

Sensitivity analysis of a bond portfolio model for the Italian market

by

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Abstract: Management of bond portfolio is formulated as a multiperiod scenario-based stochastic program with random recourse. The former results on sensitivity analysis of its optimal value with respect to the strategy applied in selection of input scenarios are extended and applied to a real life problem from the Italian bond market. The numerical study provides details on this application and illustrates also the impact of the utility function chosen and of the size of transaction costs.

Keywords: stochastic program, scenarios, sensitivity, application in finance.

1. Problem formulation

We shall describe a stochastic programming model for management of portfolio of fixed income securities, called bonds for brevity. The main purpose of the portfolio management is to maximize the expected utility of the wealth at the end of a given period and, depending on the specific field of investment activities, to secure the prescribed or uncertain future payments. Similar problems arise in the context of management of one person investment of fixed income securities.

of liabilities can rank from fixed prescribed or planned external outflows (or inflows) to liabilities whose value depends on various external random factors such as mortality rates. In contrast to the problem of a dedicated portfolio selection (see e.g. Hiller and Eckstein, 1993, Shapiro, 1988), we allow for an *active trading strategy*. Accordingly, we formulate constraints on conservation of holdings for each asset at each time period and, similarly as in Golub et al. (1995), we extend the constraints on cashflows for each time period to include the possibility of *rebalancing* the portfolio.

The prices of bonds and sometimes also the coupon cashflows f_t are driven by the assumed evolution of the interest rates: Given a sequence of equilibrium future forward short term interest rates r_t valid for the time interval $[t, t + 1)$, $t = 0, \dots, T - 1$ the fair price of the j -th bond at time t equals the total cashflow generated by this bond in subsequent time instances discounted to t :

$$B_{jt}(\mathbf{r}) = \sum_{\tau=t+1}^T f_{j\tau} \prod_{h=t}^{\tau-1} (1 + r_h)^{-1} \quad (1)$$

where T is greater than or equal to the time to maturity.

In reality, however, the sequence of the future short term interest rates that determines the prices (1) is not known, the sequences of interest rates are prescribed ad hoc or modeled in a probabilistic way. We shall consider a discrete distribution carried by S possible vectors \mathbf{r} of interest rates concentrated with probabilities $p_s > 0 \forall s$, $\sum_s p_s = 1$ at points $\mathbf{r}^s \in \mathbb{R}^T$, $s = 1, \dots, S$ called *scenarios*; this is the input information which is used to build the discussed model, and which influences the results.

We shall mostly use the notation introduced in Golub et al. (1995), Dupačová and Bertocchi (1996), Bertocchi, Dupačová and Moriggia (1996) and Moriggia (1997):

$j = 1, \dots, J$ are indices of the considered bonds and t_j the dates of their maturities;

$t = 0, \dots, T_1$ is the considered discretization of the planning horizon;

$\mathbf{b} \in \mathbb{R}^J$ denote the J -vector of initial holdings (in face value) of bonds;

c is the initial holding in riskless asset;

$\mathbf{f}_t^s \in \mathbb{R}^J$ are cashflows generated from bonds portfolio at time t under scenario s expressed as a fraction of their face value;

$\xi_t^s \in \mathbb{R}^J$ and $\zeta_t^s \in \mathbb{R}^J$ are the selling and purchasing price J -vectors of bonds at time t for scenario s obtained from the corresponding fair prices (1) by subtracting or adding fixed transaction costs and spread; the initial price vectors $\xi_0 \in \mathbb{R}^J$ and $\zeta_0 \in \mathbb{R}^J$ are known, i.e., scenario independent;

$L_t \in \mathbb{R}$ is liability due at time t ;

$\mathbf{x}_0 \in \mathbb{R}_+^J$ (and $\mathbf{y}_0 \in \mathbb{R}_+^J$) is the face value J -vector of bonds to be purchased (sold) at the beginning of the planning period, i.e., at $t = 0$;

$\mathbf{z}_0 \in \mathbb{R}^J$ is the face value J -vector of bonds held in portfolio after the initial

The first stage decision variables $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ are nonnegative,

$$\mathbf{y}_0 + \mathbf{z}_0 = \mathbf{b} + \mathbf{x}_0 \tag{2}$$

and

$$\overset{+}{v}_0 + \zeta_0^\top \mathbf{x}_0 = c + \xi_0^\top \mathbf{y}_0 \tag{3}$$

where the auxiliary nonnegative variable $\overset{+}{v}_0 \in \mathbb{R}_+$ denotes the surplus.

The second-stage decisions on rebalancing the portfolio, borrowing or reinvestment of the surplus depend on individual scenarios and have to fulfil constraints on conservation of holdings in each bond at each time period and for each of scenarios

$$\mathbf{z}_t^s + \mathbf{y}_t^s = \mathbf{z}_{t-1}^s + \mathbf{x}_t^s \quad \forall s, t \tag{4}$$

where $\mathbf{x}_t^s, \mathbf{y}_t^s, \mathbf{z}_t^s$ denote the face value of bonds purchased, sold, held in the portfolio at time $t, t = 1, \dots, T_1$ under scenario s , and constraints on rebalancing the portfolio at each time period $1 \leq t \leq T_1$

$$\begin{aligned} &\zeta_t^{s\top} \mathbf{y}_t^s + \mathbf{f}_t^{s\top} \mathbf{z}_{t-1}^s + (1 + r_{t-1}^s - \eta) \overset{+}{v}_{t-1}^s + \bar{v}_t^s \\ &= L_t + \zeta_t^{s\top} \mathbf{x}_t^s + (1 + \delta + r_{t-1}^s - \eta) \bar{v}_{t-1}^s + \overset{+}{v}_t^s \quad \forall s, t \end{aligned} \tag{5}$$

where $\mathbf{z}_0^s \equiv \mathbf{z}_0, \forall s, \overset{+}{v}_t^s \in \mathbb{R}_+$ denotes the surplus in cash for scenario s and $\bar{v}_t^s \in \mathbb{R}_+$ is the short position in cash for scenario s . Parameter η measures the distortion between risk-free asset and market short rate, parameter δ accounts for the positive cost of borrowing.

The optimization problem consists in maximization of the expected utility of the final wealth

$$\sum_s p_s U(W_{T_1}^s) \tag{6}$$

subject to constraints (2)–(5) and nonnegativity constraints on all variables, with

$$W_{T_1}^s = \xi_{T_1}^{s\top} \mathbf{z}_{T_1}^s + \overset{+}{v}_{T_1}^s - \alpha \bar{v}_{T_1}^s \quad \forall s \tag{7}$$

The multiplier α should be fixed according to the problem area. For instance, a pension plan assumes repeated application of the model with rolling horizon and values $\alpha > 1$ take into account the debt service in the future.

