**Control and Cybernetics**

vol. **29** (2000) No. 4

# Sensitivity analysis for optimal control of problems governed by semilinear parabolic equations

by

**M. Bergounioux and N. Merabet**

Département de Mathématiques
UMR 6628 – MAMPO – Université d'Orléans
B.P. 6759 – 45067 Orléans Cedex 2, France
E-mail: Maitine.Bergounioux@labomath.univ-orleans.fr
E-mail: Nora.Merabet@labomath.univ-orleans.fr

**Abstract:** We investigate an optimal control problem governed by a semilinear parabolic equation with perturbed initial data. We perform some sensitivity analysis: under polyhedricity assumption and second order optimality conditions we derive second order expansion of the optimal value function.

**Keywords:** optimal control, second order optimality conditions, sensitivity analysis.

## 1. Introduction and assumptions

Many physical phenomena are modelled by more or less complicated partial differential equations systems. For instance, meteorological previsions may be deduced from systems based on Navier-Stokes equations. The state function that describes the system is observed at time $t = 0$, via measurements or previous computations from another model. For many situations, it is necessary to control such systems (so that the temperature or the velocity of a fluid is not too high, for example). Therefore, we have to deal with systems governed by partial differential equations involving control functions, whose initial data are not well known (noise, measurements or computation errors...) This is related to the more general question: how to perform a stability and sensitivity analysis with respect to a parameter (perturbation or unknown data) appearing in the data? Stability and/or sensitivity has been studied by many authors, especially in the ODE context. Let us mention papers by Malanowski (1995), and Malanowski & Maurer (1996), where the perturbation belongs to a Banach space: the main ingredients are first and second order optimality assumptions and strict complementarity. In Maurer & Pesch (1994) the solution's differentiability with respect to a finite dimensional parameter is studied. In the PDE context, let us mention differential operator and of the source term as well: sufficient conditions for a directional differentiability of the solution are given with applications to shape sensitivity analysis. More recently, Tröltzsch (1996, 1997) and Malanowski & Tröltzsch (1999) have established Lipschitz stability theorems for the solution to nonlinear parabolic optimal control problems. At last we have to mention the work of Bonnans & Cominetti (1996a, 1996b) and Bonnans (1998a). We have used these authors' techniques to establish the results presented here.

In this paper, we focus on the (simple) model case where the system is described by a semilinear parabolic equation and the control function is a distributed one. The initial value function is not well known and may belong to the neighborhood of a fixed value, say $\bar{g}$; therefore, we can view it as a system perturbation. Of course, we could consider boundary controls (or both distributed and boundary controls), but the analysis would be the same: the main tool is the state function regularity which allows to deal with the two-norm discrepancy phenomenon.

In adddition, we have considered smooth perturbations (that is perturbations in $W_o^{1,p}$) to ensure good regularity properties of the state function. Of course a more realistic approach should involve quite general perturbations (for example $L^\infty$ functions or measures as in Ahmed & Xiang, 1997).

Now we present the problem. Let $\Omega$ be a bounded open domain in $\mathbb{R}^N$ ($N \geq 2$) of class $C^{2+\gamma}$, for some $\gamma$ satisfying $0 < \gamma \leq 1$. We denote by $\Gamma$ its boundary and set $Q = \Omega \times ]0, T[$, $\Sigma = \Gamma \times ]0, T[$ where $T$ is a positive real number.

Next, we consider a system whose state $y$ is the solution of

$$\begin{cases} \partial_t y + Ay + f(y) = u \ \text{ in } Q \\ \qquad\qquad\qquad y = 0 \ \text{ on } \Sigma \\ \qquad\qquad\quad y(0) = g \ \text{ in } \Omega, \end{cases} \tag{1.1}$$

and an optimal control problem ($g$ being fixed)

$$\begin{cases} \min J(u, y) \\ y = y[u, g] \text{ solution to (1.1)} \\ u \in \mathcal{K}. \end{cases} \tag{$\mathcal{P}_g$}$$

The cost functional $J$ and the control constraint set $\mathcal{K}$ will be made precise in the sequel. Problem $(\mathcal{P}_g)$ has at least one solution $u^*(g)$ under appropriate assumptions that are quoted thereafter. We would like to describe the (local) behavior of $u^*(g)$ with respect to $g$ as well as the behavior of the optimal value function $g \mapsto J(u^*(g), y[u^*(g), g])$. Under second order optimality conditions, we shall give some continuity results and a local expansion of the optimal value function in a neighborhood of a fixed value $\bar{g}$. As mentioned, most of the techniques we use are due to J.F. Bonnans (1998a) who has considered a problem governed by elliptic equations and a linear perturbation of the desired state in-

(and quadratic) with respect to the perturbation so that second order expansions were exact. A direct extension of this situation to the parabolic case is studied in Merabet (2000), Chapter 3. We present here a further generalization.

