Control and Cybernetics

vol. **29** (2000) No. 4

Optimality conditions and a classification of duality schemes in vector optimization

by

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Abstract: Vector minimization of a relation F valued in an ordered vector space under a constraint A consists in finding $x_0 \in A$, $w_0 \in Fx_0$ such that w_0 is minimal in FA. To a family of vector minimization problems $\text{minimize}_{x \in X} F(x, y)$, $y \in Y$, one associates a Lagrange relation $L(x, \xi, y_0) = \bigcup_{y \in Y}(F(x, y) - \xi(y) + \xi(y_0))$ where ξ belongs to an arbitrary class Ξ of mappings. For this type of problem, there exist several notions of solutions. Some useful characterizations of existential solutions are established and, consequently, some necessary conditions of optimality are derived. One result of intermediate duality is proved with the aid of the scalarization theory. Existence theorems for existential solutions are given and a comparison of several exact duality schemes is established, more precisely in the convex case it is shown that the majority of exact duality schemes can be obtained from one result of S. Dolecki and C. Malivert.

Keywords: vector optimization, minimality, existential solutions, duality.

1. Introduction

A basic notion of vector optimization is minimality (or maximality) with respect to a partial ordering. Let C be a convex cone of a real vector space W. Suppose that C is pointed (i.e. $C \cap (-C) = \{0\}$) so that the relation

$$w_0 \leq w_1 \Leftrightarrow w_1 \in w_0 + C \tag{1}$$

defines an order (strict or broad) on W. In vector optimization there exist several notions of optimality. All of them may be expressed in terms of minimality with respect to a properly chosen cone defined with the aid of C. An element a_0 of a subset A of W is a *minimal*[1] point of A (with respect to the order (1)),

[1] Such points are called "efficient" in vector optimization.

whenever

$$(a_0 - C_0) \cap A = \emptyset,$$

where $C_0 = C \setminus \{0\}$. We denote by $\min_C A$ the set of minimal points of A. It turns out to be very useful to represent this set as

$$\min_C A = A \cap \downarrow A, \tag{2}$$

where $\downarrow A = \{w : (w - C_0) \cap A = \emptyset\}$, Dolecki and Malivert (1988). In the main part of this paper, we shall be concerned with the family of vector minimization problems

$$\underset{x \in X}{\text{minimize}}\, F(x, y), \;\; y \in Y, \tag{3}$$

where F is a relation from $X \times Y$ to W ($F \subset (X \times Y) \times W$). Let τ be a topology on X, $\imath$ the discrete topology on Y and θ_C the cyrtology on W generated by the lower closure with respect to C_0. A family θ of subsets of a set W is called a *cyrtology*, if, for every subfamily $\mathcal{A}$ of θ, $\bigcap \mathcal{A} \in \theta$. An element $x_0 \in X$ is called a *τ-local existential solution* of (3) at y_0 if there exists a τ-neighborhood Q of x_0 such that $F(x_0, y_0) \cap \min_C F(Q, y_0) \neq \emptyset$, where $F(Q, y_0)$ stands for $\bigcup_{x \in Q} F(x, y_0)$. The *epigraph* of a relation $\Omega \subset Z \times W$ (with respect to C) is the set

$$\operatorname{epi} \Omega = \{(z, w) \in Z \times W : w \in \Omega z + C\}.$$

It is known (from Dolecki 1980A, B, and 1988), in the univocal and scalar case, that a point x_0 of X is a τ-local solution of (3) at y_0 if and only if there exists a τ-neighborhood Q of x_0 such that $(y_0, F(x_0, y_0)) \in \operatorname{Fr}_{\imath \times \nu}((\operatorname{epi} F)Q)$, where $\operatorname{Fr} A$ denotes the boundary of A and ν is the usual topology on the real line $\mathbb{R}$. We extend this result to the multivocal and vector case. We prove (Theorem 2.1) that x_0 is a τ-local existential solution of (3) at y_0 if and only if there exists a τ-neighborhood Q of x_0 such that

$$(\{y_0\} \times F(x_0, y_0)) \cap \operatorname{Fr}_{\imath \times \theta_C}((\operatorname{epi} F)Q) \neq \emptyset.$$

We give some sufficient conditions for the existence of a locally convex topology satisfying a similar characterization (Proposition 2.4). The latter, besides some conditions which guarantee the separation of Hahn-Banach, enables us to get necessary conditions of optimality (Proposition 3.1). We provide also some characterizations of those convex cones C with $\operatorname{int}_a C \neq \emptyset$ and for that (W, C) is a Riesz space, Corollary 2.4 and Corollary 2.5. The set $\operatorname{int}_a C$ denotes the algebraic interior of C.

In Section 4, we introduce a new duality concept, namely intermediate duality. The *marginal relation* Φ for (3) is a multifunction from Y to W defined by

$$\Phi y = F(X, y) = \bigcup_{x \in X} F(x, y). \tag{4}$$

Let Ξ be a family of mappings from Y to W. The (*definite*) *lagrangean* at y_0 (of F with respect to Ξ) is defined by

$$L(x, \xi, y_0) = \bigcup_{y \in Y} (F(x, y) - \xi(y) + \xi(y_0)), \tag{5}$$

where ξ belongs to Ξ. The (*definite*) *marginal lagrangean* at y_0 is defined by

$$\Lambda(\xi, y_0) = \bigcup_{y \in Y} (\Phi y - \xi(y) + \xi(y_0)), \tag{6}$$

where Φ is the marginal relation of (3). Observe that for each $\xi \in \Xi$,

$$\Lambda(\xi, y_0) = L(X, \xi, y_0) \text{ and } \Phi y_0 \subset \Lambda(\xi, y_0).$$

The duality approach consists in comparing the sets $\min_C \Phi y_0$ and $\min_C \Lambda(\xi, y_0)$ and similarly the sets $\inf_C \Phi y_0$ and $\inf_C \Lambda(\xi, y_0)$. In order to define a notion of infimum (resp. supremum), we need a notion of closure of a set with respect to order (1), introduced by S. Dolecki and C. Malivert (Dolecki, 1990; Dolecki and Malivert, 1988, 1993). We shall denote

$$\mathrm{cl}_C^+ A = \uparrow\downarrow A \text{ and } \mathrm{cl}_C^- A = \downarrow\uparrow A,$$

where $\uparrow A = \{w : (w + C_0) \cap A = \emptyset\}$. Accordingly, $w_0 \in \mathrm{cl}_C^+ A$ if and only if, for every $w \in w_0 + C_0$, $(w - C_0) \cap A \neq \emptyset$ and symmetrically for $\mathrm{cl}_C^- A$. If $A = \mathrm{cl}_C^+ A$ (resp. $A = \mathrm{cl}_C^- A$), then A is said to be *upper closed* (resp. *lower closed*). We define the *infimal* points of A as the minimal points of the upper closure of A:

$$\inf_C A = \min_C(\mathrm{cl}_C^+ A).$$

Analogously, $\sup_C A = \max_C(\mathrm{cl}_C^- A) = -\inf_C(-A)$ is the set of *supremal* points of A (where $\max_C A = A \cap \uparrow A$).

In the case of one dimensional space $W = \mathbb{R}$, each bounded set A has upper bound and $\sup A \in \mathrm{cl}_\nu A$, where $\mathrm{cl}_\tau A$ denote the closure of A with respect to the topology τ, thus $\sup A$ is a limit of a sequence of elements of A. Hence when, for instance, $\min \Phi y_0 = \sup_{\xi \in \Xi} \inf \Lambda(\xi, y_0)$ then there exists a sequence $(\xi_n) \subset \Xi$ such that $\min \Phi y_0 = \lim_n \inf \Lambda(\xi_n, y_0)$. We suggest here to extend this result to vector case. For that, we define different notions of Ξ-intermediate duality with the aid of the cyrtology θ_C and our contribution to this problem is the Theorem 4.1. Our approach is based on the theory of scalarization, on some results of convex analysis and some results of lower semicontinuity of intersection.

Finally, we shall pay special attention to exact duality. Using classical approaches such as lagrangean and conjugate approaches, several duality results for problems satisfying some constraint qualification and convexity assumptions were obtained (see for instance Nakayama, 1984; The Luc Dinh, 1989; Li and Wang, 1994; and Song, 1998, and the references therein) and also some results of Ξ-exact duality according to Dolecki and Malivert (1993), formulated in terms of vector Ξ-subdifferentiability. The Luc Dinh (1989) introduced the axiomatic duality. In particular, the lagrangean duality considered by S. Dolecki and C. Malivert (1993) is an axiomatic duality. The exact axiomatic duality results require assumptions for the existence of some optimal solutions or some constraint qualifications, like Slater type, Abadie type, Cottle type or Mangasarian-Fromovitz for linear problems, see Meada (1994). We obtain some results on the existence of existential solutions of (3), Proposition 5.1, generalizing in this way the result of The Luc Dinh (Corollary 5.1) and we derive also some existence results of weak (resp. proper) existential solutions of (3), Corollary 5.2. In the sequel we shall deal with a special form of families of vector minimization problems, namely vector mathematical programming

$$\operatorname*{minimize}_{\substack{x \in X \\ g(x)\cap(y-K)\neq\emptyset}} \quad f(x),\ y \in Y; \tag{7}$$

where $f \subset X \times W$, $g \subset X \times Y$ and K is a pointed convex cone of Y with nonempty interior. The aim of Proposition 5.2 is to explore the relationships between assumptions of Dolecki and Malivert (1993, Thm. 3.4) which guarantee existence and positivity of the lagrangean multiplier of the objective multifunction and constraint qualifications existing in the literature. Hence, the exact duality results of The Luc Dinh (1989, Thm. 3.6) and of H. Nakayama (1984, Thm. 2.1) follows directly from Proposition 5.2 and Theorem 3.4 of Dolecki and Malivert (1993).