Thanks to the possibility of reinvestments and of unlimited borrowing, the problem has always a feasible solution. The existence of optimal solutions is guaranteed for a large class of utility functions that are *increasing and concave*, what will be assumed henceforth. From the point of view of stochastic

relatively complete recourse and with additional nonlinearities due to the choice of the utility function. Moreover, due to strict inequalities of elements of prices vectors $\xi_{jt}^s < \zeta_{jt}^s \forall j, t, s$ and $\delta > 0$, the optimal solutions fulfil the following conditions

$$\begin{aligned} y_j \cdot x_j &= 0 \quad \forall j \\ y_{jt}^s \cdot x_{jt}^s &= 0 \quad \forall s, j, 1 \leq t \leq T_0 \\ v_t^+ \cdot v_t^- &= 0 \quad \forall s, 1 \leq t \leq T_0 \end{aligned}$$

whose interpretation is straightforward: to maximize the gain one has to *avoid unnecessary trading and borrowing*.

The resulting large scale deterministic program (2)–(7) with a concave objective function and numerous linear constraints can be solved, e.g., by GAMS. Its size as well as the numerical values of the coefficients result from the choice of the considered bonds, their characteristics (initial prices and cashflows) and initial holdings, from the scheduled stream of liabilities, from the choice of utility function, from the used model of interest rates and the market data used to fit the model, and from how a modest number of scenarios has been selected out of the whole population. This is the input. The main outcome is the optimal value of the objective function (the maximal expected utility of the final wealth) and the optimal values of the first-stage variables $\mathbf{x}_0, \mathbf{y}_0$ (and \mathbf{z}_0).

In this study we shall assume that the liabilities are fixed and that the interest rate scenarios have been generated according to Black-Derman-Toy (1990) and we shall analyze the sensitivity of the optimal value of (6) for the selected scenarios of interest rates. This is an important task because there is an arbitrariness in constructing the probability distribution of the interest rates; there are scenarios designed only by experts or required by local authorities (e.g. the New York State regulation #126 requires evaluation of bonds for 7 specific scenarios), those based on a continuous time stochastic model, and scenarios obtained by the binomial lattice techniques, e.g., Black, Derman, Toy (1990), that will be used in our report. Moreover, due to the size of the resulting problem, not all scenarios can be used and a sampling procedure has to be used to get a manageable number of scenarios out of the fitted binomial lattice. One of possibilities is the nonrandom sampling strategy by Zenios and Shtilman (1993) applied in the context of the Black-Derman-Toy model. A natural question is *the impact of the chosen sampling strategy and the influence of including additional or out-of-sample scenarios on the output* based on an initial manageable sample of scenarios. The first question will be treated in the subsequent section and we refer to Dupačová (1999) for a postoptimality technique with respect to additional scenarios and to Dupačová, Bertocchi, Moriggia (1997), (1998) for an application of these results to the bond portfolios management problem. A selection of numerical results related to application of model (2)–(7) to the Italian market, including the suggested sensitivity analysis comparisons of results for various

2. Sampling strategy

The basic assumptions of the Black-Derman-Toy model can be summarized as follows:

- The short rate is the only factor that drives the bond prices, it can move up or down with equal probability over the next time period; the sequences of “up-down” and “down-up” moves from any fixed stage at a time point t result into the same value of interest rate at the time point $t + 2$ (the path independence property).
- The expected returns on all securities over one period are equal, short rates are lognormally distributed with the volatility of their logarithms depending only on time.
- The input is the yield curve and yield volatilities valid for zero-coupon governmental bonds at a given date; this input should be available for all maturities.
- The securities are valued as the expected prices one period ahead discounted by the present short rate.

As a result, at each time point t , there are $t + 1$ possible stages and for the given horizon T there are 2^{T-1} equiprobable scenarios. Each of them can be represented by a random binary fraction with $T - 1$ 0-1 digits, say

$$\omega^s = 0.\omega_1^s\omega_2^s\dots\omega_{T-1}^s$$

with $\omega_t^s = 0$ or $1 \forall t, s$ and their probabilities $p_s = 2^{-(T-1)} \forall s$. The digit 1 at the t -th position corresponds to the “up” move, the digit 0 corresponds to the “down” move of the one-period short term interest rate in the step t . This theoretical binomial lattice has to be calibrated by the existing term structure to get the base rates r_{t0} and the volatility factors k_t for all t . Pan Kang and Zenios (1992). The corresponding one-period short term interest rates for scenario s and for the time interval $(t, t + 1]$ are then given as

$$r_t^s = r_{ti(s)} \quad (8)$$

where

$$r_{ti} = r_{t0}k_t^i, \quad i_t(s) = \sum_{\tau=1}^t \omega_\tau^s \quad (9)$$

That is, $i_t(s)$ equals the number of the “up” moves for the given scenario s which occur at time points $1, \dots, t$. We denote \mathbf{r}^s the vector of components $r_t^s, \forall t$.

We shall detail now the results on sensitivity of the outcome on the non-random sampling strategy of Zenios and Shtilman (1993), presented in Dupačová and Bertocchi (1996). Our sensitivity analysis will be related to a simplified version of the deterministic sampling strategy by Zenios and Shtilman (1993):

binary fraction of length T_0 . The sample point ω^s from $[0, 1)$ is determined by one of these binary fractions and by an arbitrary continuation up to binary fraction of length $T - 1$. According to (8), (9) we build then $S = 2^{T_0}$ scenarios \mathbf{r}^s and we denote

$$i_{T_0}(s) = \sum_{t=1}^{T_0} \omega_t^s.$$

The lower and upper bounds for r_t^s with $t \geq T_0$ are evident:

$$\begin{aligned} r_t^{s-} &= r_{t_0} k_t^{i_{T_0}(s)} \leq r_t^s \leq r_{t_0} k_t^{t-T_0+i_{T_0}(s)} = r_t^{s+}, \\ t &= T_0 + 1, \dots, T - 1 \quad \forall s \end{aligned} \tag{10}$$

and for $t \leq T_0$, r_t^s are fully determined by the described choice of the path ω^s . The input of our problem (2)–(7) consists thus of $S = 2^{T_0}$ T -dimensional scenarios \mathbf{r}^s whose first component r_0 is the known initial interest rate, the next T_0 components are fixed for each scenario by the described nonrandom sampling procedure, whereas the subsequent $T - T_0 - 1$ components are subject to perturbations Δ^s such that

$$\begin{aligned} \Delta_t^s &= 0, \quad t = 1, \dots, T_0 \\ \Delta_t^s &= r_t - r_t^s, \quad t = T_0 + 1, \dots, T \end{aligned} \tag{11}$$

where r_t satisfies (10).