Let us precise assumptions:

- **(H1)** $A$ is a second order elliptic differential operator defined by

$$\begin{aligned}
&Ay = -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}y) + a_o(x)y \text{ with} \\
&a_{ij} \in \mathcal{C}^2(\overline{\Omega}),\ i,j = 1\cdots N, \\
&a_o \in L^\infty(\Omega),\ \operatorname{ess\,inf}\{a_o(x) \mid x \in \overline{\Omega}\} \geq 0. \\
&\forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^N,\ \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq M \sum_{i=1}^{N} \xi_i^2 \text{ with } M > 0.
\end{aligned} \tag{1.2}$$

- **(H2)** $f$ is a $\mathcal{C}^2$ real function from $\mathbb{R}$ to $\mathbb{R}$, nondecreasing and globally Lipschitz continuous. We denote in the same way, the real function $f$ and the Nemytskii operator $f : y \mapsto f(y)$ such that $f(y)(x,t) = f(y(x,t))$, $(x,t) \in Q$.
- **(H3)** $g \in W_o^{1,p}(\Omega)$ with $N < p$.

The paper is organized as follows. We first recall continuity and differentiability properties of the state mapping which are useful in the sequel. The subsequent section is devoted to studying the optimal control problem and "zero order" properties of the solution and the optimal value function. We also recall first and second order optimality conditions therein. Finally, we give first order (Section 4) and second order (Section 5) sensitivity analysis for the problem under the additional assumption that the set of control constraints is polyhedric.

## 2. State equation properties

In this section, we recall some continuity and differentiability properties of the state mapping which associates the state function $y$ to the control function $u$ and the initial value $g$. These results are not new, but it seems preferable to give them to make this paper more readable.

THEOREM 2.1 *Assume (H1) and (H2). For any $u \in L^p(Q)$ and $g \in W_o^{1,p}(\Omega)$ with $p > N$, equation (1.1) has a unique weak solution $y = y[u,g] \in W_2(0,T) \cap \mathcal{C}(\overline{Q})$.*

Proof. See Bergounioux & Tröltzsch (1996), p. 521 for $N \leq 3$. For the case of $N > 3$ one uses a result of Arada & Raymond (1998) and $W_o^{1,p}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ to get the continuity on the whole set $\overline{Q}$. ■

We recall that

$$W_p(0,T) = \{y \in L^p(0,T; H_o^1(\Omega)) \mid y_t \in L^{p'}(0,T; H^{-1}(\Omega))\}.$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

REMARK 2.1 *Assumption (H2) may be weakened: it is sufficient for $f$ to be $\mathcal{C}^1$ to make the previous theorem valid. One also refers to assumption A2 of Arada & Raymond (1998).*

In the sequel, we will assume that $N < p < \infty$: we do not allow $p = +\infty$ so that $W_o^{1,p}(\Omega)$ is reflexive.

We define the state space in a usual way (see Bergounioux & Tröltzsch (1996), for instance) as:

$$\mathcal{Y} = \{y \in W_p(0,T) \mid \partial_t y + Ay \in L^p(Q),\ y = 0 \text{ on } \Sigma,\ y(0) \in W_o^{1,p}(\Omega)\}.$$

$\mathcal{Y}$ is a subspace of $\mathcal{C}(\overline{Q})$ and, supplied with the norm

$$\|y\|_{\mathcal{Y}} = \|y\|_{W_p(0,T)} + \|y\|_{\mathcal{C}(\overline{Q})} + \|y_t + Ay\|_{L^p(Q)} + \|y(0)\|_{W_o^{1,p}(\Omega)},$$

it is a Banach space. From now we denote by $\|\cdot\|_V$ the norm of the space $V$. The $L^q(Q)$-norm will be denoted $\|\cdot\|_{q,Q}$ ($q = \infty$ corresponds to the uniform norm of $\mathcal{C}(\overline{Q})$) and the $L^q(\Omega)$-norm is denoted $\|\cdot\|_{q,\Omega}$. We now give some useful properties of the mapping $(u,g) \mapsto y[u,g]$ that we need in the sequel.

