end{pmatrix}, \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} \right\rangle dt, \tag{14}$$

and

$$\nabla^2 J_2(\gamma)[\vec{d}; \vec{f}] = \int_0^T \left\langle \nabla_x g, \left[\vec{d} \cdot \frac{\dot{x}\dot{f}_x + \dot{y}\dot{f}_y}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \vec{f} \cdot \frac{\dot{x}\dot{d}_x + \dot{y}\dot{d}_y}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right] \right\rangle$$
$$+ \langle \nabla_x^2 g \vec{f}, \vec{d} \rangle \sqrt{\dot{x}^2 + \dot{y}^2}$$
$$+ g \cdot \frac{(\dot{f}_x \dot{d}_x + \dot{f}_y \dot{d}_y)(\dot{x}^2 + \dot{y}^2) - (\dot{x}\dot{d}_x + \dot{y}\dot{d}_y)(\dot{x}\dot{f}_x + \dot{y}\dot{f}_y)}{\sqrt{\dot{x}^2 + \dot{y}^2}^{\,3}}\, dt. \tag{15}$$

The symmetry of $\nabla^2 J_2(\gamma)[\vec{d};\vec{f}]$ can be seen directly from (15). However, after integration by parts of the first part $I_1(\vec{d};\vec{f})$ of $\nabla^2 J_1(\gamma)[\vec{d};\vec{f}]$ we obtain (boundary terms at $t=0$ and $t=T$ vanish, because all functions are periodic in t)

$$I_1(\vec{d};\vec{f}) = \int\limits_0^T h(d_x\dot{f}_y - d_y\dot{f}_x)\,dt = \int\limits_0^T -[h\dot{d}_x]f_y + [h\dot{d}_y]f_x\,dt$$

$$= I_1(\vec{f};\vec{d}) + \int\limits_0^T \left\langle \nabla_x h, \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\right\rangle(-d_x f_y + d_y f_x)\,dt.$$

An easy calculation shows (with $I_2(\vec{d};\vec{f}) = \int\limits_0^T \langle\nabla_x h,\vec{f}\rangle\cdot\langle\vec{d},\vec{a}\rangle dt)$ that

$$\int\limits_0^T \left\langle \nabla_x h, \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\right\rangle(-d_x f_y + d_y f_x)\,dt = I_2(\vec{f};\vec{d}) - I_2(\vec{d};\vec{f}),$$

i.e., symmetry holds.

REMARK 11 As a natural method for the definition of domain variations on perturbed surfaces one may use any smooth extension of the boundary field $\vec{d}$, which is very close to the velocity field approach for autonomous velocity fields. However, this is not equivalent to the above, because it leads to

$$\widetilde{dJ}_1[\vec{d}]|_\delta = \int\limits_0^T h_\delta(t)\langle\vec{d}_\delta,\vec{a}_\delta\rangle(t)\,dt$$

$$\Rightarrow \widetilde{d^2J}_1(\gamma)[\vec{d};\vec{f}] = \nabla^2 J_1(\gamma)[\vec{d};\vec{f}] + \int\limits_0^T h(t)\left\langle \frac{d\vec{d}_\delta}{d\delta}|_{\delta=0}(t), \vec{a}_0(t)\right\rangle dt,$$

where the additional part in the derivative implies nonuniqueness (it depends on the way of extension) and destroys the symmetry of second derivatives in general.

REMARK 12 For tangential directions of perturbation $\vec{d} = \alpha(t)\vec{\tau}$, and $\vec{f} = \beta(t)\vec{\tau}$ we formally obtain

$$\nabla^2 J_1(\gamma)[\vec{d};\vec{f}] = \int\limits_0^T \alpha\beta\,\kappa h\sqrt{\dot{x}^2+\dot{y}^2}\,dt = \int\limits_\Gamma \alpha\beta\,\kappa h\,dS_\Gamma,$$

and

$$\nabla^2 J_2(\gamma)[\vec{d};\vec{f}] = \int\limits_0^T \alpha\beta\left[\frac{\partial^2 g}{\partial\tau^2} + g\kappa^2\right]\sqrt{\dot{x}^2+\dot{y}^2} + [\alpha\dot{\beta} + \dot{\alpha}\beta]\frac{\partial g}{\partial\tau}\,dt$$

$$= \int\limits_\Gamma \alpha\beta\left[\frac{\partial^2 g}{\partial\tau^2} + g\kappa^2\right] + \frac{d}{d\tau}(\alpha\beta)\frac{\partial g}{\partial\tau}\,dS_\Gamma = \int\limits_\Gamma \alpha\beta g\kappa\,dS_\Gamma.$$

For the definition of second derivatives for the normal variation approach, we use the following transformation of direction $\vec{d} = \rho \cdot \vec{n}$ onto $\Gamma_\delta := \Gamma + \delta\nu \cdot \vec{n}$: We define $\vec{d}(\gamma_\delta(t))$ by $\vec{d}(\gamma_\delta(t)) := \rho(t)\vec{n}(\gamma_\delta(t))$, where only $\rho(\cdot)$ is "unchanged translated", but the "whole direction" is perturbed. Therefore, we get

COROLLARY 2 *Let $h \in C^1(D)$, $g \in C^2(D)$ and $\Omega \in C^2$ be given. Then the functionals $J_1(\Omega)$ and $J_2(\Omega)$ are twice directionally differentiable at Ω with respect to $\rho(\cdot), \nu(\cdot) \in C^2$ with the second derivatives*

$$d^2 J_1(\gamma)[\rho;\nu] = \int_0^T \left[\rho\nu\left(h\kappa + \frac{\partial h}{\partial n}\right)\right]\sqrt{\dot{x}^2 + \dot{y}^2(t)}\,dt$$

$$= \int_\Gamma \rho\nu\left[\frac{\partial h}{\partial n} + \frac{h}{R}\right] dS_\Gamma, \tag{16}$$

and

$$d^2 J_2(\gamma)[\rho;\nu] = \int_0^T \left\{\rho\nu\left[\frac{\partial^2 g}{\partial n^2} + 2\kappa\frac{\partial g}{\partial n}\right] + g\frac{\dot{\rho}\dot{\nu}}{\dot{x}^2+\dot{y}^2}\right\}\sqrt{\dot{x}^2+\dot{y}^2}\,dt. \tag{17}$$