In our setting of the problem, the objective function (6) does not contain any coefficients depending on scenarios; such coefficients enter equations (5) and (7) and they are differentiable in \mathbf{r} :

The derivatives of the purchasing and selling prices $\zeta_{jt}^s = \zeta_{jt}(\mathbf{r}^s)$ and $\xi_{jt}^s = \xi_{jt}(\mathbf{r}^s)$ can be obtained from those of the fair prices (1); the fixed spread and additive transaction costs evidently do not enter the formulas at all whereas a fixed multiplicative rule for inclusion transaction costs, say by a factor $1 \pm \epsilon$ results in multiplication of the derivatives by the same factor.

The directional derivative of $B_{jt}(\mathbf{r}^s)$ in the direction of Δ^s is equal to the scalar product of the gradient of $B_{jt}(r^s)$ (see (1)) and the vector of increments Δ^s

$$B_{jt}^{\prime s}(0^+) = - \sum_{\tau=t+1}^T f_{j\tau}^s D_t^\tau \sum_{l=t}^{\tau-1} \frac{\Delta_l^s}{(1+r_l^s)} = - \sum_{l=t}^T \frac{\Delta_l^s}{(1+r_l^s)} \sum_{\tau=l+1}^T f_{j\tau}^s D_t^\tau \tag{12}$$

for all t ; we have used notation $D_t^\tau = \prod_{h=t}^{\tau-1} (1+r_h)^{-1}$.

We denote further $\varphi(\mathbf{r}^1, \dots, \mathbf{r}^S)$ the optimal value of (2)–(7) for the initial “input” $\mathbf{r}^1, \dots, \mathbf{r}^S$ and we indicate by asterisk the components of the corresponding optimal solution and of Lagrangean multipliers.

Besides the fixed number of scenarios, the basic assumptions that simplify

and unique Lagrangean multipliers for the initial choice of scenarios, and a fixed rank of the matrix of the system (5), (7) for all considered perturbations. If these assumptions are fulfilled, the linearly perturbed problem that corresponds to the input $\mathbf{r}^s + \mu\Delta^s$, $s = 1, \dots, S$ has an optimal solution for μ small enough and for arbitrary feasible perturbances Δ^s , there exists the directional derivative of the optimal value function at the given input $\mathbf{r}^1, \dots, \mathbf{r}^S$ in any feasible direction Δ^s , $s = 1, \dots, S$, and it equals the derivative at $\mu = 0^+$ of the Lagrange's function of the corresponding linearly perturbed problem evaluated at the initial optimal solution and multipliers

$$\begin{aligned} &\varphi'(0^+) \\ &= \frac{\partial}{\partial \mu} \mathcal{L}(\mathbf{x}^{s*}, \mathbf{y}^{s*}, \mathbf{z}^{s*}, \mathbf{W}^*; \kappa^{s*}, \lambda^{s*}; \mathbf{r}^s + \mu\Delta^s, s = 1, \dots, S) |_{\mu=0^+} \end{aligned} \tag{13}$$

(see Gol'shtein, 1970). The perturbation enters only equations (5) and (7); we denote the corresponding Lagrangean multipliers by $\kappa^s \in \mathbb{R}^{T_1}$ and $\lambda^s \in \mathbb{R}$. Using the form of (5), (7) we get for fixed additive transaction costs

$$\begin{aligned} \varphi'(0^+) &= \sum_{s=1}^S \left\{ \sum_{t=1}^{T_1} \kappa_t^{s*} \sum_j B_{jt}^{\prime s}(0^+) (y_{jt}^{s*} - x_{jt}^{s*}) \right. \\ &\left. + \sum_{t=T_0+1}^{T_1-1} \kappa_{t+1}^{s*} (\bar{v}_t^{s*} - \bar{v}_t^{-s*}) \Delta_t^s - \lambda^{s*} \sum_j B_{jT_1}^{\prime s}(0^+) z_{jT_1}^{s*} \right\} \end{aligned} \tag{14}$$

whereas for the multiplicative form of transaction costs the optimal values $y_{jt}^{s*}, x_{jt}^{s*}, z_{jT_1}^{s*}$ have to be replaced by $\hat{y}_{jt}^{s*} = (1 - \epsilon)y_{jt}^{s*}$, $\hat{x}_{jt}^{s*} = (1 + \epsilon)x_{jt}^{s*}$, $\hat{z}_{jT_1}^{s*} = (1 - \epsilon)z_{jT_1}^{s*}$.

The expression obtained is separable with respect to scenarios. Substituting (12) for $B_{jt}^{\prime s}$ and rearranging a bit we obtain separability with respect to scenarios and time periods:

$$\varphi'(0^+) = \sum_{s=1}^S \sum_{h=T_0+1}^T \frac{\Delta_h^s}{1 + r_h^s} H_h^s \tag{15}$$

where

$$\begin{aligned} H_h^s &= - \sum_{t=1}^h \kappa_t^{s*} \sum_j (y_{jt}^{s*} - x_{jt}^{s*}) \sum_{\tau=h+1}^T f_{j\tau}^s D_t^\tau(\mathbf{r}^s) \\ &+ \kappa_{h+1}^{s*} (1 + r_h^s) (\bar{v}_h^{s*} - \bar{v}_h^{-s*}), \quad T_0 < h < T_1; \\ H_h^s &= - \sum_{t=1}^{T_1} \kappa_t^{s*} \sum_j (y_{jt}^{s*} - x_{jt}^{s*}) \sum_{\tau=h+1}^T f_{j\tau}^s D_t^\tau(\mathbf{r}^s) \\ &+ \lambda^{s*} \sum z_{jT_1}^{s*} \sum_{\tau=h+1}^T f_{j\tau}^s D_\tau^\tau(\mathbf{r}^s), \quad T_1 < h < T. \end{aligned}$$

Separability of the directional derivative of the optimal value with respect to scenarios and time periods allows for a quick sensitivity analysis with respect to changes of the sampling strategy. The desired directions of changes in r_h^s for $h > T_0$ that result in decrease and/or increase of the optimal value function can be thus obtained by inspection of the *signs* of H_h^s only. The magnitude of these changes is limited by (10) (and also by the fact that this result is of a *local* character).