### 2.1. Continuity properties of the state-mapping

Since we assume $p > N$ and $\Omega$ is bounded, $W_o^{1,p}(\Omega)$ is compactly embedded in $\mathcal{C}(\overline{\Omega})$. Therefore, from Theorem 2.1 we obtain:

THEOREM 2.2 *Let be $u \in L^p(Q)$ and $g \in W_o^{1,p}(\Omega)$; there exists $C > 0$ such that*

$$\|y\|_{\infty,Q} \le C(\|u\|_{p,Q} + \|g\|_{\infty,\Omega} + 1), \tag{2.3}$$

*where $y$ is the unique solution to (1.1).*

*Moreover, $y$ is Hölder continuous on $\overline{Q}$: there exists $\nu$ such that, for any $M > 0$, there exists $C$ such that*

$$\|u\|_{p,Q} + \|g\|_{\infty,\Omega} \le M \Rightarrow \|y\|_{\mathcal{C}^{\nu,\nu/2}(\overline{Q})} \le C.$$

Proof. Estimation (2.3) is given in Theorem 3.1.i of Raymond & Zidani (1999). The above stability result of the weak solution to (1.1) with respect to the data is proved in Theorem 3.4.i) of Bergounioux & Zidani (1999). The second part of the Theorem follows from a regularity result for linear equations (see for instance Arada & Raymond, 1998, 1999). Equation (1.1) may be written as

$$\begin{cases} \partial_t y + Ay = v = u - f(y) & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = g & \text{in } \Omega \end{cases}$$

where $v \in L^p(Q)$ and $g \in W_o^{1,p}(\Omega) \subset \mathcal{C}^\sigma(\overline{\Omega})$ with $\sigma = 1 - \frac{N}{p}$, for example. We achieve the proof using assumptions on $f$. ■

The previous theorem allows to get a weak-strong continuity result of the

THEOREM 2.3 *The mapping $(u,g) \mapsto y[u,g]$ is sequentially continuous*

*(i) from $L^p(Q) \times W_o^{1,p}(\Omega)$, endowed with the weak-$L^p(Q) \times$ weak-star $L^\infty(\Omega)$ topology, into $\mathcal{C}(\overline{Q})$ (strong topology)*

*(ii) from $L^p(Q) \times W_o^{1,p}(\Omega)$ endowed with the weak topology into $\mathcal{C}(\overline{Q})$ (strong topology).*

Proof. (i) Let $(u_n, g_n)_n$ be a sequence converging to $(\tilde{u}, \tilde{g})$ in $L^p(Q)$ (weak) $\times$ $L^\infty(\Omega)$ (weak-star). Let $y_n$ be the solution to (1.1), corresponding to $(u_n, g_n)$. By (2.3), $(y_n)$ is bounded in $L^\infty(Q)$. Therefore, there exists $\tilde{y} \in L^\infty(Q)$ such that a subsequence still denoted $(y_n)$ converges to $\tilde{y}$ for the weak-star $L^\infty(Q)$ topology. In addition, by Theorem 2.2, $(y_n)$ is bounded in $\mathcal{C}^{\nu,\nu/2}(\overline{Q})$, for some $\nu > 0$. Since the embedding of $\mathcal{C}^{\nu,\nu/2}(\overline{Q})$ into $\mathcal{C}(\overline{Q})$ is compact, then $(y_n)$ converges to $\tilde{y}$ uniformly in $\overline{Q}$. On the other hand $y_n$ satisfies

$$\int_Q y_n\left(-\frac{\partial z}{\partial t} + A^* z\right) dx\, dt + \int_Q (f(y_n) - u_n) z\, dx\, dt = \int_\Omega g_n z(0)\, dx$$

for any $z \in \mathcal{C}^2(\overline{Q})$ such that $z(T) = 0$ and $z_{|\Sigma} = 0$ ($A^*$ denotes the operator adjoint to $A$). By **H2** and the Lebesgue theorem we may pass to the limit and obtain

$$\int_Q \tilde{y}\left(-\frac{\partial z}{\partial t} + A^* z\right) dx\, dt + \int_Q (f(\tilde{y}) - \tilde{u}) z\, dx\, dt = \int_\Omega \tilde{g} z(0)\, dx$$

for any $z \in \mathcal{C}^2(\overline{Q})$ such that $z(T) = 0$ and $z_{|\Sigma} = 0$. Therefore, $\tilde{y}$ is the weak solution of (1.1) associated with $(\tilde{u}, \tilde{g})$.