Proof. By making use of (8) and

$$\frac{d}{d\delta}\sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2}|_0 = \ldots = \nu(t)\langle\tau(t), \frac{d}{dt}\vec{n}(t)\rangle = \nu(t)\kappa(t)\sqrt{\dot{x}^2+\dot{y}^2}$$

we obtain (16) from

$$dJ_1(\gamma_\delta)[\rho] = \int_{\Gamma_\delta} \rho h_\delta\, dS_\Gamma$$

$$= \int_0^T h_\delta(t)\rho(t)\sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2(t)}\,dt = \int_0^T h_\delta(t)\langle\vec{d}_\delta, \vec{a}_\delta\rangle(t)\,dt$$

and

$$d^2 J_1(\gamma)[\rho;\nu] = \int_0^T h\left[\left\langle\frac{d}{d\delta}\vec{d}_\delta|_0, \vec{a}\right\rangle + \left\langle\vec{d}, \frac{d}{d\delta}\vec{a}_\delta|_0\right\rangle\right] + \langle\nabla_x h, \nu\vec{n}\rangle \cdot \langle\vec{d},\vec{a}\rangle\,dt,$$

where $\vec{d} = \rho\cdot\vec{n}$. For (17) we obtain by differentiation of $dJ_2(\gamma_\delta)[\rho]$

$$d^2 J_2(\gamma)[\rho;\nu] = \int_\Gamma \rho\left\{\frac{d}{d\delta}\left[\kappa_\delta g_\delta + \frac{\partial g}{\partial n}_\delta\right]|_0 + \nu\left(\kappa g + \frac{\partial g}{\partial n}\right)\kappa\right\} dS_\Gamma$$

$$= \int_0^T \rho\left\{\nu\kappa\langle\nabla_x g, \vec{n}\rangle + g\frac{d}{d\delta}\kappa_\delta|_0 + \nu\langle\nabla_x^2 g\vec{n}, \vec{n}\rangle\right.$$

$$+\left\langle \nabla_x g, \frac{d}{d\delta}\vec{n}_\delta|_0 \right\rangle + \nu\kappa\left(\kappa g + \frac{\partial g}{\partial n}\right)\Bigg\}\sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$= \int_0^T \rho\sqrt{\dot{x}^2+\dot{y}^2}\,dt\Bigg\{\nu\left[\frac{\partial^2 g}{\partial n^2} + 2\kappa\frac{\partial g}{\partial n}\right]$$

$$-\frac{\dot{\nu}}{\sqrt{\dot{x}^2+\dot{y}^2}}\left[\frac{\partial g}{\partial \tau} - g\cdot\frac{\dot{y}\ddot{y}+\dot{x}\ddot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}^3}\right] - \frac{\ddot{\nu}g}{\dot{x}^2+\dot{y}^2}\Bigg\},$$

by (8) and (9). For further transformations we split $d^2 J_2[\rho;\nu] = I_1(\rho;\nu) + I_2(\rho;\nu)$ into the direct symmetric part

$$I_1(\rho;\nu) = \int_0^T \rho\nu\left\{\frac{\partial^2 g}{\partial n^2} + 2\kappa\frac{\partial g}{\partial n} - 2g\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}^2+\dot{y}^2})}{\sqrt{\dot{x}^2+\dot{y}^2}^3})\right\}\sqrt{\dot{x}^2+\dot{y}^2}\,dt,$$

and the (formal) nonsymmetric part

$$I_2(\rho;\nu) = -\int_0^T \rho\left\{\dot{\nu}\frac{\partial g}{\partial\tau} + g\frac{\frac{d}{dt}\left[\dot{\nu}\sqrt{\dot{y}^2+\dot{x}^2} - 2\nu\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})\right]}{\dot{x}^2+\dot{y}^2}\right\}dt.$$

Integration by parts of $I_2(\rho;\nu)$ leads to (boundary terms vanish)

$$I_2(\rho;\nu) = \int_0^T \frac{dg}{dt}\cdot\frac{\rho\left[\dot{\nu}\sqrt{\dot{y}^2+\dot{x}^2} - 2\nu\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})\right]}{\dot{x}^2+\dot{y}^2} - \rho\dot{\nu}\frac{\partial g}{\partial\tau}$$

$$+g\frac{\left[\dot{\rho}\sqrt{\dot{y}^2+\dot{x}^2} - 2\rho\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})\right]\cdot\left[\dot{\nu}\sqrt{\dot{y}^2+\dot{x}^2} - 2\nu\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})\right]}{\sqrt{\dot{x}^2+\dot{y}^2}^3}\,dt.$$

An easy calculation shows that

$$\frac{dg(x(t),y(t))}{dt} = \left\langle \nabla_x g, \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\right\rangle = \frac{\partial g}{\partial\tau}|\sqrt{\dot{x}^2+\dot{y}^2},$$

$$\text{hence, } \frac{dg}{dt}\cdot\frac{\rho\dot{\nu}\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{x}^2+\dot{y}^2} = \rho\dot{\nu}\frac{\partial g}{\partial\tau},$$

i.e., symmetry holds for the second derivatives of J_2. Moreover, we continue with a further transformation of

$$I_2(\rho;\nu) - \int_0^T 2\rho\nu g\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}^2+\dot{y}^2})}{(\dot{x}^2+\dot{y}^2)^2}\,dt$$

$$= \int_0^T \rho\Big\{\frac{\dot{\rho}\dot{\nu}}{\ldots} - 2\frac{d}{dt}(\rho\nu)\frac{\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})}{\ldots} + \rho\nu\Big[\frac{4(\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2}))^2}{\ldots^3}$$

$$- 2\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}^2+\dot{y}^2})}{(\dot{x}^2+\dot{y}^2)^2}\Bigg]\Bigg\} - 2\rho\nu\frac{dg}{dt}\frac{\frac{d}{dt}(\sqrt{\dot{x}^2+\dot{y}^2})}{(\dot{x}^2+\dot{y}^2)^2}\,dt.$$

Integration by parts of $-2\frac{d}{dt}(\rho\nu)\{\ldots\}$ shows that all terms except of the first term vanish. The transformations are formally valid only for $\Gamma \in C^3$. However, for $\Gamma \in C^2$ we use an easy continuation argument by an approximating sequence $\{\Gamma_n\} \subset C^3$. Hence, we arrive at (17). ■

REMARK 13 Now $\frac{d\vec{d}_\delta}{d\delta}|_{\delta=0}$ is formally present, but the related term for d^2J_1 vanishes, because of $\frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} \perp \vec{n}$ (see Remark 8 and Lemma 4), whereas for d^2J_2 some of such terms have opposite sign and therefore they vanish.