2. Characterizations of existential solutions

Before we present the precise statements, let us recall some basic notions and definitions. A family θ of subsets of a set W is called a *cyrtology* if, for every subfamily $\mathcal{A}$ of θ, $\bigcap \mathcal{A} \in \theta$, where $\bigcap \mathcal{A} = \bigcap_{A\in\mathcal{A}} A$. Taking for $\mathcal{A}$ the empty family, it follows that W belongs to every cyrtology. The closure of a subset A of W, with respect to θ, is the set

$$\operatorname{cl}_\theta A = \bigcap\{F \in \theta : A \subset F\}.$$

The interior of A, with respect to θ, is defined by

$$\operatorname{int}_\theta A = (\operatorname{cl}_\theta A^c)^c,$$

where A^c stands for the complementary set of A in W, and the neighborhood system $\mathcal{N}_\theta$ is a map from W to 2^W (the set of all subsets of W) defined by

$$A \in \mathcal{N}_\theta(w) \Leftrightarrow w \in \operatorname{int}_\theta A.$$

For each $w \in W$, $\mathcal{N}_\theta(w)$ is a semifilter (a possibly empty family $\mathcal{A}$ such that $\emptyset \notin \mathcal{A}$ and $A \in \mathcal{A}$, $A \subset D$ imply $D \in \mathcal{A}$) and every element of $\mathcal{N}_\theta(w)$ contains w (Greco, 1985). A subset A of W is called *θ-closed* if $\operatorname{cl}_\theta A = A$. A subfamily φ of a cyrtology θ is an *intersectional base* for θ whenever, for every $A \subset W$,

$\mathrm{cl}_\theta A = \bigcap\{C \in \varphi : A \subset C\}$. A subfamily β of $\mathcal{N}_\theta(w)$ is called its *base* if, for each $Q \in \mathcal{N}_\theta(w)$, there exists $B \in \beta$ such that $B \subset Q$. A cyrtology θ is said to be *finer* than a cyrtology σ ($\sigma \leq \theta$), if every σ-closed set is θ-closed. In particular, every topology (identified with the family of its closed sets) is a cyrtology. The topologies are precisely those cyrtologies that contain the empty set and are stable for finite unions. Let $(W, \leq)$ be an ordered set, i.e. a set W endowed with a binary relation $\leq$ which is reflexive, anti-symmetric and transitive. An element $w_0 \in W$ is said to be a *majorant* (resp. *minorant*) of A if $a \leq w_0$ (resp. $a \geq w_0$), whenever $a \in A$. If w_0 is a majorant (resp. minorant) of A such that $w_0 \leq z$ (resp. $w_0 \geq z$) for any majorant (resp. minorant) z of A then w_0 is unique and called the *supremum* (resp. *infimum*) of A; the notation is $w_0 = \vee A$ (resp. $w_0 = \wedge A$). *Order intervals* are defined by

$$[x, y] = \{z \in W : x \leq z \leq y\} \text{ where } x, y \in W \text{ and } x \leq y.$$

A subset A of W is called *order bounded* if A is contained in some order interval. A *vector lattice* (or *Riesz space*) is an ordered vector space in which $x \vee y$ and $x \wedge y$ always exist, see Schaefer (1971) and Luxemburg and Zaanen (1971). It is well know from Schaefer (1971, Chap. V, 1.2) that if W is a real vector space ordered by a generating convex cone C such that, for each pair $(x, y) \in C \times C$, either $x \vee y$ or $x \wedge y$ exists, then (W, C) is a Riesz space. We say that a cone C of W is *generating* if $W = C - C$.

Let W be a real vector space ordered by a convex cone C and denote by $\mathcal{N}_C^-(w_0)$ the neighborhood system of w_0 generated by the lower closure cl_C^- with respect to C_0. Then the family $\mathcal{B}_{w_0} = \{w + C_0\}_{w \in w_0 - C_0}$ constitutes a base of $\mathcal{N}_C^-(w_0)$, see Dolecki (1990) and Dolecki and Malivert (1988). The topic of the following proposition is to answer the question under which assumptions the equality $\mathrm{T}^+\theta_C = \theta_C$ holds, where $\mathrm{T}^+\theta_C$ denote the *upper topology* of θ_C; that is – the coarsest topology which is finer than θ_C.

PROPOSITION 2.1 *Let W be a real vector space ordered by a generating pointed convex cone C which contains the origin and θ_C is the cyrtology on W generated by the lower closure with respect to C_0. Then, the following conditions are equivalent:*

(i) $T^+\theta_C = \theta_C$,

(ii) *(W, C) is a Riesz space.*

Proof. (i) $\Rightarrow$ (ii). Let $(c_1, c_2) \in C \times C$.

• If $c_1 = 0$ or $c_2 = 0$, then c_1 and c_2 are comparable and therefore $c_1 \vee c_2$ and $c_1 \wedge c_2$ exist.

• Suppose that $(c_1, c_2) \in C_0 \times C_0$. In view of (i), $\mathcal{B}_0 = \{-c + C_0\}_{c \in C_0}$ is a neighborhood base of 0 for the topology $\mathrm{T}^+\theta_C$. It follows that

$$(-c_1 + C_0) \cap (-c_2 + C_0) = -c + C_0 \text{ for some } c \in C_0.$$

Thus, $c = c_1 \wedge c_2$ exists and consequently $c_1 \vee c_2$ exists.

(ii) $\Rightarrow$ (i). Observe first that the inequality $\theta_C \leq \mathrm{T}^+\theta_C$ is always true. Now, let $w_0 \in W$. We need only prove that the family $\mathcal{B}_{w_0} = \{w_0 - c + C_0\}_{c \in C_0}$, which is a neighborhood base of w_0 for θ_C, is stable for finite unions. Let, for that, $c_1, c_2, \ldots, c_n \in C_0$. As (W, C) is a Riesz space, then

$$\bigcup_{i=1}^{n}(w_0 - c_i + C_0) = \bigwedge_{1 \leq i \leq n} (w_0 - c_i) + C_0 = w_0 - \bigvee_{1 \leq i \leq n} c_i + C_0.$$

It follows, since $\bigvee_{1 \leq i \leq n} c_i \in C_0$, that $\bigcup_{i=1}^{n}(w_0 - c_i + C_0) \in \mathcal{B}$. ∎

Consider now the family of vector minimization problems (3). The epigraph of the relation F (with respect to C) is the set

$$\operatorname{epi} F = \{(x, y, w) \in X \times Y \times W : w \in F(x, y) + C\}.$$

We shall look here at $\operatorname{epi} F \subset X \times Y \times W$ as a relation from X to $Y \times W$ and we shall give some useful characterizations of existential solutions.

THEOREM 2.1 *Let θ_C be the cyrtology on W generated by the lower closure with respect to C_0. If C is pointed, then an element x_0 of X is a τ-local existential solution of (3) at y_0 if and only if there exists a τ-neighborhood Q of x_0 such that*

$$(\{y_0\} \times F(x_0, y_0)) \cap \operatorname{Fr}_{\imath \times \theta_C}((\operatorname{epi} F)Q) \neq \emptyset.$$

Proof. Let Q be a τ-neighborhood of x_0 and $w_0 \in F(x_0, y_0)$ be such that $(y_0, w_0) \in \operatorname{Fr}_{\imath \times \theta_C}((\operatorname{epi} F)Q)$. Consequently, for any $w \in w_0 - C_0$, $\{y_0\} \times (w + C) \not\subset (\operatorname{epi} F)Q$. Since $(\operatorname{epi} F)Q$ has the property:

$$(y, r_0) \in (\operatorname{epi} F)Q \text{ and } r_1 \in r_0 + C \Rightarrow (y, r_1) \in (\operatorname{epi} F)Q,$$

the condition above amounts to

$$\forall_{w \in w_0 - C_0} (y_0, w) \notin (\operatorname{epi} F)Q.$$

Therefore, $(F(Q, y_0) + C) \cap (w_0 - C_0) = \emptyset$ which is equivalent to showing that $w_0 \in \min_C F(Q, y_0)$, seeing that C is pointed.

Conversely, let Q be a τ-neighborhood of x_0 such that $F(x_0, y_0) \cap \min_C F(Q, y_0)$ contains at least one element w_0. Then, $(F(Q, y_0) + C) \cap (w_0 - C_0) = \emptyset$. Now we will show that $(y_0, w_0) \in \operatorname{Fr}_{\imath \times \theta_C}((\operatorname{epi} F)Q)$. Let $w \in w_0 - C_0$. Since $w_0 \in \operatorname{cl}_{\theta_C}(w_0 - C_0)$ then $(w + C_0) \cap (w_0 - C_0) \neq \emptyset$ and let $z \in (w + C_0) \cap (w_0 - C_0)$. One has thus $(y_0, z) \in \{y_0\} \times (w + C_0)$ and $(y_0, z) \notin (\operatorname{epi} F)Q$, hence $(y_0, w_0) \notin \operatorname{int}_{\imath \times \theta_C}((\operatorname{epi} F)Q)$. On the other hand, $(y_0, w_0) \in (\operatorname{epi} F)x_0 \subset (\operatorname{epi} F)Q$. It follows that $(y_0, w_0) \in (\{y_0\} \times F(x_0, y_0)) \cap \operatorname{Fr}_{\imath \times \theta_C}((\operatorname{epi} F)Q)$. ∎

Since a *global existential solution* of (3) at y_0 amount to a o-local existential solution of (3) at y_0, where o designates the trivial topology on X, we have

COROLLARY 2.1 *Let θ_C be the cyrtology on W generated by the lower closure with respect to C_0. If C is pointed, then an element x_0 of X is a global existential solution of (3) at y_0 if and only if $(\{y_0\}\times F(x_0,y_0))\cap \mathrm{Fr}_{\imath\times\theta_C}((\mathrm{epi}\,F)X)\neq\emptyset$.*