(ii) The second point is a direct consequence of the first one since the weak-convergence in $W_o^{1,p}(\Omega)$ implies the $L^\infty$-weak-star convergence. ■

## 2.2. Differentiability properties of the state-mapping

This subsection is devoted to differentiability results for the state mapping: $(u,g) \mapsto y[u,g]$. The main tool is the implicit function theorem.

THEOREM 2.4 *The operator*

$$\begin{aligned} &\mathcal{T} : L^p(Q) \times W_o^{1,p}(\Omega) \times \mathcal{Y} \to L^p(Q) \times W_o^{1,p}(\Omega), \\ &(u, g, y) \mapsto (\partial_t y + Ay + f(y) - u, y(0) - g), \end{aligned}$$

*is of class $\mathcal{C}^2$.*

Proof. The first component of $\mathcal{T}$ has a linear part $y \mapsto \partial_t y + Ay$ which is continuous from $\mathcal{Y}$ to $L^p(Q)$; the nonlinear part is the Nemytskii operator $f$ which is known to be $\mathcal{C}^2$ (since $f$ is $\mathcal{C}^2$). The second component $(y,g) \mapsto y(0) - g$

THEOREM 2.5 *Let $(\bar{u},\bar{g}) \in L^p(Q) \times W_o^{1,p}(\Omega)$ and $\bar{y} = y[\bar{u},\bar{g}]$ be the solution to (1.1). The state mapping $(u,g) \mapsto y[u,g]$ is $\mathcal{C}^2$ in a neighborhood of $(\bar{u},\bar{g})$ and its derivative with respect to $(u,g)$ at $(\bar{u},\bar{g})$ is $\bar{z} = Dy[\bar{u},\bar{g}](v,h)$, the solution of*

$$\begin{cases} \partial_t \bar{z} + A\bar{z} + f'(\bar{y})\bar{z} = v \text{ in } Q \\ \bar{z} = 0 \text{ on } \Sigma \\ \bar{z}(0) = h \text{ in } \Omega \end{cases} \tag{2.4}$$

*We denote by $Dy$ the derivative of $y$ with respect to $(u,g)$.*

Proof. We only give a sketch of the proof. We apply the implicit function theorem to equation $\mathcal{T}(u,g,y) = 0$ in the neighborhood of the pair $(\bar{u},\bar{g}) \in L^p(Q)\times W_o^{1,p}(\Omega)$. It is easy to see that $D_y\mathcal{T}(\bar{u},\bar{g},\bar{y})$ is an isomorphism; therefore, by the implicit function theorem, there is a $\mathcal{C}^2$ function $(u,g) \mapsto y[u,g]$ defined in a neighborhood of $(\bar{u},\bar{g})$, such that $\mathcal{T}(u,g,y[u,g]) = 0$. ■

## 2.3. The adjoint equation

We end this section with a similar result for the so-called adjoint equation that appears in a natural way in optimal control theory. We consider the following linearized adjoint equation

$$\begin{cases} -\partial_t p + A^* p + f'(y[u,g])p = y[u,g] - z_d \text{ in } Q \\ p = 0 \text{ on } \Sigma \\ p(T) = 0 \text{ in } \Omega \end{cases} \tag{2.5}$$

where $(u,g) \in L^p(Q) \times W_o^{1,p}(\Omega)$ and $z_d \in L^p(Q)$.

PROPOSITION 2.1 *For any $g \in W_o^{1,p}(\Omega)$ and $u \in L^p(Q)$, there exists a unique solution $p \in W_2(0,T) \cap \mathcal{C}(\overline{Q})$ to (2.5). The mapping $(u,g) \to p[u,g]$ is $\mathcal{C}^1$ from $L^p(Q)\times W_o^{1,p}(\Omega)$ to $\mathcal{C}(\overline{Q})$ and the derivative $Dp[u,g](v,h) := q$ is the solution of*

$$\begin{cases} -\partial_t q + A^* q + f'(y[u,g])q = (1 - f''(y[u,g]p)z \text{ in } Q \\ q = 0 \text{ on } \Sigma \\ q(T) = 0 \text{ in } \Omega \end{cases} \tag{2.6}$$