REMARK 14 Formula (17) can be rewritten as

$$d^2J_2(\gamma)[\rho;\nu] = \int_\Gamma \rho\nu\left[\frac{\partial^2 g}{\partial n^2} + 2\kappa\frac{\partial g}{\partial n}\right] + g\frac{d\rho}{ds}\frac{d\nu}{ds}\,dS_\Gamma.$$

Therefore, a conjecture for an extension for $N > 2$ may be the following

$$d^2J_2(\gamma)[\rho;\nu] = \int_\Gamma \rho\nu\left[\frac{\partial^2 g}{\partial n^2} + 2\kappa\frac{\partial g}{\partial n}\right] + g\langle\nabla_\Gamma\rho,\nabla_\Gamma\nu\rangle\,dS_\Gamma,\ \Omega \subset \mathbb{R}^N.$$

Due to the definition of velocity fields on D, second derivatives in the sense of

$$d^2J_i(\Omega)[V_1;V_2] = \lim_{t\to 0}\frac{dJ_i(\Omega_{tV_2})[V_1] - dJ_i(\Omega)[V_1]}{t},\ i = 1,2,$$

can be obtained straightforwardly by using the unitary extension $\mathcal{N}_0$ of the unit normal field $\vec{n}$ on Γ.

COROLLARY 3 *Let h, g and Ω be sufficiently smooth. The second directional derivatives of the functionals J_1 and J_2 at Ω with respect to autonomous vector fields V_1, V_2 are given by*

$$d^2J_1(\Omega)[V_1;V_2] = \int_{\Gamma_0}\langle V_2,\vec{n}\rangle\operatorname{div}[h\cdot V_1]\,dS_\Gamma$$

$$= \int_\Omega \operatorname{div}[\operatorname{div}[h\cdot V_1]\cdot V_2]\,dx, \tag{18}$$

and

$$d^2J_2(\Omega)[V_1;V_2] = \int_\Gamma V_2\cdot\nabla_x\{(V_1\cdot\nabla_x g) + g(\operatorname{div}V_1 - [DV_1\mathcal{N}_0,\mathcal{N}_0])\}$$

REMARK 15 For nonautonomuos velocity fields additional terms from $\frac{\partial V}{\partial t}|_{t=0}$ occur in the formula. Moreover, d^2J_i contain a symmetric part and one from $\frac{dV_1(\Omega_{tV_2})}{dt}$ (see Remark 11).

Some examples. For the volume $J_1 = \int_\Omega dx$ of a domain we have

- $d^2J_1[r_1; r_2] = \int_0^{2\pi} r_1(\phi)r_2(\phi)\, d\phi,$
- $d^2J_1[\vec{d}; \vec{f}] = \int_0^T 1 \cdot (d_x(t)\dot{f}_y(t) - d_y(t)\dot{f}_x(t))\, dt,$
- $d^2J_1[\rho; \nu] = \int_0^T \rho\nu\kappa\sqrt{\dot{y}^2 + \dot{x}^2}dt = \int_\Gamma \rho\nu\kappa\, dS_\Gamma,$
- $d^2J_1(V_1; V_2) = \int_\Omega \text{div}[\text{div}\, V_1 \cdot V_2]\, dx.$

The second derivative of the volume does not depend on the reference domain in the first two formulae, hence, third derivatives will vanish (for 2D-domains). This is not the case for the normal perturbation approach, because the boundary variations depend on the domain. For the velocity method the nonsymmetric part "destroys" the independence. Especially for $V_1 = \vec{d} = (1,0)^T$ (parallel shifting in x-direction) and $V_2 = \vec{f} = (0.5x^2, 0)^T$ ("blow up/shrinking" in x-direction) we get

$$0 = d^2J_1(\vec{d}; \vec{f}) = d^2J_1(\vec{f}; \vec{d}),$$

$$\text{whereas } 0 = d^2J_1(V_1; V_2) < d^2J_1(V_2; V_1) = \int_\Omega dx,$$

holds for the velocity method. Similarly for the perimeter $J_2 = \int_\Gamma dS_\Gamma$ we obtain

- $d^2J_2[r_1; r_2] = \int_0^{2\pi} \frac{(r_1r_2 + r_1'r_2')(r^2 + r'^2) - (rr_1 + r'r_1')(rr_2 + r'r_2')}{\sqrt{r^2 + r'^2}^3}\, d\phi,$
- $d^2J_2[\vec{d}; \vec{f}] = \int_0^T \frac{(\dot{f}_x\dot{d}_x + \dot{f}_y\dot{d}_y)(\dot{x}^2 + \dot{y}^2) - (\dot{x}\dot{d}_x + \dot{y}\dot{d}_y)(\dot{x}\dot{f}_x + \dot{y}\dot{f}_y)}{\sqrt{\dot{x}^2 + \dot{y}^2}^3}\, dt,$
- $d^2J_1[\rho; \nu] = \int_0^T \frac{\dot{\rho}\dot{\nu}}{\dot{x}^2 + \dot{y}^2}\sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_\Gamma \frac{d\rho}{ds}\frac{d\nu}{ds}\, dS_\Gamma.$

A more general formulation in terms of boundary integrals seems to be not

4. Optimality conditions for shape functionals

In the first two subsections we shall study only the case of free minima. We denote by Ω_0 a local minimum, where related neighbourhoods are meant in the sense of C_p^1 for domain integrals, and in the sense of C_p^2 for boundary functionals, respectively. Moreover, the subscript "0" denotes in the sequel all quantities $(r_0, \Gamma_0 \ldots)$, connected to Ω_0, whereas directions of domain- or boundary perturbations, like r, $\vec{d}$ and ρ are used without any subscript.

4.1. Volume functionals

Whereas necessary optimality conditions can be easily obtained by using directional derivatives of first and second order, the situation for sufficient conditions is generally more complicated in shape optimization. Due to the special approach for starshaped domains, standard methods are applicable. From the standard necessary condition it follows immediately ("all $r \in C^1$ are admissible") that

$$dJ_1(\Omega_0)[r] = \nabla J_1(r_0)[r] = \int_0^{2\pi} r_0(\phi) r(\phi) h(r_0, \phi)\, d\phi = 0 \Rightarrow h|_{\Gamma_0} \equiv 0. \quad (20)$$

Moreover, according to (12) we get for a domain, satisfying the necessary condition

$$\nabla^2 J_1(r_0)[r; r] = \int_0^{2\pi} r^2(\phi) r_0(\phi) \frac{\partial h}{\partial \vec{r}}|_0(\phi)\, d\phi. \quad (21)$$

Optimality can be guaranteed often by some coercivity of the second Fréchet-derivative. However, it is impossible to have coercivity with respect to C^1 (the "space of differentiation"), only an estimate

$$\nabla^2 J_1(r_0)[r, r] \geq c_0 \|r\|_{L_2}^2, \text{ (where } c_0 > 0 \text{ is ensured by } \frac{\partial h}{\partial \vec{r}}|_0(\phi) > 0,\ \forall \phi)$$

can be expected. This is known from other control problems as the so-called "two-norm-discrepancy".