COROLLARY 2.2 *Let ζ be an arbitrary topology on Y and θ_C be the cyrtology on W generated by the lower closure with respect to C_0. If C is pointed and x_0 is a τ-local existential solution of (3) at y_0, then there exists a τ-neighborhood Q of x_0 such that*

$$(\{y_0\}\times F(x_0,y_0))\cap \mathrm{Fr}_{\zeta\times\theta_C}((\mathrm{epi}\,F)Q)\neq\emptyset.$$

Proof. Since $\zeta\times\theta_C\leq \imath\times\theta_C$ then $\mathrm{Fr}_{\imath\times\theta_C}\subset\mathrm{Fr}_{\zeta\times\theta_C}$. ■

COROLLARY 2.3 *Let θ_C be the cyrtology on W generated by the lower closure with respect to C_0. If C is pointed and if there exists a τ-neighborhood Q of x_0 such that $(\{y_0\}\times F(x_0,y_0))\cap \mathrm{Fr}_{\imath\times\mathrm{T}^{+}\theta_C}((\mathrm{epi}\,F)Q)\neq\emptyset$, then x_0 is a τ-local existential solution of (3) at y_0.*

Proof. The result stem from $\imath\times\theta_C\leq\imath\times\mathrm{T}^{+}\theta_C$ and Theorem 2.1. ■

We now are interested in conditions that assure the existence of a minimal topology σ satisfying the equivalence; an element x_0 of X is a τ-local existential solution of (3) at y_0 if and only if there exists a τ-neighborhood Q of x_0 such that $(\{y_0\}\times F(x_0,y_0))\cap\mathrm{Fr}_{\imath\times\sigma}((\mathrm{epi}\,F)Q)\neq\emptyset$. Proposition 2.2 states that, in the case of Riesz spaces, there exists a family consisting of order intervals and generating a locally convex topology. It turns out that such topology, which we denote by σ_C, satisfies the above characterization for local existential solutions of (3) with respect to the cone $\widehat{C_a}=\mathrm{int}_a\,C\cup\{0\}$, where $\mathrm{int}_a\,C$ designate the algebraic interior of C. We recall that the *algebraic interior* of a subset A of W is the set

$$\mathrm{int}_a\,A=\{a\in A:\ \underset{w\in W}{\forall}\ \underset{\lambda_0>0}{\exists}\ \underset{\lambda\in[0,\lambda_0]}{\forall}\ a+\lambda w\in A\}. \tag{8}$$

An element x_0 of X is called a *τ-local existential weak solution* of (3) at y_0 if it is a τ-local existential solution of (3) at y_0 with respect to the cone $\widehat{C_a}$. Denote by $\widehat{\theta_C}$ the cyrtology on W generated by the lower closure $\mathrm{cl}^{-}_{\widehat{C_a}}$.

For the promised existence result of the topology σ_C we need the following notions and definitions. A family $\mathcal{B}$ of subsets of a set W is a *filter base* if $\mathcal{B}\neq\emptyset$ and $\emptyset\notin\mathcal{B}$, and if $B_1\in\mathcal{B}$ and $B_2\in\mathcal{B}$ there exists $B_3\in\mathcal{B}$ such that $B_3\subset B_1\cap B_2$. Let now A and B be subsets of a real vector space W. We say that A *absorbs* B if there exists $\lambda_0>0$ such that $B\subset\lambda A$ whenever $\lambda\geq\lambda_0$. A subset A of W is called *radial* if A absorbs every finite subset of W. A subset A of W is *circled* if $\lambda A\subset A$ whenever $\lambda\in[-1,+1]$. Recall that if A is a nonempty subset of W, then

1) A is convex $\Rightarrow$ ($\text{int}_a A$ is convex and $\text{int}_a(\text{int}_a A) = \text{int}_a A$).
2) $0 \in \text{int}_a A \Leftrightarrow$ (A absorbs $\{w\}$ whenever $w \in W$).
3) A is radial $\Leftrightarrow 0 \in \text{int}_a A$.
4) $0 \in \text{int}_a A \Longrightarrow \text{cone}(A) = W$,

where $\text{cone}(A)$ denotes the cone generated by the set A, i.e. $\text{cone}(A) = \bigcup_{\lambda \geq 0} \lambda A$. Suppose, now, that W is a real vector space ordered by a convex cone C. In this case, the order interval between x and y (with respect to C) can be written as $[x, y] = (x + C) \cap (y - C)$. Every order interval is convex, and every order interval of the form $[-x, x]$ is circled. An element $e \in W$ such that $[-e, e]$ is radial is called an *order unit* of W.

REMARK 2.1 *If* $\text{int}_a C \neq \emptyset$, *then every element of* $\text{int}_a C$ *is an order unit of* W *and* C *is generating in* W.

Indeed, let $e \in \text{int}_a C$ then $0 \in (-e + \text{int}_a C) \cap (e - \text{int}_a C) = \text{int}_a[-e, e]$.

The following proposition, asserting the existence of locally convex topology on every Riesz space W ordered by a convex cone C such that $\text{int}_a C \neq \emptyset$.

PROPOSITION 2.2 *Let* W *be a Riesz space ordered by a convex cone* C *such that* $\text{int}_a C \neq \emptyset$. *Then, the family* $\mathcal{B} = \{\text{int}_a[-e, e] : e \in \text{int}_a C\}$ *is a neighborhood base of* 0 *for a unique locally convex topology* σ_C *such that* (W, σ_C) *is a real locally convex space.*

Proof. Observe, initially, that $\text{int}_a[-e, e] = (-e + \text{int}_a C) \cap (e - \text{int}_a C)$. We verify the assumptions of theorem from Schaefer (1971, Chap. I, 1.2).

– First, we show that the family $\mathcal{B}$ is a filter base. Since $0 \in \text{int}_a[-e, e]$ for any $e \in \text{int}_a C$, $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$. Let, now, e_1 and e_2 be two arbitrary elements of $\text{int}_a C$ and put $e = e_1 \wedge e_2$. We check, at first, that $e \in \text{int}_a C$. For that, let $w \in W$, then there exist $\lambda_1, \lambda_2 > 0$ such that, for each $i = 1, 2$, we have

$$e_i + \lambda w \in C \text{ for all } \lambda \in [0, \lambda_i].$$

If we pick $\lambda_0 = \min(\lambda_1, \lambda_2) > 0$, then for any $i \in \{1, 2\}$ and any $\lambda \in [0, \lambda_0]$ we obtain $e_i + \lambda w \in C$. Hence, we get for any $\lambda \in [0, \lambda_0]$, $e + \lambda w \in C$. On the other hand, $[-e, e] \subset [-e_1, e_1] \cap [-e_2, e_2]$ and consequently $\text{int}_a[-e, e] \subset \text{int}_a[-e_1, e_1] \cap \text{int}_a[-e_2, e_2]$.

– Now, take any $e \in \text{int}_a C$. It is obvious that $e/2 \in \text{int}_a C$ and since $[-e/2, e/2] + [-e/2, e/2] \subset [-e, e]$, we conclude that $\text{int}_a[-e/2, e/2] + \text{int}_a[-e/2, e/2] \subset \text{int}_a[-e, e]$.

– Next, let $e \in \text{int}_a C$. As $[-e, e]$ is convex, the set $\text{int}_a[-e, e]$ is convex and $\text{int}_a(\text{int}_a[-e, e]) = \text{int}_a[-e, e]$ so it contains 0 and therefore $\text{int}_a[-e, e]$ is radial. Besides, $[-e, e]$ is circled so it's the same for $\text{int}_a[-e, e]$. Thus, we obtain the desired result with theorem from Schaefer (1971, Chap. I, 1.2). ■

The following proposition explores the relationships between the cyrtology $\widehat{\theta_C}$ and the topology σ_C.

PROPOSITION 2.3 *Let W be a Riesz space ordered by a convex cone C such that $\mathrm{int}_a C \neq \emptyset$. Then, $\widehat{\theta_C} \leq \sigma_C$ and consequently*

$$\sigma_C = \mathrm{T}^+\widehat{\theta_C} \vee \mathrm{T}^+\widehat{\theta_{(-C)}} = \widehat{\theta_C} \vee \widehat{\theta_{(-C)}}.$$

Proof. In this setting, we notice that the cone C is pointed and generating. First, we shall show that $\widehat{\theta_C} \leq \sigma_C$. To see this, let $A \subset W$ and $w_0 \in \mathrm{cl}_{\sigma_C} A$. Before, note that $w_0 + \mathrm{int}_a[-e, e] = \mathrm{int}_a[w_0 - e, w_0 + e]$ and therefore the family $\{\mathrm{int}_a[w_0 - e, w_0 + e] : e \in \mathrm{int}_a C\}$ is a neighborhood base of w_0 for σ_C. Take now $e \in \mathrm{int}_a C$, then by hypothesis

$$\emptyset \neq \mathrm{int}_a[w_0 - e, w_0 + e] \cap A \subset (w_0 - e + \mathrm{int}_a C) \cap A.$$

Hence, $w_0 \in \mathrm{cl}_{\widehat{\theta_C}} A$ which prove that every $\widehat{\theta_C}$-closed set is σ_C-closed.

Next, it suffices to prove that $\sigma_C \leq \widehat{\theta_C} \vee \widehat{\theta_{(-C)}}$ because, in view of the Proposition 2.1 and the first part of this proof, we have $\mathrm{T}^+\widehat{\theta_C} = \widehat{\theta_C}$, $\mathrm{T}^+\widehat{\theta_{(-C)}} = \widehat{\theta_{(-C)}}$ and $\widehat{\theta_C} \vee \widehat{\theta_{(-C)}} \leq \sigma_C$. From Proposition 2.1, we deduce that $\widehat{\theta_C}$ and $\widehat{\theta_{(-C)}}$ are topologies. Besides, by definition, $\mathcal{B}_1 = \{-e + \mathrm{int}_a C\}_{e \in \mathrm{int}_a C}$ (resp. $\mathcal{B}_2 = \{e - \mathrm{int}_a C\}_{e \in \mathrm{int}_a C}$) is a neighborhood base of 0 for $\widehat{\theta_C}$ (resp. $\widehat{\theta_{(-C)}}$). It follows that the family $\mathcal{B} = \{\mathrm{int}_a[-e, e] : e \in \mathrm{int}_a C\}$ is a neighborhood base of 0 for the topology $\widehat{\theta_C} \vee \widehat{\theta_{(-C)}}$ and consequently $\sigma_C = \widehat{\theta_C} \vee \widehat{\theta_{(-C)}}$. ■

REMARK 2.2 *The equality $\mathrm{T}^+\widehat{\theta_C} = \sigma_C$ does not always hold because the topology σ_C is separated whenever the cone C is pointed but (in general) that is not the case for the topology $\mathrm{T}^+\widehat{\theta_C}$.*

Using the topology σ_C, we get the following characterization of local existential weak solutions of (3).