For multiple optimal solutions or Lagrangean multipliers belonging to compact sets \mathcal{M} , Λ , respectively, we get

$$\begin{aligned} \varphi'(0^+) = & \max_{\mathcal{M}} \min_{\Lambda} \sum_{s=1}^S \left\{ \sum_{t=1}^{T_0} \kappa_t^{s*} \sum_j B_{jt}^{\prime s} (0^+) (y_{jt}^{s*} - x_{jt}^{s*}) \right. \\ & \left. + \sum_{t=T_0+1}^{T_1-1} \kappa_{t+1}^{s*} (\bar{v}_t^{s*} - \underline{v}_t^{s*}) \Delta_t^s - \lambda^{s*} \sum_j B_{jT_1}^{\prime s} (0^+) z_{jT_1}^{s*} \right\}. \end{aligned} \quad (16)$$

An application of this result is rather involved. One possibility is to reformulate the bond portfolio management problem (2)–(7) to another form, with constraints independent of r . In such a case, a variant of (16) does no more include Lagrangean multipliers and with the right hand side evaluated at an *arbitrary* optimal solution it provides a bound for the derivative $\varphi'(0^+)$. This idea has been detailed in Dupačová (1999) and exploited also in scenario sensitivity simulation studies, e.g. Abaffy et al. (1999).

Notice that by allowing cashflows dependent on scenarios we are able to extend the results of Dupačová and Bertocchi (1996) to portfolios that include bonds with call or put options. This means to mark the scenarios for which the option on a bond is likely to be exercised and to update the cashflows and prices that correspond to the bonds with options and to the corresponding marked scenarios accordingly. Clearly, both the program (2)–(7) and the subsequent sensitivity analysis have to be based on the updated cashflows and prices.

The coefficients of the resulting mathematical program depend on the model input and they are influenced both by the market data, the chosen submodels and the related estimation and sampling procedures. The crucial question is the precision or robustness of the optimal trading strategy in the first time period and of the optimal value of the objective function. Some of these problems have been discussed in Dupačová (1999), in Bertocchi, Dupačová and Moriggia (1996), in Abaffy et al. (1999) and in Dupačová, Bertocchi and Moriggia (1997 and 1998).

3. Numerical results

3.1. Inputs of the model

The application of the model to the Italian Government bonds market improved

utility functions with a portfolio of Italian Treasury bonds and puttable bonds along one year time horizon ($T_1 = 12$). September 1st 1994 was used as the starting date of dynamic management. Portfolio contains five Treasury coupon bonds (called BTP) maturing between 2 (1Oct1996) and 29 years (1Nov2023) with semi-annual coupons and two puttable coupon bonds (called CTO) maturing after 6 and 8 years with put option maturing after 3 and 4 years respectively. Table 1 summarizes main features of considered portfolio. Coupons and redemption prices are after tax.

Bonds	Qt	coupon	payment dates	exercise	redemp.	maturity
BTP36658	10	3.9375	01Apr & 01Oct		100.187	01Oct96
BTP36631	20	5.0312	01Mar & 01Sep		99.531	01Mar98
BTP12687	15	5.2500	01Jan & 01Jul		99.231	01Jan02
BTP36693	10	3.7187	01Aug & 01Feb		99.387	01Aug04
BTP36665	5	3.9375	01May & 01Nov		99.218	01Nov23
CTO13212	20	5.2500	20Jan & 20Jul	20Jan95	100.000	20Jan98
CTO36608	20	5.2500	19May & 19Nov	19May95	99.950	19May98

Table 1. Portfolio composition

Time horizon is monthly discretized and ends at T defined as follows:

$$T = \max_{j=1, \dots, J} t_j = 350$$

where t_j is the maturity of bond j . Our empirical application does not consider in-the-period liabilities nor future liabilities, formally: $L_t = 0$, for $t = 1, \dots, T$, in equation (5) and $\alpha = 1$ in equation (7). The interest rate spread (see eq. (5)) equals $\delta = 0.20833\%$, or, equivalently, 2.5% annually, and the penalty of the risk-free asset equals $\eta = 0.0005$.

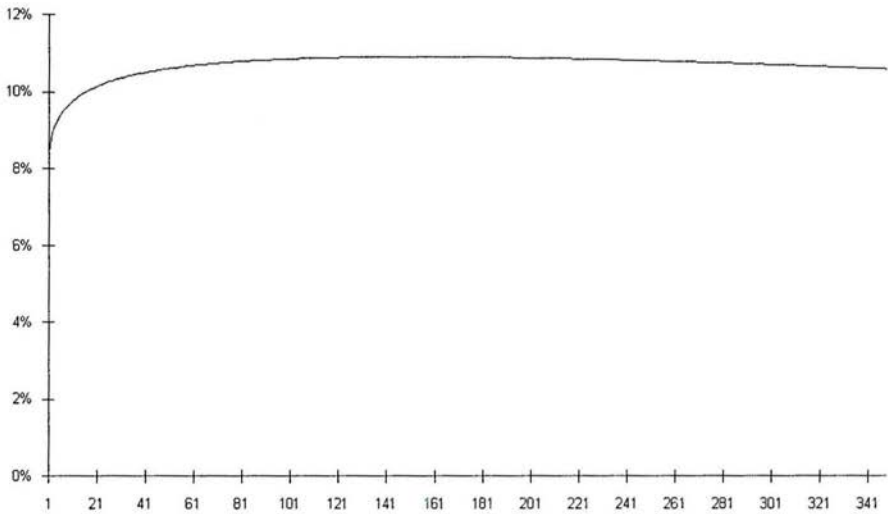
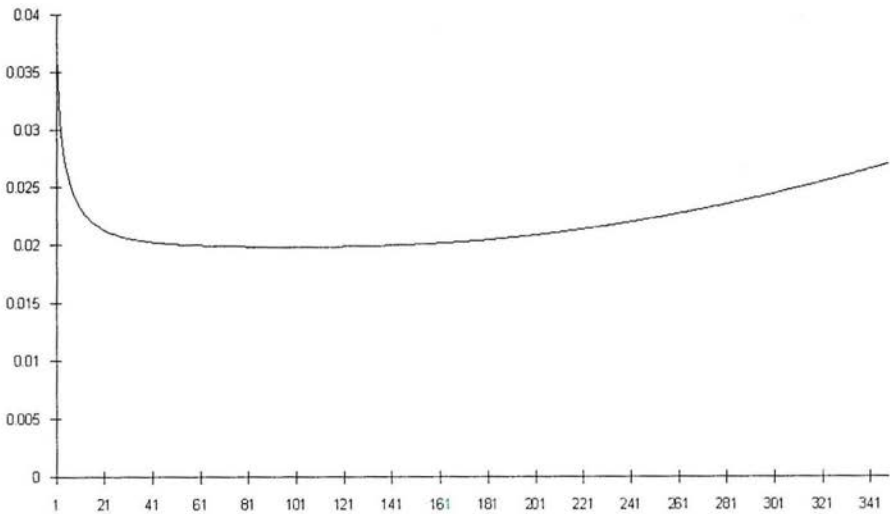
The development environment is composed of four workstations Digital 5000/240, running Ultrix 4.3 operating system. Source programs in C language have been compiled with system compiler and the optimization platform is the General Algebraic Modeling System (GAMS) ver. 2.25.062 (GAMS, 1992).