REMARK 16 The conditions $\frac{\partial h}{\partial \vec{r}}|_0 > 0$ and $\frac{\partial h}{\partial n}|_0 > 0$ are equivalent for starshaped domains (we have $(\vec{e}_r, \vec{n}) > 0\ \forall \phi$ and $\frac{\partial h}{\partial \vec{\tau}}|_0 = 0 \Rightarrow \frac{\partial h}{\partial \vec{r}}|_0 = \frac{\partial h}{\partial n}|_0 (\vec{e}_r, \vec{n})$).

THEOREM 2 *For $\Omega_0 \in C^1 (r_0 \in C_p^1[0, 2\pi]$ and $h \in C^2$ the conditions $h|_{\Gamma_0} \equiv 0$*

∂h

Proof. We have (from differential calculus): $J_1(r_0+r)-J_1(r_0) = \frac{1}{2}[d^2J_1(r_0)[r,r] + \vartheta_2(r),]$, where $\frac{|\vartheta_2(r)|}{\|r\|_{C^1}^2} \to 0$ for $\|r\|_{C^1} \to 0$, but this is not enough to ensure optimality. Nevertheless, by a more careful estimate of the remainder $\vartheta_2(r) = d^2J_1(r_\nu)[r,r] - d^2J_1(r_0)[r,r]$ (where $r_\nu := r_0 + \nu r$) it follows that

$$\begin{aligned}
|\vartheta_2(r)| &= \left| \int_0^{2\pi} r^2 \left[h(r_\nu, \phi) - 0 + (r_\nu)\frac{\partial h}{\partial \vec{r}}|_\nu - r_0 \frac{\partial h}{\partial \vec{r}}|_0 \right] d\phi \right| \\
&\le \max |r(\phi)| \int_0^{2\pi} r^2 [c_1(h,\eta) + c_2(h,\eta) + c_3(h,\eta)]\, d\phi \\
&\le c(h,\eta)\|r\|_C \|r\|_{L_2}^2, \text{ with } \|r\|_C < \eta.
\end{aligned}$$

We arrive (for η sufficiently small) at

$$J_1(r_0 + r) - J_1(r_0) \ge \frac{c_0}{2}\|r\|_{L_2}^2, \text{ if } \|r\|_C < \eta,$$

which ensures the optimality of Ω_0. ■

REMARK 17 The easy situation allows an interpretation as follows: From the necessary and sufficient condition we have for the data field h

(i) $h|_{\Gamma_0} = 0$,
(ii) $h(x) > 0, \ \forall x \in U_\delta(\Gamma_0) \setminus \bar{\Omega}_0$,
(iii) $h < 0, \ \forall x \in U_\delta(\Gamma_0) \cap \Omega_0$.

Therefore, each perturbation of the boundary increases the functional value. In spite of being intuitively trivial, this shows that sometimes the results, obtained for a restrictive approach, can be valid also for more general situations.

REMARK 18 The same discussion is obviously possible using the second derivatives for normal variation. After the transformation of the second derivatives for the smooth field approach we see that

$$\begin{aligned}
d^2J_1(\gamma_0)[\vec{d};\vec{d}] &= \int_0^T \langle \nabla_x h_0, \vec{d}\rangle \cdot d_n(\sqrt{\dot{y}_0^2 + \dot{x}_0^2}dt) \\
&= \int_0^T d_n \left(\frac{\partial h}{\partial n}|_0 d_n + \frac{\partial h}{\partial \tau}|_0 d_\tau \right)(\sqrt{\dot{y}_0^2 + \dot{x}_0^2}dt) = \int_0^T d_n^2 \frac{\partial h}{\partial n}|_0(\sqrt{\dot{y}_0^2 + \dot{x}_0^2}dt),
\end{aligned}$$

because of $\frac{\partial h}{\partial \tau}|_0 \equiv 0$. Hence, second order sufficient conditions are similar for a

4.2. Boundary functionals

The necessary condition for a free minimum can be directly seen from the derivative for the case of normal boundary variation, if we additionally assume $\Omega_0 \in C^2$.

$$dJ_2(\gamma_0)[\rho] = \int_{\Gamma_0} \rho \cdot \left(\frac{\partial g}{\partial n} + \frac{g}{R}\right) dS_\Gamma = 0 \Rightarrow \left[\frac{\partial g}{\partial n} + g\kappa\right]|_0 \equiv 0.$$

Nevertheless, this follows also from derivative formulae for the other approaches. We have

$$\nabla J_2(r_0)[r] = \int_0^{2\pi} r\sqrt{r_0^2 + r'^2_0}\frac{\partial g}{\partial \vec{r}}|_0 + g_0\,\frac{r_0 r + r'_0 r'}{\sqrt{r_0^2 + r'^2_0}}\,d\phi$$

$$= \int_{\Gamma_0} r\left\{\left[\frac{r_0^2}{\sqrt{r_0^2 + r'^2_0}}\frac{\partial g}{\partial \vec{r}}|_0 - \frac{r'_0}{\sqrt{r_0^2 + r'^2_0}}\frac{\partial g}{\partial \phi}|_0\right] + g_0 r_0 \frac{r_0^2 + 2r'^2_0 - r_0 r''_0}{\sqrt{r_0^2 + r'^2_0}^3}\right\} dS_\Gamma,$$

where $\left[\frac{r_0^2}{\sqrt{r_0^2 + r'^2_0}}\frac{\partial g}{\partial \vec{r}}|_0 - \frac{r'_0}{\sqrt{r_0^2 + r'^2_0}}\frac{\partial g}{\partial \phi}|_0\right] = r_0\frac{\partial g}{\partial n}|_0,$

and analogously

$$\nabla J_2(\gamma_0)[\vec{d}] = \int_0^T g_0\frac{\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} + \left(\nabla_x g_0 \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\,dt$$

$$= \int_0^T g_0(\vec{\tau} \cdot \frac{d}{dt}\vec{d}) + (\nabla_x g_0 \cdot [d_n\vec{n} + d_\tau\vec{\tau}])\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\,dt$$

$$= \int_{\Gamma_0} d_n\left(g_0\kappa_0 + \frac{\partial g}{\partial n}|_0\right) dS_\Gamma.$$