PROPOSITION 2.4 *A point x_0 of X is a τ-local existential weak solution of (3) at y_0 if and only if there exists a τ-neighborhood Q of x_0 such that*

$$(\{y_0\} \times F(x_0, y_0)) \cap \mathrm{Fr}_{\imath \times \sigma_C}((\mathrm{epi}\, F)Q) \neq \emptyset.$$

Proof. $\Leftarrow$) It results from $\imath \times \widehat{\theta_C} \leq \imath \times \sigma_C$ and Theorem 2.1.

$\Rightarrow$) Let Q be a τ-neighborhood of x_0 such that $F(x_0, y_0) \cap \min_{\widehat{C_a}} F(Q, y_0)$ contains at least one element w_0. Then, $(F(Q, y_0) + C) \cap (w_0 - \mathrm{int}_a C) = \emptyset$. Now we prove that $(y_0, w_0) \in \mathrm{Fr}_{\imath \times \sigma_C}((\mathrm{epi}\, F)Q)$. Let $e \in \mathrm{int}_a C$. Since $w_0 \in \mathrm{cl}_{\widehat{\theta_C}}(w_0 - \mathrm{int}_a C)$ then $(w_0 - e + \mathrm{int}_a C) \cap (w_0 - \mathrm{int}_a C) \neq \emptyset$ and let $z \in (w_0 - e + \mathrm{int}_a C) \cap (w_0 - \mathrm{int}_a C)$. On the other hand, $w_0 - \mathrm{int}_a C \subset w_0 + e - \mathrm{int}_a C$ it ensues that

$$(y_0, z) \in \{y_0\} \times \mathrm{int}_a[w_0 - e, w_0 + e] \text{ and } (y_0, z) \notin (\mathrm{epi}\, F)Q$$

proving that $(y_0, w_0) \notin \mathrm{int}_{\imath \times \sigma_C}((\mathrm{epi}\, F)Q)$. Moreover, $(y_0, w_0) \in (\mathrm{epi}\, F)x_0 \subset (\mathrm{epi}\, F)Q$ and therefore $(y_0, w_0) \in (\{y_0\} \times F(x_0, y_0)) \cap \mathrm{Fr}_{\imath \times \sigma_C}((\mathrm{epi}\, F)Q)$. ■

REMARK 2.3 *A point x_0 of X is a τ-local existential weak solution of (3) at y_0 if and only if there exists $w_0 \in F(x_0, y_0)$ such that (x_0, y_0, w_0) is a $(\tau, \imath \times \sigma_C)$-*singular *point of* epi F *that is not a $(\tau, \imath \times \sigma_C)$-regular point of* epi F. *We say that a point (x_0, y_0) is a* regular *point of a relation $\Omega \subset X \times Y$ whenever the inverse relation Ω^- ($\Omega^- = \{(y, x) : (x, y) \in \Omega\}$) is lower semicontinuous at (y_0, x_0). A relation $\Omega \subset X \times Y$ is called* lower semicontinuous *(l.s.c.) at (x_0, y_0) if, for each neighborhood V of y_0, $\Omega^- V$ is a neighborhood of x_0.*

At this stage, it may be helpful to characterize the convex cones C of W satisfying $\text{int}_a C \neq \emptyset$ and for which W is a Riesz space. In the case of locally convex separated spaces, G. Choquet (1969) provided some examples with the aid of the simplex theory. Let B be a compact convex subset of W and putting $\tilde{B} = \text{cone}(B \times \{1\})$ that is the cone generated by the set $B \times \{1\}$ in $W \times \mathbb{R}$. We say that B is a *simplex* if the vector space $\tilde{B} - \tilde{B}$ ordered by $\tilde{B}$ is a Riesz space. Recall that a *base* of a convex cone C is a convex subset B of C such that $C = \text{cone}(B)$ and $0 \notin \text{cl}\, B$. Let C and D be two convex cones. We say that C is *isomorphic* to D if there exists a one-to-one mapping $\varphi : C \to D$ such that $\varphi(\lambda c) = \lambda \varphi(c)$ whenever $\lambda > 0$, $\varphi(c_1 + c_2) = \varphi(c_1) + \varphi(c_2)$ and $c_2 - c_1 \in C \Leftrightarrow \varphi(c_2) - \varphi(c_1) \in D$.

Let us quote the following result of G. Choquet:

PROPOSITION 2.5 (Choquet, 1969, Chap. 6, § 28.3) *Let W be a real locally convex separated space and $C \subset W$ a convex cone with a compact base B. Then, C_0 is isomorphic to $\tilde{B}$. Especially, the vector space $C - C$ is a Riesz space if and only if B is a simplex.*

COROLLARY 2.4 *Under the assumptions of Proposition 2.5, if* $\text{int}_a C \neq \emptyset$, *then W is a Riesz space if and only if B is a simplex.*

Proof. $\Rightarrow$) Since W is a Riesz space, then C is generating and further, in view of Proposition 2.5, B is a simplex.

$\Leftarrow$) Since $\text{int}_a C \neq \emptyset$ then C is generating. ■

It is worthy of note that, in finite dimensional space, for instance on $\mathbb{R}^n$, a simplex has a more simple and concrete geometric property. A *simplex* S of $\mathbb{R}^n$ is the convex hull of $(n+1)$ points $a_0, a_1, \ldots, a_n$ such that $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ are linearly independent. The points a_i are called the vertices of S. For example, a subset S of $\mathbb{R}^2$ is a simplex if and only if it is a triangle. A simplex of $\mathbb{R}^n$ has always a nonempty interior (Ioffe and Tihomirov, 1979, Chap. 3, § 5.2). A closed *half space* of W is a set of the form $H = \{w \in W : l(w) \geq 0\}$, for some $l \in W^*$ (where W^* is the topological dual space of W). A *polyhedral cone* of $\mathbb{R}^n$ is the intersection of a finite number of closed half spaces. A subset A of W is called an *affine subspace* of W if $A = a + L$, where $a \in W$ and L is a vector subspace of W. The *dimension* of A is, by definition, equal to the dimension of L. The set $\text{aff}(A)$ denotes the affine subspace generated by A that is the intersection of all affine subspaces of W that contains A and the dimension of A is equal to the dimension of $\text{aff}(A)$.

A characterization of Riesz spaces, in finite dimensional case, is given by

COROLLARY 2.5 *Suppose that $\mathbb{R}^n$ is ordered by a convex cone C with a closed bounded base B. Then, $\mathbb{R}^n$ is a Riesz space if and only if B is a simplex if and only if C is a polyhedral cone of dimension n.*

At the end of this section, let us consider the special form of families of vector minimization problems

$$\underset{x \in \Gamma y}{\text{minimize}}\, f(x), \; y \in Y; \tag{9}$$

where $f \subset X \times W$ and $\Gamma \subset Y \times X$. Formula (9) constitutes a special case of (3) with

$$F(x,y) = \begin{cases} f(x) & \text{if } x \in \Gamma y \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear, in this case, that

$$\text{epi}\, F = \{(x,y,w) \in X \times Y \times W : x \in \Gamma y \text{ and } w \in f(x) + C\}.$$

In other words, $(\text{epi}\, F)x = \Gamma^{-}x \times (f(x) + C)$ and therefore $(\text{epi}\, F)X = (\text{epi}\, f)\Gamma$, the composition of the relations Γ and f. In the general setting (3), the relation $\text{epi}\, F$ is intimately related to the marginal relation Φ of (3) by $(\text{epi}\, F)X = \Phi + \{0\} \times C$.

3. Optimality conditions

Suppose that (Y, ζ) is a real locally convex separated space and W is a Riesz space with order corresponding to a convex cone C such that $\text{int}_a\, C \neq \emptyset$. Consider the family of vector minimization problems (3) and the class Ξ of (ζ, σ_C)-continuous linear operators from Y to W (i.e. $L(Y,W)$). We denote by C^* the topological dual cone of C in W^*, that is

$$C^* = \{\theta \in W^* : \theta(c) \geq 0 \text{ for all } c \in C\}.$$

Using the topology σ_C, the Proposition 2.4 and the fact that $\zeta \times \sigma_C \leq \imath \times \sigma_C$ we can get the following Ξ-exact duality result (necessary optimality conditions) which is valid for any convex cone C such that $\text{int}_a\, C \neq \emptyset$ and not necessarily open.

PROPOSITION 3.1 *Suppose that $(\text{epi}\, F)X$ is convex and $\{y_0\} \times (\Phi y_0 + C) \cap \text{int}_{\zeta \times \sigma_C}((\text{epi}\, F)X) \neq \emptyset$. If an element x_0 of X is a global existential weak solution of (3) at y_0, then there exists $\xi_0 \in \Xi$ such that x_0 is a global existential weak solution of* $\text{minimize}_{x \in X}\, L(x, \xi_0, y_0)$.