3.1.1. Interest rate evolution

To obtain the yield curve we solved the non-linear regression for the Bradley and Crane (1972) function, applied to BTP's prices of Milan market on September 1st 1994. Estimated coefficients of that function gave the following function:

$$y_t = 0.098671 \cdot t^{0.065017} \cdot e^{-0.005058 \cdot t}.$$

The volatility curve has been obtained by Dupačová, Abaffy et al. (1997). Figs. 1 and 2 draft the yield curve and the volatility curve, respectively. Similar

Figure 1. Yield curve on September 1st 1994Figure 2. Volatility curve on September 1st 1994

Interest rate evolution has been simulated by the Black, Derman and Toy model (1990). Fig. 3 drafts upper and lower bounds of binomial tree built with

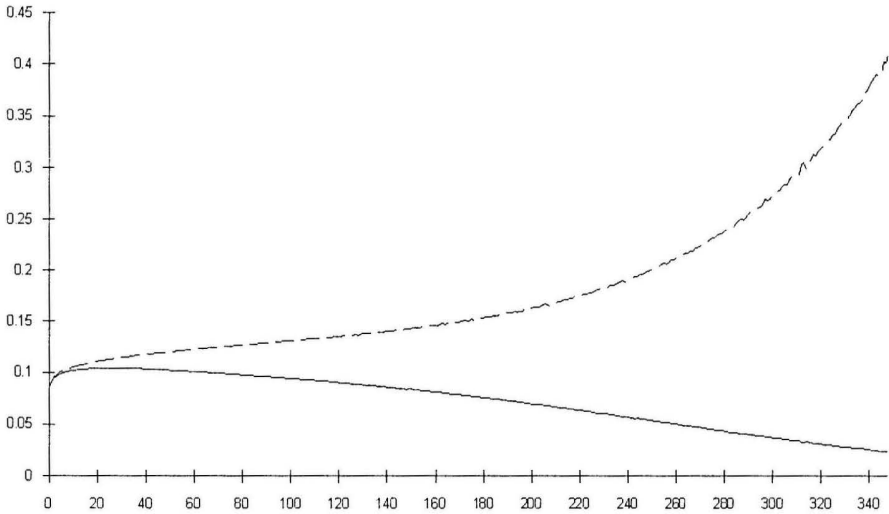
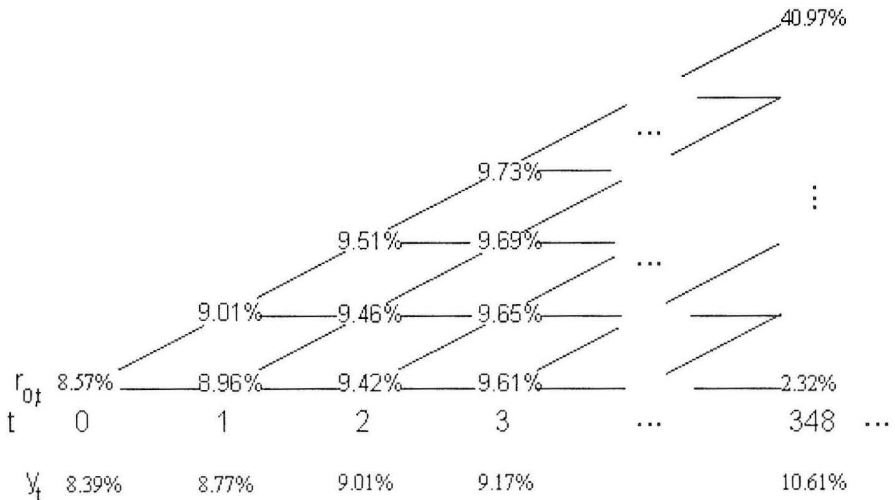


Figure 3. Upper and lower bounds of interest rate evolution

Lower bound represents the base rate, i.e. the interest rate of all down movements, whereas upper bound is given by all up movements of binomial lattice of interest rate evolution. Fig. 4 reports a part of binomial lattice.



3.1.2. Case studies

The sampling strategy allowed us to consider four diverse case studies, named A, B, C and D, with a fixed sample size of 32, 8, 16, and 4 scenarios, respectively (see Table 2).

case	T_0	no. of scenarios
A	5	$2^5 = 32$
B	3	$2^3 = 8$
C	4	$2^4 = 16$
D	2	$2^2 = 4$

Table 2. Scenario sample size

The sampling strategy considered all possible paths in $[1, T_0]$ and defined a path choice strategy for range $(T_0, T]$. Depending on that path strategy we considered six subcases as Table 3 shows.

subcase	$[1, T_0]$	$T_0 + 1$	$[T_0 + 2, T_1]$	$T_1 + 1$	$[T_1 + 2, T]$
1	all paths	down	down	up	up & down
2	all paths	down	down	down	up & down
3	all paths	down	up	up	up & down
4	all paths	up	up	up	up & down
5	all paths	up	down & up	down	up & down
6	all paths	repeat	repeat	repeat	repeat

Table 3. Path choice strategy

Case A3, e.g., regards 32 scenarios (see Table 2) that cover all possible paths for $t = 1, \dots, 5$ (Table 3) and move down in $t = 6$, then move up 7 times ($t = 7, \dots, 13$) and then alternate up and down movements along time horizon $t = 14, \dots, 350$. Subcase no. 6 repeats the pattern of the range $[1, T_0]$ along the whole sub-horizon $(T_0, T]$ (see also Fig. 5).