For the derivation of sufficient condition we investigate the second derivative $d^2J_2(\Omega_0)[r;r] = \nabla^2 J_2(r_0)[r;r]$.

$$\nabla^2 J_2(r_0)[r;r] = \int_0^{2\pi} r^2\sqrt{r_0^2 + r'^2_0}\frac{\partial^2 g}{\partial \vec{r}^2}$$

$$+ 2r\frac{\partial g}{\partial \vec{r}}\frac{r_0 r + r'_0 r'}{\sqrt{r_0^2 + r'^2_0}} + g\frac{(r'_0 r - r_0 r')^2}{\sqrt{r_0^2 + r'^2_0}^3}\,d\phi.$$

By integration by parts of the "mixed terms" $r'r \cdot f(\phi)$ we arrive at

$$\nabla^2 J_2(r_0)[r;r] = \int_0^{2\pi} r^2 \cdot f_1(\nabla_x^2 g, \nabla_x g, g, r_0) + {r'}^2 \cdot f_2(g, r_0)\, d\phi, \tag{22}$$

where f_1 and f_2 are given by

$$f_1(\nabla_x^2 g, \nabla_x g, g, r_0)(\phi) = r_0 \frac{\partial}{\partial n}\left(\frac{\partial g}{\partial \vec{r}}\right) + \frac{\partial g}{\partial \phi} \frac{r_0 r_0'}{\sqrt{r_0^2 + {r_0'}^2}^3}$$
$$+ \frac{\partial g}{\partial \vec{r}} \frac{2r_0^3 + 4{r_0'}^2 r_0 - r_0^2 r_0''}{\sqrt{r_0^2 + {r_0'}^2}^3} + g \frac{2{r_0'}^4 + r_0^3 r_0'' - 2r_0 {r_0'}^2 r_0'' - r_0^2 {r'}_0^2}{\sqrt{r_0^2 + {r_0'}^2}^5}$$
$$\text{and } f_2(g, r_0)(\phi) = \frac{r_0^2 g}{\sqrt{r_0^2 + {r_0'}^2}^3}.$$

REMARK 19 Here, only a H^1-estimate

$$\nabla^2 J_2(r_0)[r;r] \geq c_0 \|r\|_{H^1}^2, \text{ with some } c_0 > 0 \tag{23}$$

is possible. For the verification of such an estimate a Riccati equation technique may be used.

THEOREM 3 *For $\Omega_0 \in C_p^2[0, 2\pi]$ and $g \in C^3$ the condition $\left[\frac{\partial g}{\partial n} + g\kappa\right]|_0 \equiv 0$ and estimate (23) are sufficient for optimality.*

Proof. Similar to the volume case we have to estimate $\vartheta_2(r) = d^2 J_2(r_\nu)[r,r] - d^2 J_2(r_0)[r,r]$. From (22) it follows that

$$|\vartheta_2(r)| \leq \int_0^{2\pi} r^2 |f_1^\nu(\phi) - f_1^0(\phi)| + {r'}^2 |f_2^\nu(\phi) - f_2^0(\phi)|\, d\phi,$$

where $f_1^\nu(\phi) = f_1(\nabla^2 g, \nabla g, g, r_\nu)(\phi)$ and $f_2^\nu(\phi) = f_2(g, r_\nu)(\phi)$, respectively. Moreover, with $g \in C^3$ and (22) we get (because of

$$|f_1^\nu(\phi) - f_1^0(\phi)| \leq \left| \frac{r_\nu^2}{\sqrt{r_\nu^2 + {r'}_\nu^2}} \frac{\partial^2 g}{\partial \vec{r}^2}|_\nu - \frac{r_0^2}{\sqrt{r_0^2 + {r'}_0^2}} \frac{\partial^2 g}{\partial \vec{r}^2}|_0 \right| + \ldots$$
$$+ \left| \frac{2r_\nu^3 + 4{r'}_\nu^2 - r_\nu^2 r_\nu''}{\sqrt{\cdots}^3} \frac{\partial g}{\partial \vec{r}}|_\nu - \frac{2r_0^3 + 4{r'}_0^2 - r_0^2 r_0''}{\sqrt{\cdots}^3} \frac{\partial g}{\partial \vec{r}}|_0 \right| + \ldots$$

and so on)

$$|\vartheta_2(r)| \leq \int_0^{2\pi} r^2\{c_1|r| + c_2|r'| + c_3|r''|\} + {r'}^2 c_4|r|\, d\phi,$$

with $c_i = c_i(g, r_0, \eta)$, $i = 1(1)4$.

$$\leq \tilde{c}(g, r_0, \eta) \cdot \|r\|_{C^2} \cdot \|r\|^2_{H^1}, \text{ for } \|r\|_{C^2} < \eta.$$

Summarizing up, we are able to estimate (for sufficiently small $\eta > 0$)

$$J_2(r_0 + r) - J_2(r_0) \geq \frac{c_0}{2}\|r\|^2_{H^1}, \text{ for } \|r\|_{C^2} < \eta.$$

■

REMARK 20 If $g < 0$ holds somewhere on the boundary Γ_0, Ω_0 cannot be optimal.

The similarity of the sufficient conditions can be seen by the following transformation of $\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}]$. We use

$$\vec{d} = \langle\vec{n}, \vec{d}\rangle\vec{n} + \langle\vec{\tau}, \vec{d}\rangle\vec{\tau} = d_n\vec{n} + d_\tau\vec{\tau},\ (\dot{d}_n = \frac{d}{dt}\langle\vec{n}, \vec{d}\rangle,\ \dot{d}_\tau = \frac{d}{dt}\langle\vec{\tau}, \vec{d}\rangle)$$
$$\Rightarrow \dot{d}_n = \kappa\sqrt{\dot{y}^2 + \dot{x}^2}d_\tau + \left\langle\vec{n}, \frac{d}{dt}\vec{d}\right\rangle,\ \dot{d}_\tau = -\kappa\sqrt{\dot{y}^2 + \dot{x}^2}d_n + \left\langle\vec{\tau}, \frac{d}{dt}\vec{d}\right\rangle$$