Proof. We adopt a similar proof to that of Theorem 3.4 of Dolecki and Malivert (1993). Let $w_0 \in F(x_0, y_0)$ be such that $(y_0, w_0) \in \mathrm{Fr}_{\zeta\times\sigma_C}((\mathrm{epi}\, F)X)$, then $(y_0, w_0) \notin \mathrm{int}_{\zeta\times\sigma_C}((\mathrm{epi}\, F)X)$. By virtue of the Hahn-Banach theorem, there exist $\lambda \in Y^*$, $\mu \in W^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\mu(w - w_0) + \lambda(y - y_0) \geq 0 \text{ for all } (y, w) \in (\mathrm{epi}\, F)X, \text{ and} \tag{10}$$

$$\mu(w - w_0) + \lambda(y - y_0) > 0 \text{ for all } (y, w) \in \mathrm{int}_{\zeta\times\sigma_C}((\mathrm{epi}\, F)X). \tag{11}$$

By putting $y = y_0$ in (10) and since $w_0 \in \Phi y_0$, we get

$$\mu(c) = \mu(w_0 + c - w_0) \geq 0 \text{ for all } c \in C.$$

On the other hand, there exists $w \in \Phi y_0 + C$ with $(y_0, w) \in \mathrm{int}_{\zeta\times\sigma_C}((\mathrm{epi}\, F)X)$. Thus, by (11), we have $\mu(w - w_0) > 0$ and consequently $\mu \in C^* \setminus \{0\}$. Therefore we can choose $e \in \mathrm{int}_a C$ such that $\mu(e) = 1$ and define $\xi_0 \in \Xi$ by $\xi_0(y) = -\lambda(y)e$. Note that $w_0 \in F(x_0, y_0) \subset L(x_0, \xi_0, y_0) \subset \Lambda(\xi_0, y_0)$. It remains to prove that

$$\Lambda(\xi_0, y_0) \cap (w_0 - \mathrm{int}_a C) = \emptyset.$$

If not, there exist $y_1 \in Y$ and $w_1 \in \Phi y_1$ such that $w_1 - \xi_0(y_1) + \xi_0(y_0) \in w_0 - \mathrm{int}_a C$. Since $\mu(e) = 1$, $\mu \in C^* \setminus \{0\}$ and $\mathrm{int}_a C \subset \{w \in W : \theta(w) > 0$ for all $\theta \in C^* \setminus \{0\}\}$, we get $\mu(w_1 - w_0) + \lambda(y_1 - y_0) < 0$ which contradicts (10) and hence $w_0 \in \min_{\widehat{C_a}} \Lambda(\xi_0, y_0)$. ■

4. Intermediate duality

Consider the family of vector minimization problems (3). When the Ξ-exact duality lacks at a point y_0 of Y, then it may be useful to look for the existence of a filter $\mathcal{F}$ on Ξ such that every element of $\min_C \Phi y_0$ may be approximated arbitrarily well by some elements of $\{\min_C \Lambda(\xi, y_0)\}_{\mathcal{F}}$ (resp. $\{\inf_C \Lambda(\xi, y_0)\}_{\mathcal{F}}$, $\{\downarrow\Lambda(\xi, y_0)\}_{\mathcal{F}}$). Another interesting question is under which assumptions, for each element w_0 of $\min_C \Phi y_0$, there exists a filter $\mathcal{F}_{w_0}$ on Ξ such that w_0 may be approximated arbitrarily well by some elements of $\{\min_C \Lambda(\xi, y_0)\}_{\mathcal{F}_{w_0}}$ (resp. $\{\inf_C \Lambda(\xi, y_0)\}_{\mathcal{F}_{w_0}}$, $\{\downarrow\Lambda(\xi, y_0)\}_{\mathcal{F}_{w_0}}$). We underline the fact that the intermediate duality we consider is formulated in terms of the cyrtology θ_C and of an arbitrary class Ξ of mappings from Y to W. Let $\mathcal{F}$ be a filter on Ξ. A relation $\Delta \subset \Xi \times W$ *θ_C-adheres* (with respect to $\mathcal{F}$) to a subset A of W if, for each $w_0 \notin A$, there exist $w \in w_0 - C_0$ and $F \in \mathcal{F}$ such that $(w + C_0) \cap \Delta F = \emptyset$, where $\Delta F = \bigcup_{\xi\in F} \Delta\xi$. We denote by $\mathrm{Lim}_{(\mathcal{F}^-, \theta_C^-)} \Delta$ the smallest subset of W to which Δ θ_C-adheres. We say that an *existential* (resp. *universal*, *strict*) *Ξ-intermediate duality*[2] holds at y_0 whenever there exists a filter $\mathcal{F}$ on Ξ such that

$$\min_C \Phi y_0 \cap \mathrm{Lim}_{(\mathcal{F}^-, \theta_C^-)}(\inf_C \Lambda(., y_0)) \neq \emptyset, \tag{12}$$

[2] We may contemplate other concepts of Ξ-intermediate duality by substituting, in formulae (12), (13) and (14), $\inf_C \Lambda(., y_0)$ by $\min_C \Lambda(., y_0)$ or $\downarrow\Lambda(., y_0)$. If we replace minimality by weak (resp. proper) minimality, we get the weak (resp. proper) Ξ-intermediate duality.

(resp.

$$\min_C \Phi y_0 \subset \operatorname{Lim}_{(\mathcal{F}^-, \theta_C^-)}(\inf_C \Lambda(., y_0)), \tag{13}$$

$$\min_C \Phi y_0 = \operatorname{Lim}_{(\mathcal{F}^-, \theta_C^-)}(\inf_C \Lambda(., y_0))). \tag{14}$$

We say also that the *approximate Ξ-intermediate duality*[3] holds at y_0 if, for each $w_0 \in \min_C \Phi y_0$, there exists a filter $\mathcal{F}_{w_0}$ on Ξ such that

$$w_0 \in \operatorname{Lim}_{(\mathcal{F}_{w_0}^-, \theta_C^-)}(\inf_C \Lambda(., y_0)).$$

In order to formulate our intermediate duality result, we begin by listing various useful scalarization results in the convex case. Suppose that W is a real locally convex separated space ordered by a non-trivial convex cone C with nonempty interior. We use $\operatorname{int} C$ to denote the topological interior of C. A subset A of W is *C-convex* if $A + C$ is convex. A function $\theta : W \to \mathbb{R}$ is called *strictly increasing* (resp. *strictly decreasing*) (with respect to C) if

$$w_1 \in w_0 + \operatorname{int} C \Rightarrow \theta(w_1) > \theta(w_0) \text{ (resp. } \theta(w_1) < \theta(w_0)).$$

LEMMA 4.1 *If $\theta \in C^* \setminus \{0\}$, A is a nonempty subset of W and $\widehat{C} = \operatorname{int} C \cup \{0\}$, then $\theta^{-1}(\min \theta A) \cap A \subset \min_{\widehat{C}} A$.*

Proof. Let $w \in \theta^{-1}(\min \theta A) \cap A$. Then, $w \in A$ and $\theta(w) = \min \theta A$. If $w \notin \min_{\widehat{C}} A$, then there exists $z \in (w - \operatorname{int} C) \cap A$. Since θ is strictly increasing, $\theta(z) < \theta(w)$ which is a contradiction. ∎

As a direct consequence, we get $\theta^{-1}(\inf \theta A) \cap \operatorname{cl}^+_{\widehat{C}} A \subset \inf_{\widehat{C}} A$.

By applying Lemma 4.1, we obtain the following result.

PROPOSITION 4.1 *Let A be a nonempty subset of W. If A is C-convex, then*

$$\min_{\widehat{C}} A = \bigcup_{\theta \in C^* \setminus \{0\}} \mathrm{S}(\theta, A),$$

where $\mathrm{S}(\theta, A) = \{w \in A : \theta(w) = \min \theta A\}$.

Proposition 4.1 has also been proved by J. Jahn (1986, Corollary 5.29) in the algebraic setting. For a generalization of this proposition, we refer reader to Song (1998, Theorem 2.1) and the references therein.

We next provide an analogous of Lemma 4.1 for Benson's notion of proper minimality. We use $\text{PB-}\min_C A$ to denote the set of all *proper minimal* points (with respect to C) of A in the sense of Benson (1979). That is, $a_0 \in \text{PB-}\min_C A$ if $a_0 \in A$ and $\operatorname{cl} \operatorname{cone}(A - a_0) \cap (-C_l) = \emptyset$, where $C_l = C \setminus l(C)$ and $l(C) = C \cap (-C)$. The *quasi-interior* of C^* is the following probably empty set

$$\text{q-int}\, C^* = \{\theta \in W^* : \theta(c) > 0 \text{ for all } c \in C_0\}.$$

[3]We have, of course, analogous counterparts of that concept as for the Ξ-intermediate duality.

LEMMA 4.2 *If $\theta \in$ q-int C^*, C is pointed and A is a nonempty subset of W, then*

$$\theta^{-1}(\min \theta A) \cap A \subset \text{PB-min}_C\, A.$$

Proof. Let $a \in \theta^{-1}(\min \theta A) \cap A$, then $a \in A$ and $\theta(a) = \min \theta A$. If $a \notin \text{PB-min}_C\, A$, then there exists $w_1 \in \operatorname{cl}\operatorname{cone}(A-a) \cap (-C_0)$. Thus, $\theta(w_1) < 0$. Since θ is continuous, there exists a neighborhood U of w_1 such that, for every $u \in U$, $\theta(u) < 0$. On the other hand, $w_1 \in \operatorname{cl}\operatorname{cone}(A-a)$. It follows that $\operatorname{cone}(A-a) \cap U \neq \emptyset$ and let $w_2 \in \operatorname{cone}(A-a) \cap U$. Consequently, $\theta(w_2) < 0$ and $w_2 = \lambda(x-a)$ with $\lambda \geq 0$ and $x \in A$. Because of $\theta(a) = \min \theta A$, we have $\theta(w_2) = \lambda(\theta(x) - \theta(a)) \geq 0$ which is impossible. ■