We decided to consider three different levels of additive transaction costs depending on three different classes of traders. For that reason we studied all the subcases shown in Table 4.

subcase	transaction cost
a	$100 \cdot 0.0001 = 0.01$
b	$100 \cdot 0.0010 = 0.10$
c	$100 \cdot 0.0100 = 1.00$

Table 4. Transaction costs

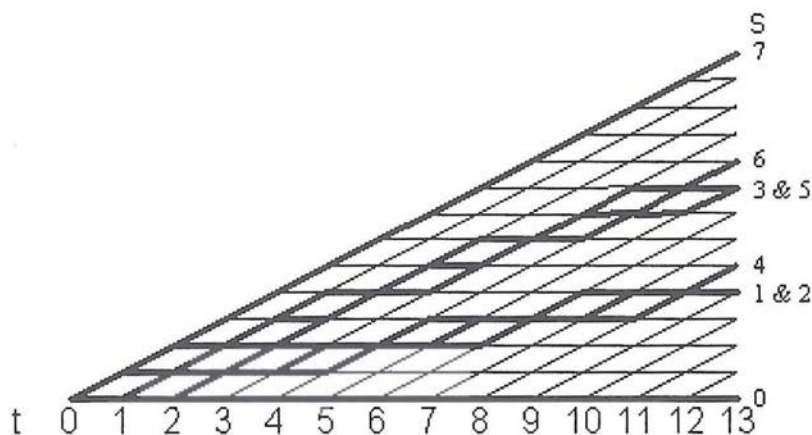


Figure 5. Scenario sample of case B6

3.1.3. Fair prices

Given the sample of interest rate scenarios, we used the following backward algorithm to compute fair prices of bonds:

1. Compute

$$B_{jt}^s \equiv B_{jt}(\mathbf{r}^s) = \sum_{\tau=t+1}^T f_{j\tau} \prod_{h=t}^{\tau-1} (1+r_h^s)^{-1}$$

for any bond $j = 1, \dots, J$, at any stage $t = 1, \dots, T$ and for each scenario $s = 1, \dots, S$.

2. For $\forall j, t, s$, if $\exists K_{jt}$ exercise price and $K_{jt} > B_{jt}^s$, then

- (a) compute

$$f_{j\tau}^s = \begin{cases} f_{j\tau} & \tau < t \\ K_{jt} & \tau = t \\ 0 & \tau > t \end{cases}$$

- (b) compute revised prices as follows:

$$B_{jt}^s = \frac{1}{1+r_t^s} (f_{j,t+1}^s + B_{j,t+1}^s)$$

for $j = 1, \dots, J$, $t = 1, \dots, T$ and $s = 1, \dots, S$.

3. Stop.

3.1.4. Utility functions

Most of utility functions used in finance are rather flat growing to infinity. This fact could cause some inconveniences when we are maximizing the final wealth. To avoid the dependence on magnitude of invested amount (initial wealth) we maximized the utility function of the final wealth standardized with respect to the initial one:

$$\Psi^s \equiv \Psi(W_{T_1}^s) = \frac{W_{T_1}^s}{W_0}$$

where W_0 is the initial wealth, i.e. the amount invested in the risk-free asset c and in the initial portfolio \mathbf{b} at the current price \mathbf{B}_0 :

$$W_0 \equiv c + \mathbf{b}^\top \mathbf{B}_0$$

We defined nine subcases for the following utility functions:

- case α : $U(\Psi^s) = \ln(2 + \Psi^s)$.
- case β : $U(\Psi^s) = \ln(\frac{1}{2} + \Psi^s)$.
- case γ : $U(\Psi^s) = -\exp(-\frac{\Psi^s}{2})$.
- case δ : $U(\Psi^s) = 2\sqrt{\Psi^s}$.
- case ϵ : $U(\Psi^s) = 10(\Psi^s)^{\frac{1}{10}}$.
- case ζ : $U(\Psi^s) = \Psi^s$.
- case η : $U(\Psi^s) = -10(\Psi^s)^{-\frac{1}{10}}$.
- case θ : $U(\Psi^s) = -\frac{2}{\sqrt{\Psi^s}}$.
- case ι : $U(\Psi^s) = \sqrt{2\Psi^s + 2}$.

3.2. Portfolio management

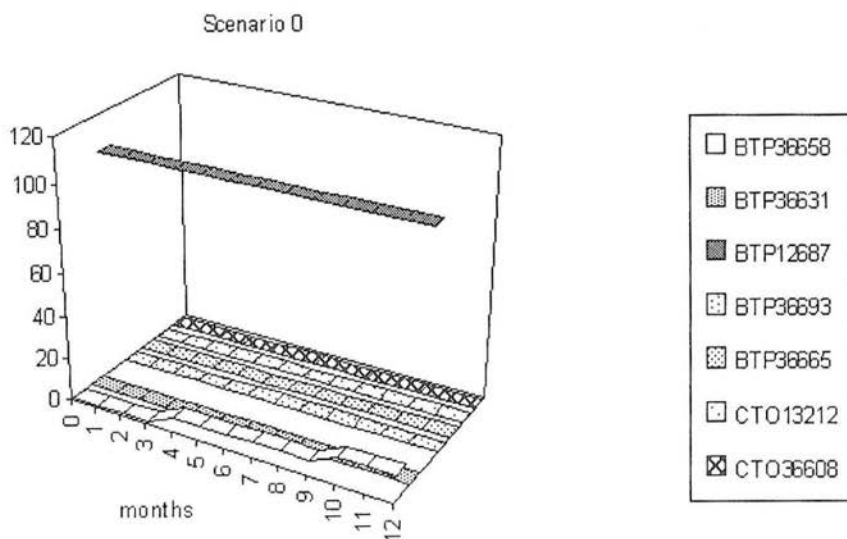
Considering the results of numerical solution of the problem (2)–(7), we comment on case B1a α . Table 5 shows that the (locally) optimal investment for scenario 0 consists of 104.58 units of BTP 12687 along the whole time horizon, whereas the payments of matured coupons (in $t = 4$, i.e. 1Jan95 and in $t = 10$, i.e. 1Jul95) are reinvested in bond BTP 36658. Note that even the coupon cash of this last bond (in $t = 7$, i.e. 1Apr95) is reinvested in itself. Fig. 6 summarizes that investment strategy.