and obtain

$$\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}] = \int_0^T \langle\nabla^2_x g_0\, \vec{d}, \vec{d}\rangle\sqrt{\dot{x}_0^2 + \dot{y}_0^2}$$
$$+ 2\langle\nabla_x g_0, \vec{d}\rangle \cdot \left\langle\vec{\tau}, \frac{d}{dt}\vec{d}\right\rangle + g_0 \cdot \frac{(\dot{x}_0\dot{d}_y - \dot{y}_0\dot{d}_x)^2}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}^3}\, dt$$
$$= \int_0^T \left(d_n^2\frac{\partial^2 g_0}{\partial n^2} + 2d_n d_\tau\frac{\partial^2 g_0}{\partial n\partial\tau} + d_\tau^2\frac{\partial^2 g_0}{\partial\tau^2}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2}$$
$$+ \frac{g_0(\dot{d}_n - \kappa_0\sqrt{\dot{x}_0^2 + \dot{y}_0^2}d_\tau)^2}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}$$
$$+ 2(\dot{d}_\tau + \kappa_0\sqrt{\dot{x}_0^2 + \dot{y}_0^2}d_n)\left(d_n\frac{\partial g_0}{\partial n} + d_t au\frac{\partial g_0}{\partial\tau}\right)dt$$
$$= \int^T I_1 + I_2 + I_3\, dt.$$

Here we introduced

$$I_1 = d_n^2\left(\frac{\partial^2 g_0}{\partial n^2} + 2\kappa_0 \frac{\partial g_0}{\partial n}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} + \dot{d}_n^2 \frac{g_0}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}},$$

$$I_2 = d_\tau^2\left(\frac{\partial^2 g_0}{\partial \tau^2} + \kappa_0^2 g_0\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} + 2\dot{d}_\tau d_\tau \frac{\partial g_0}{\partial \tau},$$

$$I_3 = 2d_n d_\tau\left(\frac{\partial^2 g_0}{\partial n \partial \tau} + \kappa_0 \frac{\partial g_0}{\partial \tau}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} + 2\dot{d}_\tau d_n \frac{\partial g_0}{\partial n} - 2d_n \dot{d}_\tau \kappa_0 g_0.$$

By using the necessary optimality condition $\kappa_0 g_0 + \dfrac{\partial g_0}{\partial n} \equiv 0$ on Γ_0, we immediately get

$$\int_0^T I_2\, dt = 0 \text{ and } \int_0^T I_3\, dt = 0,$$

because of

$$\int_0^T 2\dot{d}_\tau\, d_\tau\, \frac{\partial g_0}{\partial \tau}\, dt = -\int_0^T d_\tau^2\left(\frac{\partial^2 g_0}{\partial \tau^2} - \kappa_0 \frac{\partial g_0}{\partial n}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt,$$

for the second part, and the third part vanishes by

$$2\dot{d}_\tau d_n \frac{\partial g_0}{\partial n} - 2d_n \dot{d}_\tau \kappa_0 g_0 = 2\frac{d}{dt}[d_\tau d_n]\frac{\partial g_0}{\partial n} \text{ and}$$
$$\left(\frac{\partial^2 g_0}{\partial n \partial \tau} + \kappa_0 \frac{\partial g_0}{\partial \tau}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} - \frac{d}{dt}\frac{\partial g_0}{\partial n} \equiv 0.$$

Hence, we arrive (for a domain Ω_0, satisfying the necessary condition) at

$$\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}]$$
$$= \int_0^T \left[d_n^2\left(\frac{\partial^2 g_0}{\partial n^2} + 2\kappa_0 \frac{\partial g_0}{\partial n}\right) + (\dot{d}_n)^2 \frac{g_0}{\dot{x}_0^2 + \dot{y}_0^2}\right]\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt. \qquad (24)$$

The same can be directly obtained from (17).

REMARK 21 The equivalence between (24) and (22) is also obvious for star-shaped domains. Moreover, coercivity holds simultaneously.

4.3. Problems with equality constraints

The "standard results" for free local minima can be extended to problems with finitely many (mixed) equality constraints like (C) $\{\bar{J}(\Omega) \to \inf$, subject to domain or boundary integral type with sufficiently smooth data fields. Whereas this can be done along the lines of standard techniques (for example, see Casas, Tröltzsch, Unger, 1996), we present it for the sake of completeness. To this aim we assume that Ω_0 is a regular solution of the system of the first order necessary conditions, i.e. there exists a $\lambda^0 = (\bar{\lambda}^0, \lambda_1^0, \ldots, \lambda_k^0)^T \neq \mathbf{0}$, satisfying together with $\Omega_0(r_0)$

$$\nabla L(r_0, \lambda^0)[r] = 0,\ \forall r \in C^2,\ J_i(\Omega_0) = 0,\ i = 1(1)k.$$

Here the Lagrangian $L(\Omega, \lambda) = L(r, \lambda) = \bar{\lambda}\bar{J}(r) - \sum_{i=1}^{k} \lambda_i J_i(r)$ is defined as usual, and regularity means that

- $\bar{\lambda}^0 = 1$ holds, i.e., the necessary condition is of Kuhn-Tucker and not of Fritz-John type. We do not discuss this assumption in detail, sometimes for special applications it can be shown explicitly (see the section below).
- The gradients of the constraints are linearly independent at r_0, implying that $\nabla \vec{J}(r_0)[\cdot] := (\nabla J_1(r_0)[\cdot], \ldots, \nabla J_k(r_0)[\cdot])^T$ is a mapping from C^2 **onto** $\mathbb{R}^k$. Moreover, this is sufficient for the coincidence of the tangent cone and the linearizing cone $T_c(r_0) = \{r \in C^2 \mid \nabla J_1(r_0)[r] = \ldots = \nabla J_k(r_0)[r] = 0\}$.

COROLLARY 4 *Let Ω_0 be a regular stationary point of problem (C). Then the condition*

$$\nabla^2 L(r_0; \vec{\lambda})[r, r] \geq c_0 \|r\|_{L_2}^2, \text{ for all } r \in T_c(r_0), \tag{25}$$

is sufficient for the optimality of Ω_0 if only domain integrals occur in problem (C). For a "mixed" formulation, the coercivity condition (25) has to be required with respect to H^1.