We need also the definitions of upper and lower limits in the sense of Kuratowski, see Dolecki (1982). We restrict ourselves intentionally to the framework of locally convex spaces. Let $\mathcal{F}$ be a filter in a set I and let $\mathcal{A} = \{A_i\}_{i \in I}$ be a family of subsets of a real locally convex separated space W. The *upper limit* of $\mathcal{A}$ (filtered by $\mathcal{F}$) is defined by

$$\limsup \mathcal{A} = \limsup_{\mathcal{F}} A_i = \bigcap_{F \in \mathcal{F}} \operatorname{cl}\Big(\bigcup_{i \in F} A_i\Big).$$

We say that $\mathcal{A}$ *upper converges* to a subset A of W, whenever $\limsup \mathcal{A} \subset A$. Recall that the *grill* $\ddot{\mathcal{F}}$ of a family $\mathcal{F}$ of subsets of I consists of all these subsets of I which meet every member of $\mathcal{F}$. The *lower limit* of $\mathcal{A}$ is, by definition,

$$\liminf \mathcal{A} = \liminf_{\mathcal{F}} A_i = \bigcap_{F \in \ddot{\mathcal{F}}} \operatorname{cl}\Big(\bigcup_{i \in F} A_i\Big).$$

If A is a subset of the lower limit, we say that $\mathcal{A}$ *lower converges* to A. We write also $\liminf_{i \to i_0}$ (resp. $\limsup_{i \to i_0}$), when $\mathcal{F}$ is a neighborhood filter of i_0. If a sequence $(A_n)_n$ of subsets of W lower converges to A in the strong topology and $(A_n)_n$ sequentially upper converges to A in the weak topology (that is if $(x_k \in A_{n_k}$ and (x_k) weak converges to $x)$ then $x \in A)$, we say that $(A_n)_n$ *Mosco converges* to A and we denote $A = \text{M-lim}\, A_n$ or $A_n \xrightarrow{M} A$.

LEMMA 4.3 *If $\theta \in C^* \setminus \{0\}$ and (r_n) is a convergent sequence of elements of $\mathbb{R}$, then*

$$\theta^{-1}(\lim r_n) = \text{M-lim}\, \theta^{-1}(r_n).$$

Proof. The function θ admits a closed graph thus, for any compatible topology, we have $\limsup \theta^{-1}(r_n) \subset \theta^{-1}(\lim r_n)$. On the other hand, θ is an open mapping hence θ^{-1} is lower semicontinuous and consequently $\theta^{-1}(\lim r_n) \subset \liminf \theta^{-1}(r_n)$ for the strong topology. ■

From now on X will be an arbitrary set, Y and W will be two real locally convex separated spaces and W will be ordered by a non-trivial pointed convex cone C with nonempty interior. The family Ξ stands for the class of continuous linear operators from Y to W. Let $F \subset (X \times Y) \times W$ a relation and consider the family of vector minimization problems (3). For each $\theta \in C^* \setminus \{0\}$, we associate the following family of scalar minimization problems with (3)

$$\underset{x \in X}{\text{minimize}}\, \theta F(x,y),\ y \in Y. \tag{15}$$

We have $\theta F \subset (X \times Y) \times \mathbb{R}$. The *marginal function* of (15) is the function $\varphi_\theta : Y \to \overline{\mathbb{R}}$ defined by $\varphi_\theta(y) = \inf_{w \in \Phi y} \theta(w)$, where Φ is the marginal relation of (3). Let us introduce the family $\Theta = \{\theta\xi : \xi \in \Xi\}$. It is obvious that $\Theta \subset Y^*$. The definite lagrangean at y_0 of θF (with respect to Θ) is the relation $\mathcal{L}_\theta(.,.,y_0) \subset (X \times \Theta) \times \mathbb{R}$ defined by

$$\mathcal{L}_\theta(x, \theta\xi, y_0) = \bigcup_{y \in Y} (\theta F(x,y) - (\theta\xi)(y) + (\theta\xi)(y_0)).$$

It is easy to see that $\mathcal{L}_\theta(x,\theta\xi,y_0) = \theta L(x,\xi,y_0)$ and therefore $\mathcal{L}_\theta(X,\theta\xi,y_0) = \theta\Lambda(\xi,y_0)$. Note that when the marginal relation Φ of (3) is a convex subset in $Y \times W$, $\Lambda(\xi,y)$ is convex for all $(\xi,y) \in \Xi \times Y$. The *domain* of Φ is the set $\operatorname{Dom}\Phi = \{y \in Y : \Phi y \neq \emptyset\}$. For $A \subset W$, $\Downarrow A = \{w \in W : (w - \operatorname{int} C) \cap A = \emptyset\}$. The *dual relation* for F (with respect to $\widehat{C}$), one considers here, is the relation $D \subset (\Xi \times Y) \times W$ defined by $D(\xi,y) = \Downarrow\Lambda(\xi,y)$.

The following lemma is a key to the next theorem.

LEMMA 4.4 *Let $\theta \in C^* \setminus \{0\}$, $r \in \mathbb{R}$, Φ be the marginal relation of (3) and $y_0 \in \operatorname{Dom}\Phi$. Assume that*
(i) *Φ is convex in $Y \times W$ and $\Phi y_0 = \Phi y_0 + C$,*
(ii) *$\theta^{-1}(r) \cap \min_{\widehat{C}} \Phi y_0 \neq \emptyset$,*
(iii) *$\forall_{\xi \in \Xi} \min_{\widehat{C}} \Phi y_0 \cap \inf_{\widehat{C}} \Lambda(\xi, y_0) = \emptyset$.*
Then $\theta^{-1}(r) \cap \operatorname{int} B \neq \emptyset$, where B is the set $\operatorname{cl}(\bigcap_{\xi\in\Xi} \Lambda(\xi,y_0))$.

Proof. First, we remark that the operator $A \mapsto \Downarrow A$ is decreasing. Take any $\xi \in \Xi$. By virtue of (2) and (iii), we get $\min_{\widehat{C}} \Phi y_0 \cap \inf_{\widehat{C}} \Lambda(\xi,y_0) = \Phi y_0 \cap D(\xi,y_0) = \emptyset$. As $\Downarrow B = \Downarrow(\bigcap_{\xi\in\Xi} \Lambda(\xi,y_0)) = D(\Xi,y_0)$ and $\Phi y_0 \subset B$, it follows that

$$\min_{\widehat{C}} \Phi y_0 \cap \min_{\widehat{C}} B = \Phi y_0 \cap \Downarrow B = \bigcup_{\xi \in \Xi} (\Phi y_0 \cap D(\xi,y_0)) = \emptyset. \tag{16}$$

Next, let $w_0 \in \theta^{-1}(r) \cap \min_{\widehat{C}} \Phi y_0$. The formula (16) entails that $w_0 \notin \min_{\widehat{C}} B$. Since $w_0 \in B$, we obtain $(w_0 - \operatorname{int} C) \cap B \neq \emptyset$. Let now $\varepsilon \in \operatorname{int} C$ be such that $w_0 - \varepsilon \in B$. Since $\Phi y_0 = \Phi y_0 + C \subset B$ and B is convex by the convexity of Φ in $Y \times W$, we have $w_0 - \varepsilon + C \subset B$. Hence,

$$(w_0 - \varepsilon + C) \cap (w_0 + \varepsilon - C) \subset B. \tag{17}$$

As $\operatorname{int} C \neq \emptyset$ and $w_0 - \varepsilon \in w_0 + \varepsilon - \operatorname{int} C$, the lemma from Jahn (1986, Lemma 1.22, b) enables us to assert that

$$w_0 \in \operatorname{int}_a((w_0 - \varepsilon + C) \cap (w_0 + \varepsilon - C)). \tag{18}$$

On the other hand, we have $\operatorname{int} \Phi y_0 \neq \emptyset$ because $\operatorname{int} C \neq \emptyset$, $\Phi y_0 = \Phi y_0 + C$ and $\Phi y_0 + \operatorname{int} C \subset \operatorname{int} \Phi y_0$. Hence, $\operatorname{int} B \neq \emptyset$. Since B is convex and $\operatorname{int} B \neq \emptyset$, we obtain with (Jahn, 1986, Lemma 1.32) that $\operatorname{int}_a B = \operatorname{int} B$. Accordingly, by (17) and (18), we have $w_0 \in \operatorname{int} B$ and thus $w_0 \in \theta^{-1}(r) \cap \operatorname{int} B$. ■

The following theorem is the main result of this section. A real function $\varphi : Y \to \mathbb{R}$ is said to be Ξ-*convexoid* at y_0 (with respect to a family Ξ of mappings from Y to $\mathbb{R}$), Dolecki and Kurcyusz (1978), if

$$\varphi(y_0) = \sup_{\xi - d \leq \varphi} (\xi(y_0) - d).$$

Of course, this amounts to that, for every $r < \varphi(y_0)$, there exists $(\xi, d) \in \Xi \times \mathbb{R}$ such that

$$\xi - d \leq \varphi \text{ and } \xi(y_0) - d > r.$$

THEOREM 4.1 *Let Φ be the marginal relation of (3) and $y_0 \in \operatorname{Dom} \Phi$. Assume that*

(i) *Φ is convex in $Y \times W$,*

(ii) *$\min_{\widehat{C}} \Phi y_0 \neq \emptyset$, $\Phi y_0 = \Phi y_0 + C$ and*

(iii) *φ_θ is Θ-convexoid at y_0, for all $\theta \in C^* \setminus \{0\}$,*

then, for every $w_0 \in \min_{\widehat{C}} \Phi y_0$, there exists a sequence $(\xi_n) \subset \Xi$ such that $w_0 \in \liminf(\inf_{\widehat{C}} \Lambda(\xi_n, y_0))$.