When utility function has not been linear and concave, our model has been very stable. In fact, it often chose the same investment strategy along almost every scenario. When utility function is linear and the scenario sample includes extreme scenarios, i.e., all up and all down movements of interest rates (see Fig. 5), investment strategies change along different scenarios. Figs. 7 through 9 show how case B6a ζ leads to different strategies along different scenario. Finally, increasing in transaction costs forces a static behaviour of our model, i.e., the

CASE: /usr/users/marida/d5240/gams/B1
 BID: 0001 ASK: 9999 FUN: log RAF: 2
 optimal solution: 1.14
 scenario: 0
 final wealth: 11530.66 initial wealth: 10295.23 11.81%

time	BTP36658	BTP36631	BTP12687	BTP36693	BTP36665	CT013212	CT036608	yPLUS	yMINUS
0	0	0	104.58	0	0	0	0	0	0
1	0	0	104.58	0	0	0	0	0	0
2	0	0	104.58	0	0	0	0	0	0
3	0	0	104.58	0	0	0	0	0	0
4	5.6	0	104.58	0	0	0	0	0	0
5	5.6	0	104.58	0	0	0	0	0	0
6	5.6	0	104.58	0	0	0	0	0	0
7	5.83	0	104.58	0	0	0	0	0	0
8	5.83	0	104.58	0	0	0	0	0	0
9	5.83	0	104.58	0	0	0	0	0	0
10	11.38	0	104.58	0	0	0	0	0	0
11	11.38	0	104.58	0	0	0	0	0	0
12	11.38	0	104.58	0	0	0	0	0	0

Table 5. Optimal solution of case study B1 α on scenario 0



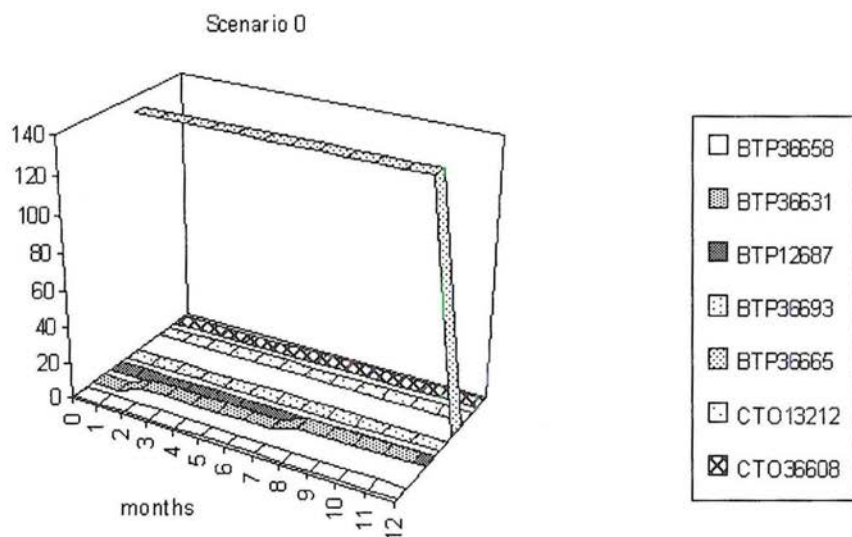
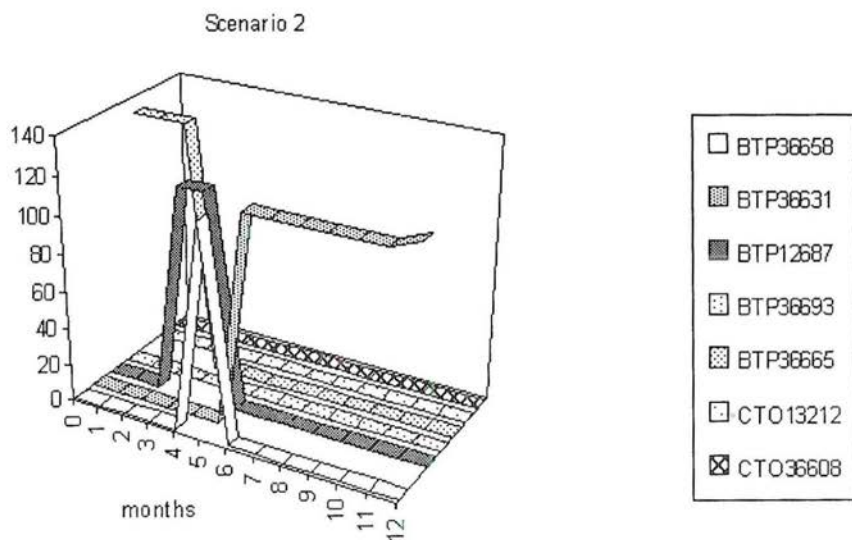


Figure 7. Optimal solution of case study B6aζ on scenario 0



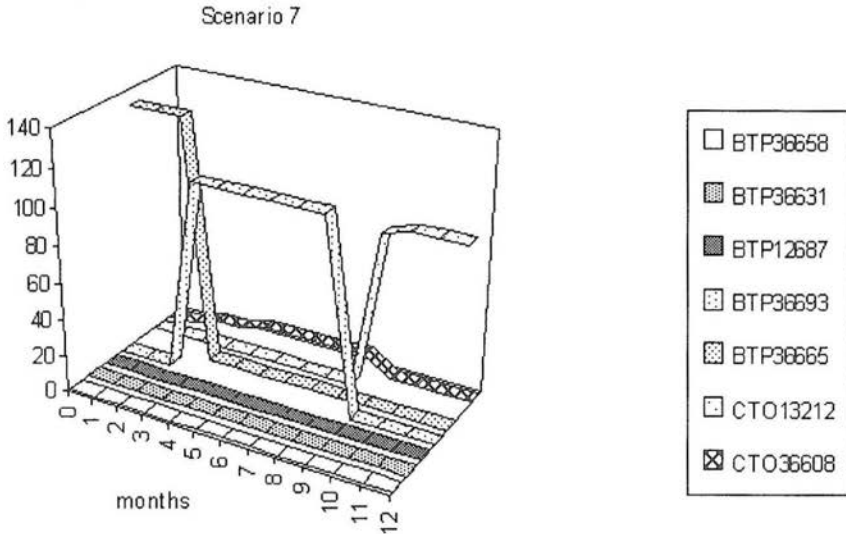


Figure 9. Optimal solution of case study B6a ζ on scenario 7

3.3. Sensitivity analysis

In the previous section equation (15) showed that the sensitivity of the optimal value of the objective function (6) with respect to sample of scenarios depends on signs of H_h^s . In particular, positive values of these H_h^s indicate positive effects on changes Δ_h^s of interest rate r_h^s . In other words, when interest increases, first derivative $\varphi'(0^+)$ is positive and, consequently, optimal value of objective function increases. We get opposite results for either negative H_h^s or negative changes of interest rate.¹ Finally, note that when Lagrangean multipliers κ^s and λ^s are zeros, our portfolio management model does not depend on interest rate changes.

Let us see, for example, what are H_h^s values for scenarios 0 and 3 of cases B4a. Figs. 10–13 show how much sensitivity changes with respect to utility function and scenario of interest rate. The diagrams show that, although sensitivity depends on utility function, the trend is the same. In fact, we have constant and negative values close to zero on interval (T_0, T_1) , whereas in $[T_1, T)$ we have an initial positive section, followed by an all-zero section. Since the model is stable, sensitivity analysis is stable too, with exception of linear utility function case. Actually, the linear case ζ in scenario 0 (Fig. 9) has a different trend in H_h^s with respect to scenario 3 (Fig. 11).