Proof. The main "difficulty" of the constraint case is as follows: For some admissible r ($J_i(r) = 0$, $i = 1(1)k$) from a neighbourhood $B_\delta(r_0)$ we have in general $r - r_0 \notin T_c(r_0)$. Consequently, we need for the comparison of $\bar{J}(r) - \bar{J}(r_0)$ the existence of a $v \in T_c(r_0)$ satisfying in addition to $\|v - (r - r_0)\|_{C^2} = o(\|r - r_0\|_{C^2})$ for $\|r - r_0\|_{C^2} \to 0$

$$\frac{\|v - (r - r_0)\|_{L_2}}{\|r - r_0\|_{L_2}} \to 0,$$

$$\text{or, in the "mixed" case } \frac{\|v - (r - r_0)\|_{H^1}}{\|r - r_0\|_{H^1}} \to 0. \tag{26}$$

This can be obtained by using the first order remainder of the constraints. We have: $0 = J_i(r) = J_i(r_0) + \nabla J_i(r_0)[r - r_0] + \vartheta_1^i(r - r_0)$, $i = 1(1)k$, and define a $r_\vartheta \in C^2$ (and related $v := r - r_0 + r_\vartheta$) as a solution of (note that $\nabla \vec{J} : C^2 \xrightarrow{\text{onto}} \mathbb{R}^k$)

$$\nabla \vec{J}(r_0)[r_\vartheta] = \vec{\vartheta}_1(r - r_0) := (\vartheta_1^1, \ldots \vartheta_1^k)^T$$

Moreover, by the concrete structure, the remainders satisfy for boundary integrals (after integration by parts)

$$|\vartheta_1^i(r - r_0)| = |\nabla\{J_i(r_{\nu_i}) - J_i(r_0)\}[r - r_0]| \le c_i \|r - r_0\|_{C^2} \|r - r_0\|_{L_2}$$

($r_{\nu_i} = r_0 + \nu_i(r - r_0)$, $\nu_i \in (0,1)$), whereas for domain integrals the related estimate is

$$|\vartheta_1^i(r - r_0)| = |\nabla\{J_i(r_{\nu_i}) - J_i(r_0)\}[r - r_0]| \le c_i \|r - r_0\|_{C} \|r - r_0\|_{L_2}.$$

Consequently, for all $i = 1(1)k$ we obtain in addition

$$|\vartheta_1^i(r - r_0)| \le c_i \|r - r_0\|_{C^2} \|r - r_0\|_{H^1}.$$

These estimates carry over to $\|r_\vartheta\|$, because $\nabla \vec{J}$ is also a continuous mapping (more precisely: $\nabla \vec{J}$ can be continuously extended) with respect to L_2 or H^1, i.e., it holds (26). At the end we present a short outline of the remaining estimates for the "mixed" case (replace H^1-norm by L_2-norm for the other case with ϑ_2^L denoting the second order remainder of the Laplacian — see Sections 4.1 and 4.2)

$$\begin{aligned}
\bar{J}(r) - \bar{J}(r_0) &= \frac{1}{2}\nabla^2 L(r_0, \lambda^0)[r - r_0; r - r_0] + \vartheta_2^L,\ r - r_0 = v - r_\vartheta, \\
&= \frac{1}{2}\nabla^2 L^0[v, v] - \nabla^2 L^0[v; r_\vartheta] + \frac{1}{2}\nabla^2 L^0[r_\vartheta; r_\vartheta] + \vartheta_2^L \\
&\ge \frac{c_0}{2}\|v\|_{H^1}^2 - c\|v\|_{H^1}\|r_\vartheta\|_{H^1} - \hat{c}\|r_\vartheta\|_{H^1} - |\vartheta_2^L| \\
&\ge \frac{c_0}{3}\|v\|_{H^1}^2 - |\vartheta_2^L|,\ \text{if } \|r - r_0\|_{C^2} \le \eta \\
&= \frac{c_0}{3}\|r - r_0 + r_\vartheta\|_{H^1}^2 - |\vartheta_2^L| \\
&\ge \frac{c_0}{3}\|r - r_0\|_{H^1}^2 \left\{1 - \frac{\|r_\vartheta\|_{H^1}}{\|r - r_0\|_{H^1}} - \frac{|\vartheta_2^L|}{\|r - r_0\|_{H^1}^2}\right\} \ge \frac{c_0}{4}\|r - r_0\|_{H^1}^2,
\end{aligned}$$

where the last inequality holds once again for $\|r - r_0\|_{C^2} \le \eta$. ■

REMARK 22 For the stronger norm-requirement in the mixed case we get also a stronger estimate for the difference of the functional values.

5. The Dido problem

As an illustrating example we want to apply the foregoing investigations to the Dido problem of maximizing the volume (area) of a domain subject to a given length of the perimeter. There are two elementary proofs known for the optimality of the circle (see, for example, Tichomirow, 1990). One of them is mainly based on investigations of Zenodorus in ancient Greece. The second proof of the problem are given in the calculus of variation (see Ioffe and Tichomirow, 1979). If we restrict our considerations to starshaped domains only, the problem seems to become

$$
\text{(P)} \quad \begin{cases} \bar{J}(r) = \int\limits_{\Omega} -1\,dx = \int\limits_{0}^{2\pi} -\frac{1}{2} r^2(\phi)\,d\phi \to \inf, \\ \text{subject to} \\ J_1(r) = \int\limits_{\Gamma} 1\,dS_\Gamma = \int\limits_{0}^{2\pi} \sqrt{r^2(\phi) + r'^2(\phi)}\,d\phi = l_0. \end{cases}
$$

However, the problem is invariant with respect to parallel shifting. Hence, for the investigation of **sufficient** condition we additionally fix the baricentre, for convenience at the origin, which "forbids" the parallel shifting and does not influence the original problem otherwise. We arrive at the following modified problem

$$
\text{(PM)} \quad \begin{cases} \bar{J}(r) = \int\limits_{\Omega} -1\,dx = \int\limits_{0}^{2\pi} -\frac{1}{2} r^2(\phi) d\phi \to \inf, \\ \text{subject to} \\ J_1(r) = \int\limits_{\Gamma} 1\,dS_\Gamma - l_0 = \int\limits_{0}^{2\pi} \sqrt{r^2(\phi) + r'^2(\phi)}\,d\phi - l_0 = 0, \\ J_2(r) = \int\limits_{\Omega} x_1\,dx = \int\limits_{0}^{2\pi} \cos\phi \int\limits_{0}^{r(\phi)} \rho^2\,d\rho\,d\phi = 0, \\ J_3(r) = \int\limits_{\Omega} x_2\,dx = \int\limits_{0}^{2\pi} \sin\phi \int\limits_{0}^{r(\phi)} \rho^2\,d\rho\,d\phi = 0. \end{cases}
$$