Proof. Let $w_0 \in \min_{\widehat{C}} \Phi y_0$. By Proposition 4.1, there exists $\theta \in C^* \setminus \{0\}$ such that $w_0 \in \mathrm{S}(\theta, \Phi y_0)$ what amounts to $\varphi_\theta(y_0) = \min \theta \Phi y_0 = \theta(w_0)$. In view of (iii) and the theorem from Dolecki and Kurcyusz (1978, Thm. 7.9),

$$\varphi_\theta(y_0) = \sup_{\xi \in \Xi} \inf \theta \Lambda(\xi, y_0) = \theta(w_0) \in \mathbb{R}.$$

Consequently, there exists a sequence $\{\xi_n\} \subset \Xi$ such that $\theta(w_0) = \lim(\inf \theta \Lambda(\xi_n, y_0))$. By setting $r_n = \inf \theta \Lambda(\xi_n, y_0)$ and $r = \lim r_n = \theta(w_0)$ and with Lemma 4.3, one has $w_0 \in \min_{\widehat{C}} \Phi y_0 \cap \text{M-lim}\, \theta^{-1}(r_n)$. At this stage, we distinguish two cases:

$$\exists_{\xi_0 \in \Xi} \min_{\widehat{C}} \Phi y_0 \cap \inf_{\widehat{C}} \Lambda(\xi_0, y_0) \neq \emptyset, \tag{\dagger}$$

that is exactly the definition of existential Ξ-exact duality at y_0 with respect to the cone $\widehat{C}$, thus there's nothing to prove.

$$\forall_{\xi \in \Xi} \min_{\widehat{C}} \Phi y_0 \cap \inf_{\widehat{C}} \Lambda(\xi, y_0) = \emptyset. \tag{\ddagger}$$

We notice that $\min_{\widehat{C}} \Phi y_0 \subset \Phi y_0 \subset \bigcap_{\xi \in \Xi} \Lambda(\xi, y_0)$ hence the idea of considering the set $B = \mathrm{cl}(\bigcap_{\xi \in \Xi} \Lambda(\xi, y_0))$ and of applying one result of lower semicontinuity of the intersection, see for instance Jourani (1996), Lechicki and Spakowski (1985), Rolewicz (1980) and Sonntag (1982), to the sequences of subsets (B) and $(\theta^{-1}(r_n))$. From (i), we deduct that B is a nonempty closed convex subset of W. Since $\theta^{-1}(r)$ and $\theta^{-1}(r_n)$ are nonempty closed convex subsets of W such that $\theta^{-1}(r) = \text{M-lim}\, \theta^{-1}(r_n)$ then, in view of Lemma 4.4, (Sonntag, 1982, 4.16, P. IV. 13) and Lemma 4.1, we have

$$\theta^{-1}(r) \cap B \subset \liminf(\theta^{-1}(r_n) \cap B) \subset \liminf(\theta^{-1}(r_n) \cap \mathrm{cl}\, \Lambda(\xi_n, y_0))$$
$$\subset \liminf(\theta^{-1}(r_n) \cap \mathrm{cl}^{+}_{\widehat{C}} \Lambda(\xi_n, y_0)) \subset \liminf(\inf_{\widehat{C}} \Lambda(\xi_n, y_0)). \qquad \blacksquare$$

5. Comparison of some exact duality results

In this section, we shall be concerned with the axiomatic duality introduced by The Luc Dinh (1989). Given a general vector minimization problem

$$\underset{x \in X}{\text{minimize}}\, P(x), \tag{$\mathcal{P}$}$$

where X is a nonempty set and $P \subset X \times W$ a relation, a vector maximization problem

$$\underset{\pi \in \Pi}{\text{maximize}}\, D(\pi), \tag{$\mathcal{D}$}$$

where Π is a nonempty set and $D \subset \Pi \times W$, is said to be an *axiomatic dual* problem of $(\mathcal{P})$ if the following relation called *weak duality axiom* holds

$$D(\pi) \cap (P(x) + C_0) = \emptyset, \text{ for each } x \in X \text{ and } \pi \in \Pi. \tag{WDA}$$

A simple computation show that (WDA) is equivalent to $\mathrm{cl}^{+}_{C} P(X) \subset \uparrow D(\Pi)$. We say that $(\mathcal{D})$ is an *exact axiomatic dual* problem of $(\mathcal{P})$ if $P(X) \cap D(\Pi) \neq \emptyset$. It is well know from The Luc Dinh (1989, Chap. 5, Prop. 3.3) that if $(\mathcal{D})$ is an exact axiomatic dual problem of $(\mathcal{P})$, then both problems possess global existential solutions and $\min_C P(X) \cap \max_C D(\Pi) \neq \emptyset$.

The exact axiomatic duality results requires the existence assumptions of some optimal solutions.

5.1. Existence of solutions

Consider the family of vector minimization problems (3) and a family Ξ of mappings from Y to W. The dual relation for (3), that one deals with here, is the subset D of $\Xi \times Y \times W$ defined by $D(\xi, y) = \downarrow\Lambda(\xi, y)$.

The following proposition gives sufficient conditions for the existence of solutions of (3).

PROPOSITION 5.1 *Let $x_0 \in X$ and $y_0 \in Y$ are such that $F(x_0, y_0) \cap \max_C D(\Xi, y_0) \neq \emptyset$. Then, x_0 is a global existential solution of (3) at y_0.*

Proof. By Dolecki and Malivert (1993, Prop. 2.1), we have

$$D(\Xi, y_0) \subset \downarrow \Phi y_0 \text{ and } \Phi y_0 \subset \uparrow D(\Xi, y_0).$$

It follows that,

$$F(x_0, y_0) \cap \max_C D(\Xi, y_0) \subset F(x_0, y_0) \cap \min_C \Phi y_0. \qquad \blacksquare$$

The Luc Dinh has proved the result for vector mathematical programming problems (7) at $y_0 = 0$ i.e.

$$\underset{\substack{x \in X \\ g(x) \cap (-K) \neq \emptyset}}{\text{minimize}} \; f(x), \tag{19}$$

where $f \subset X \times W$, $g \subset X \times Y$, Y and W are real separated topological vector spaces, $C \subset W$ and $K \subset Y$ are convex pointed cones with nonempty interior. Suppose that the class $\Xi = \{\xi \in L(Y, W) : (-\xi)(K) \subset C\}$, where $L(Y, W)$ is the family of all continuous linear operators from Y to W. He has called the subset $Ł(., ., y_0) \subset (X \times \Xi) \times W$ defined by

$$Ł(x, \xi, y_0) = f(x) - \xi(g(x)) + \xi(y_0)$$

the *lagrangean relation* of (7). It is clear that $Ł(x, \xi, y_0) \subset L(x, \xi, y_0) \subset Ł(x, \xi, y_0) + C$; where, in this case, $L(x, \xi, y_0) = f(x) - \xi(g(x) + K) + \xi(y_0)$. If we set $D_{\min}(\xi, y) = \min_C \Lambda(\xi, y)$, then we get $D_{\min}(\xi, y) = \min_C Ł(X, \xi, y)$ provided that C is pointed.

As first consequence, we obtain the following result of The Luc Dinh (1989, Chap. 5, Coro. 1.3):

COROLLARY 5.1 *Let $x_0 \in X$ be such that $g(x_0) \cap (-K) \neq \emptyset$ and $f(x_0) \cap \max_C(\bigcup_{\xi \in \Xi} \min_C Ł(X, \xi, 0)) \neq \emptyset$. Then, x_0 is a global existential solution of (19).*

Proof. Notice that $D_{\min}(\xi, 0) = \min_C Ł(X, \xi, 0)$ and $D_{\min}(\xi, 0) \subset D(\xi, 0)$. In view of Dolecki and Malivert (1993, Prop. 2.1), we have

$$f(x_0) \subset \Phi 0 \subset \uparrow D(\Xi, 0) \subset \uparrow D_{\min}(\xi, 0).$$

Consequently

$$f(x_0) \cap \max_C D_{\min}(\Xi, 0) \subset f(x_0) \cap \max_C D(\Xi, 0)$$

and the assertion follows from Proposition 5.1. $\blacksquare$

Further existence results for weakly and properly solutions of (3) are given by the corollary given below. Before, recall that an element a_0 of a subset A of W is a *proper minimal* point (with respect to C) of A according to Henig (1982), if there exists a non-trivial convex cone D such that $C_l \subset \operatorname{int} D$ and $a_0 \in \min_D A$. We denote by PH-$\min_C A$ the set of these points.

COROLLARY 5.2 *Let* $x_0 \in X$ *and* $y_0 \in Y$ *are such that* $F(x_0, y_0) \cap \max_{\widehat{C}} D(\Xi, y_0) \neq \emptyset$ *(resp.* $F(x_0, y_0) \cap \text{PH-max}_C D(\Xi, y_0) \neq \emptyset$*). Then,* x_0 *is a global existential weak (resp. Henig proper) solution of (3) at* y_0.

Proof. The first part is evident. Let, now, $w_0 \in F(x_0, y_0) \cap \text{PH-max}_C D(\Xi, y_0)$. Then, there exists a non-trivial convex cone K such that $C_0 \subset \text{int}\, K$ and $w_0 \in \max_K D(\Xi, y_0)$. Since $F(x_0, y_0) \cap \max_K D(\Xi, y_0) \subset F(x_0, y_0) \cap \min_K \Phi y_0$, hence $w_0 \in F(x_0, y_0) \cap \text{PH-min}_C \Phi y_0$. ■

In the sequel, we shall be confronted with the comparison of various exact duality results for the vector mathematical programming problems (7).

5.2. Comparison of constraint qualifications

In this subsection, we will especially be concerned with the vector mathematical programming problems (7). Put $A_y = \{x \in X : g(x) \cap (y - K) \neq \emptyset\}$. The marginal relation for (7) is the subset Φ of $Y \times W$ defined by

$$\Phi y = \bigcup_{x \in A_y} f(x) = f g^{-1}(y - K).$$

Suppose that $\Xi = L(Y, W)$ the class of continuous linear operators from Y to W. We designate by Q the relation from X to $Y \times W$ defined by $Q(x) = (f(x) + C) \times (g(x) + K)$. A relation $\Phi \subset Y \times W$ is said to be *C-convex* provided that, for every $(y_1, y_2) \in Y^2$ and $\alpha \in [0, 1]$,

$$\alpha \Phi y_1 + (1 - \alpha)\Phi y_2 \subset \Phi(\alpha y_1 + (1 - \alpha) y_2) + C.$$

This is equivalent to the convexity of $\Phi + \{0\} \times C$ in $Y \times W$.