In all cases we studied, we found negative values of H_h^s in (T_0, T_1) and positive or zero in $[T_1, T)$.

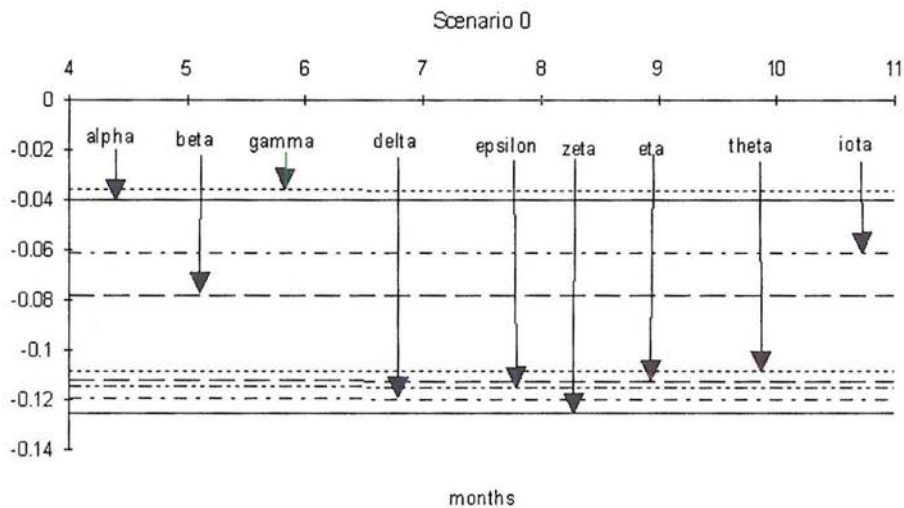
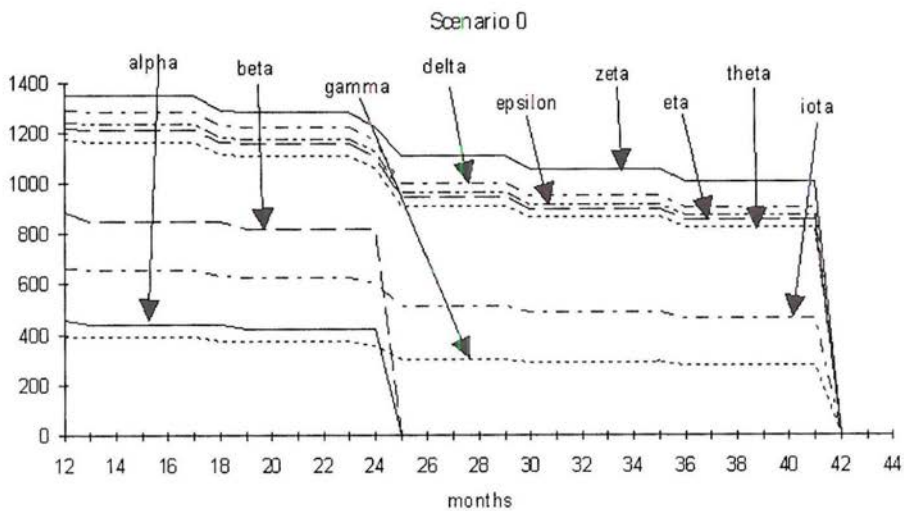
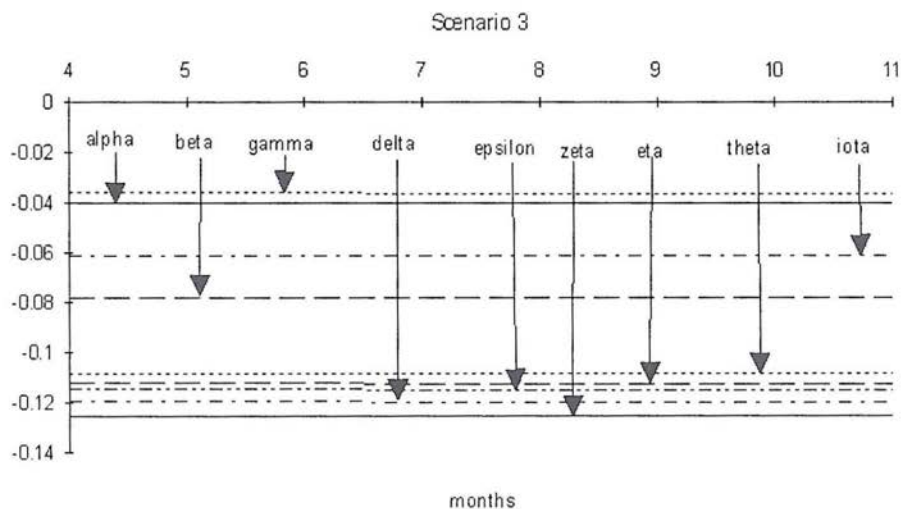
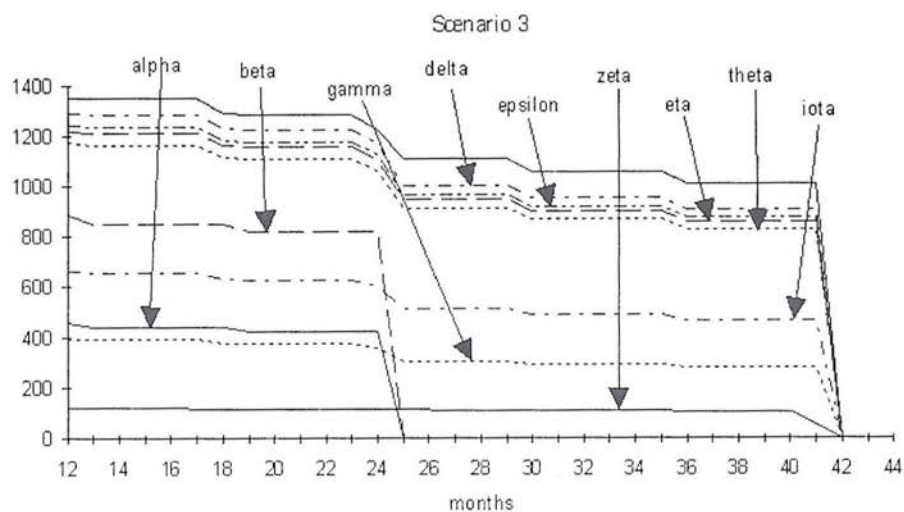


Figure 10. Sensitivity analysis of cases B4a in (T_0, T_1)



Figure 12. Sensitivity analysis of cases B4a in (T_0, T_1) 

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