Whereas the discussion of necessary conditions is known from calculus of variation, we repeat it in terms of shape funtionals. We define the Lagrangian

$$
L(r;\lambda) = \bar{J}(r) - \sum_{k=1}^{3} \lambda_k J_k(r),
$$

and obtain for $r \in C_p^2$

$$
dL(r;\lambda)[r_1] = \int\limits_{0}^{2\pi} -r(\phi) r_1(\phi)[1 + \lambda_2 r(\phi)\cos\phi + \lambda_3 r(\phi)\sin\phi]
$$

$$
- \lambda_1 \frac{r r_1 + r' r_1'}{\ldots}(\phi)\,d\phi
$$

$$= \int_0^{2\pi} -r(\phi) r_1(\phi)(1 + \lambda_1 \cdot \kappa(\phi) + \lambda_2 \cos\phi r(\phi)$$

$$+ \lambda_3 \sin\phi r(\phi))\, d\phi \stackrel{!}{=} 0,$$

$$\Rightarrow 1 + \lambda_1 \cdot \kappa(\phi) + \lambda_2 \cos\phi r(\phi) + \lambda_3 \sin\phi r(\phi) = 0,\ \phi \in [0, 2\pi].$$

With $\lambda_2^0 = \lambda_3^0 = 0$ and according to our constraints, we get

$$\kappa_0 \equiv \text{const.} \neq 0 \Rightarrow r_0(\phi) \equiv r_0,\ \lambda_1^0 = -\kappa_0^{-1} = -r_0 = -\frac{l_0}{2\pi}.$$

REMARK 23 The assertion $\lambda_2^0 = \lambda_3^0 = 0$ makes sense, because the optimal value function is obviously constant with respect to a variation of the value of the second and third constraint. Moreover, a vanishing Lagrange multiplier of the objective (i.e., $\bar{\lambda} = 0$) implies $\lambda_1 = 0$ or $\kappa_0 \equiv 0$. Therefore, regularity of the Lagrangian can be assumed.

REMARK 24 The additional constraints are formally not needed for the necessary condition. Also for Problem (P) we obtain

$$\kappa_0 \equiv \text{const.} \neq 0 \text{ and } \lambda_1^0 = -\kappa_0^{-1} = -\frac{l_0}{2\pi}.$$

However, we cannot conclude uniquely $r_0(\phi) \equiv r_0$, because all "shifted" circle with centre at $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$ satisfies the necessary condition for $\varepsilon_1^2 + \varepsilon_2^2 < r_0^2$ $(\Rightarrow r_\varepsilon(\phi) = \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi + \sqrt{r_0^2 - \varepsilon_1^2 \sin^2\phi - \varepsilon_2^2 \cos^2\phi - \varepsilon_1 \varepsilon_2 \sin 2\phi})$.

For the validity of a sufficient second order condition we need

$$\nabla^2 L(r_0, \lambda^0)[r; r] \geq c_0 \|r\|_{H^1}^2,$$

for all r from the tangent cone T_c^0 at Ω_0 of the constraints. Due to the regularity, the tangent cone coincides with the linearizing cone, i.e., according to the derivatives of J_k,

$$T_c^0 = T_c(\Omega_0)$$

$$= \left\{ r \in C^2 \,\middle|\, \int_0^{2\pi} r(\phi) d\phi = 0,\ \int_0^{2\pi} r(\phi) \cos\phi d\phi = 0,\ \int_0^{2\pi} r(\phi) \sin\phi d\phi = 0 \right\}.$$

LEMMA 6 *It is true that*

$$\nabla^2 L(r_0, \lambda^0)[r; r] \ \geq\ \frac{3}{5} \|r\|_{H^1}^2,$$

for all $r \in T_c^0$, ensuring that a sufficient second order condition is satisfied for

Proof. An easy calculation yields

$$\nabla^2 L(r_0, \lambda^0)[r; r] = \int_0^{2\pi} r'^2(\phi) - r^2(\phi)\, d\phi.$$

Moreover, the system of trigonometric functions $\{\mathbf{1}, \cos n\phi, \sin n\phi, n \geq 1\}$ is complete in C^2 and a orthonormal basis in H^1, hence,

$$\|r\|_{H^1}^2 = \int_0^{2\pi} r'^2(\phi) + r^2(\phi)\, d\phi = \mu_0^2(r) + (1 + n^2) \sum_{n=1}^{\infty} \mu_n^2(r) + \nu_n^2(r).$$

The Fourier-coefficients of r are given as usual $\mu_0(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} r(\phi)\, d\phi$,

$$\nu_n(r) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} r(\phi) \sin n\phi\, d\phi,\ \mu_n(r) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} r(\phi) \cos n\phi\, d\phi.$$

Furthermore, the tangent cone is contained in the closure of the linear hull of $\{\cos n\phi, \sin n\phi, n \geq 2\}$. Therefore, we are able to estimate as follows for $r \in T_c^0$

$$\int_0^{2\pi} r'^2(\phi) - r^2(\phi)\, d\phi = \sum_{n=2}^{\infty} (n^2 - 1)[\mu_n^2(r) + \nu_n^2(r)]$$

$$\geq \sum_{n=2}^{\infty} \frac{3(n^2+1)}{5} [\mu_n^2(r) + \nu_n^2(r)] = \frac{3}{5} \|r\|_{H^1}^2.$$

Hence, we have the desired coercivity of $\nabla^2 L(r_0, \lambda^0)[r; r]$. ■

REMARK 25 From calculus of variation the validity of

$$\nabla^2 L(r_0, \lambda^0)[r; r] \geq 0,\ \forall r \in T_{c2}^0 = \left\{ r \in C^2 \,\middle|\, \int_0^{2\pi} r(\phi) d\phi = 0 \right\}$$

is known. However, this is directly clear from the discussion above. Moreover, the functions $r_1(\phi) = \cos \phi$ and $r_2(\phi) = \sin \phi$ are associated with the "linearized directions of parallel shifting" at Ω_0 with respect to x_1 and x_2, respectively.

REMARK 26 Sufficient conditions for shape functionals only are not too important, because some of the results are obviously or intuitively clear. Nevertheless, it can be a first step for the study of more interesting shape optimization problems. For example, it seems to be possible to combine the presented tech-

Hackbusch, 1989) for the computation of shape derivatives for elliptic equations (Potthast, 1994a, 1994b, Fujii and Goto, 1994, Eppler, 1998a), also related to investigations of Fujii (Fujii, 1986, 1990, 1994, Belov and Fujii., 1997). This will be discussed in a forthcoming paper.

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