In order to develop the implications among the various exact duality results, the relationships between Slater constraint qualification and assumptions of Theorem 3.4 in Dolecki and Malivert (1993) are first established in the following proposition.

PROPOSITION 5.2 *Assume that* $\text{Dom}\, f = \text{Dom}\, g = X$.

(i) *If* $Q(X)$ *is convex, then* Φ *is C-convex.*

(ii) *If* $g(X) \cap (y_0 - \text{int}\, K) \neq \emptyset$, *then* $\{y_0\} \times (\Phi y_0 + C_0) \cap \text{int}(\Phi + \{0\} \times C_0) \neq \emptyset$.

Proof.

(i) Let $(y_1, y_2) \in Y^2$, $\alpha \in [0, 1]$ and $w_i \in \Phi y_i$ for $i = 1, 2$. Then, for each $i \in \{1, 2\}$, there exists $x_i \in g^{-1}(y_i - K)$ such that $w_i \in f(x_i) \subset f(x_i) + C$. It follows that, for each $i \in \{1, 2\}$, we have $g(x_i) \cap (y_i - K) \neq \emptyset$ thus $y_i \in g(x_i) + K$ and consequently $(w_i, y_i) \in Q(x_i) \subset Q(X)$. Since $Q(X)$ is convex, $\alpha(w_1, y_1) + (1 - \alpha)(w_2, y_2) \in Q(X)$ and therefore

$(\alpha w_1 + (1 - \alpha) w_2, \alpha y_1 + (1 - \alpha) y_2) \in Q(x)$ for a certain $x \in X$,

which amounts to

$\alpha w_1 + (1 - \alpha) w_2 \in f(x) + C$ and $g(x) \cap (\alpha y_1 + (1 - \alpha) y_2 - K) \neq \emptyset$.

Hence, $\alpha w_1 + (1 - \alpha) w_2 \in \Phi(\alpha y_1 + (1 - \alpha) y_2) + C$.

(ii) Let $x_0 \in X$, $z_0 \in g(x_0) \cap (y_0 - \operatorname{int} K)$, $c \in \operatorname{int} C = \operatorname{int} C_0$ and $w_0 \in f(x_0)$. Now we prove that $(y_0, w_0 + c) \in \operatorname{int}(\Phi + \{0\} \times C_0)$. Since $y_0 - z_0 \in \operatorname{int} K$ and $c \in \operatorname{int} C_0$, there exists a neighborhood U of 0 in Y and a neighborhood V of 0 in W such that

$$y_0 - z_0 + U \subset K \text{ and } c + V \subset C_0.$$

Accordingly, $w_0 + c + V \subset w_0 + C_0 \subset f(x_0) + C_0$ and $y_0 + U \subset z_0 + K \subset g(x_0) + K$. It ensues, for every $y \in y_0 + U$ and every $w \in w_0 + c + V$, that $w \in \Phi y + C_0$ which is equivalent to $(y_0 + U) \times (w_0 + c + V) \subset \Phi + \{0\} \times C_0$. We infer that $\{y_0\} \times (f(x_0) + \operatorname{int} C) \subset \operatorname{int}(\Phi + \{0\} \times C_0)$. On the other hand, we have $\{y_0\} \times (f(x_0) + \operatorname{int} C) \subset \{y_0\} \times (\Phi y_0 + C_0)$ because $g(x_0) \cap (y_0 - \operatorname{int} K) \neq \emptyset$. ■

Let us quote the following exact duality result of S. Dolecki and C. Malivert (1993, Theorem 3.4):

THEOREM 5.1 *Suppose that C is open, Φ is C-convex and $\{y_0\} \times (\Phi y_0 + C) \cap \operatorname{int}(\Phi + \{0\} \times C) \neq \emptyset$. If $w_0 \in \min_C \Phi y_0$, then there exists $\xi_0 \in \Xi$ such that $w_0 \in \min_C \Lambda(\xi_0, y_0)$.*

Using the problem denoted by $(\mathcal{D}_l)$, which is really an axiomatic dual problem of (19),

$$\underset{\xi \in \Xi}{\text{maximize}} \min_C \text{Ł}(X, \xi, 0), \qquad (\mathcal{D}_l)$$

we get the following results as consequences of Proposition 5.2 and Theorem 5.1.

COROLLARY 5.3 (The Luc Dinh, 1989, Thm. 3.6) *Assume that $Q(X)$ is convex and $g(X) \cap (-\operatorname{int} K) \neq \emptyset$. If PH-$\min_C \Phi 0 \neq \emptyset$, then $(\mathcal{D}_l)$ is an exact axiomatic dual problem of (19).*

Proof. Let $w_0 \in$ PH-$\min_C \Phi 0$. Then, there exists a non-trivial convex cone $D \subset W$ such that $C_0 \subset \operatorname{int} D$ and $w_0 \in \min_D \Phi 0 \subset \min_{\widehat{D}} \Phi 0$, where $\widehat{D} = \operatorname{int} D \cup \{0\}$. With both hypotheses we conclude, in view of Proposition 5.2, that Φ is C-convex and $\{0\} \times (\Phi 0 + C_0) \cap \operatorname{int}(\Phi + \{0\} \times C_0) \neq \emptyset$. Since $C_0 \subset \operatorname{int} D$, Φ is $(\operatorname{int} D)$-convex and $\{0\} \times (\Phi 0 + C_0) \cap \operatorname{int}(\Phi + \{0\} \times C_0) \subset \{0\} \times (\Phi 0 + \operatorname{int} D) \cap \operatorname{int}(\Phi + \{0\} \times \operatorname{int} D)$. By Theorem 5.1, one gets $e \in \operatorname{int} D$ and $\mu \in K^*$ such that

$$w_0 \in \min_{\widehat{D}} \Lambda(\xi_0, 0) \text{ with } \xi_0 = \mu(.)e. \qquad (20)$$

As $C_0 \subset \operatorname{int} D$, it follows from (20) that $w_0 \in \min_C \Lambda(\xi_0, 0) = D_{\min}(\xi_0, 0) = \min_C \text{Ł}(X, \xi_0, 0)$. Thus, $w_0 \in$ PH-$\min_C \Phi 0 \cap \min_C \Lambda(\xi_0, 0) \subset \Phi 0 \cap \min_C \text{Ł}(X, \xi_0, 0)$ which proves the assertion. ■

The following corollary furnishes a sufficient condition for the existence of the Henig properly minimal points, that is, the strong domination property introduced by E. Bednarczuk (1994). We say that a subset A of W has the *strong domination property* (SDP) if there exists a closed convex cone D such that $D_0 \subset \operatorname{int} C$ (where $D_0 = D \setminus \{0\}$), $A \subset \min_C A + D$ and the cone D has the property that, for each neighborhood V of 0, there exists a neighborhood U of 0 such that $D \setminus V + U \subset C$.

COROLLARY 5.4 *Suppose that* $Q(X)$ *is convex,* $g(X) \cap (-\operatorname{int} K) \neq \emptyset$ *and* $\Phi 0$ *has the property (SDP). Then,* $(\mathcal{D}_l)$ *is an exact axiomatic dual problem of (19).*

Proof. Since $\Phi 0$ has the property (SDP), $\min_C \Phi 0 \neq \emptyset$ and $\min_C \Phi 0 =$ PH-$\min_C \Phi 0$ and therefore the result follows from Corollary 5.3. ■

H. Nakayama (1984) investigated also the vector problem (19) under the following assumptions. Let X be a compact convex subset of $\mathbb{R}^n$, $Y = \mathbb{R}^m$, $W = \mathbb{R}^r$, $f : \mathbb{R}^n \to \mathbb{R}^r$ is continuous and C-convex, $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and K-convex, $C \subset \mathbb{R}^r$ and $K \subset \mathbb{R}^m$ are pointed closed convex cones with nonempty interior and $\mathcal{U} = \{U \in \mathcal{M}(r,m) : U(K) \subset C\}$, where $\mathcal{M}(r,m)$ designate the set of all (r,m)-matrix. Note, in this case, that the condition $g(x) \cap (-K) \neq \emptyset$ amounts to $g(x) \in -K$. We use Φ to indicate the marginal relation for (7). We have $\Phi 0 = f(A_0)$ with $A_0 = \{x \in X : g(x) \in -K\}$. Let us observe that every closed pointed convex cone C in $\mathbb{R}^r$ has a compact base and consequently $\operatorname{int} C^* \neq \emptyset$ and $\operatorname{int} C^* =$ q-int C^*.

COROLLARY 5.5 (Nakayama, 1984, Thm. 2.1) *If* $x_0 \in A_0$ *be such that* $f(x_0) \in \mathrm{S}(\theta, \Phi 0)$ *for some* $\theta \in \operatorname{int} C^*$ *and if there exists* $\overline{x} \in X$ *such that* $g(\overline{x}) \in -\operatorname{int} K$, *then there exists* $U_0 \in \mathcal{U}$ *such that* $f(x_0) \in \min_C L(X, -U_0, 0)$ *and* $U_0 g(x_0) = 0$.

Proof. Since f is C-convex, g is K-convex and X is convex, $Q(X)$ and $\Phi 0 = f(A_0)$ are convex. On the other hand, $f(x_0) \in A_0 \cap \theta^{-1}(\min \theta \Phi 0) \subset$ PB-$\min_C \Phi 0 =$ PH-$\min_C \Phi 0$ and the result follows from Corollary 5.3. ■

Acknowledgement. I wish to express my appreciation to my thesis director, Professor Szymon Dolecki of the Bourgogne University of Dijon, for suggesting the problem, for his support and for many stimulating conversations that helped to improve the paper. Further thanks are due to the two referees for their constructive comments and suggestions, and for their active interest in the publication of this paper.

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