*where $z = Dy[u,g](v,h)$.*

Proof. Again we use the implicit function theorem. ■

COROLLARY 2.1 *The mapping $(u,g) \to p[u,g]$ is sequentially continuous from $L^p(Q) \times W_o^{1,p}(\Omega)$ endowed with the weak topology to $\mathcal{C}(\overline{Q})$ endowed with the strong topology.*

Proof. It is a direct consequence of the above Proposition and Theorem 2.3. We use the (strong) continuity of $f'$ and a stability result for the linear equa-

## 3. The optimal control problem

Given $g \in W_o^{1,p}(\Omega)$, we call $(\mathcal{P}_g)$ the following optimal control problem

$$\begin{cases} \min J(u,y) := \dfrac{1}{2}\displaystyle\int_Q (y - z_d)^2 \, dx \, dt + \dfrac{\alpha}{2}\displaystyle\int_Q u^2 \, dx \, dt \\ y = y[u,g] \text{ solution to (1.1)} \\ u \in \mathcal{K} \end{cases} \tag{$\mathcal{P}_g$}$$

where $\mathcal{K}$ is a nonempty, convex, bounded and closed subset of $L^p(Q)$. Such a problem $(\mathcal{P}_g)$ admits (at least) a solution for any $g \in W_o^{1,p}(\Omega)$ and we call $\mathcal{S}_g$ the set of solutions of $(\mathcal{P}_g)$.

### 3.1. Regularity of the cost functional

We would like to describe the (local) behavior of the optimal value function for $(\mathcal{P}_g)$ in a neighborhood of some fixed $\bar{g}$. First, we have to establish some differentiability results for the cost functional with respect to both the control $u$ and the perturbation $g$. This cost functional $F$ is defined by

$$L^p(Q) \times W_o^{1,p}(\Omega) \to \mathbb{R}, \; u, g \mapsto F(u,g) = J(u, y[u,g]), \tag{3.7}$$

where $y[u,g]$ is the unique solution to (1.1). Indeed, problem $(\mathcal{P}_g)$ may be written equivalently as

$$\min \quad \{F(u,g) \mid u \in \mathcal{K}\}.$$

We first give continuity properties for $F$ which are deduced from Theorem 2.3.

THEOREM 3.1 *The mapping $F : L^p(Q) \times W_o^{1,p}(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous (lsc).*

*Moreover, for any $u$ in $L^p(Q)$, the mapping $g \mapsto F(u,g)$ is weakly continuous from $W_o^{1,p}(\Omega)$ to $\mathbb{R}$.*

Proof. Let $(u_k, g_k)$ be a sequence of $L^p(Q) \times W_o^{1,p}(\Omega)$ weakly convergent to $(\bar{u}, \bar{g})$ in $L^p(Q) \times W_o^{1,p}(\Omega)$. Theorem 2.3 yields that $y_k = y[u_k, g_k]$ strongly converges to $y[\bar{u}, \bar{g}]$ in $\mathcal{C}(\overline{Q})$ (and in $L^2(Q)$). Moreover, the mapping $\cdot \mapsto \|\cdot\|_{2,Q}^2$ is convex continuous on $L^2(Q)$. Therefore, $F$ is weakly lsc and the mapping $g \mapsto F(u,g)$ is weakly continuous on $W_o^{1,p}(\Omega)$ (for every fixed $u$). ∎

We get also some differentiability properties for $F$.

THEOREM 3.2 *$F$ is a $\mathcal{C}^2$ mapping on $L^p(Q) \times W_o^{1,p}(\Omega)$. Moreover, for any $(v,h) \in L^p(Q) \times W_o^{1,p}(\Omega)$,*

$$F'(u,g)(v,h) = (p + \alpha u, v)_{2,Q} - (p(0), h)_{2,\Omega} \tag{3.8}$$

$$F''(u,g)((v,h),(v,h)) = \alpha\|v\|_{2,Q}^2 + \|z\|_{2,Q}^2$$

$$- \int p[u,g](x,t) f''(y[u,g]) z^2(x,t) \, dx \, dt - 2 \int q(0)(x) h(x) \, dx \tag{3.9}$$

*where $p$ is the adjoint-state defined by (2.5), whereas $z = Dy[u,g](v,h)$ and $q = Dp[u,g](v,h)$ are given by (2.4) and (2.6) respectively.*

Proof. The mapping

$$L^p(Q) \times W_o^{1,p}(\Omega) \to \mathcal{Y},\ u, g \mapsto y = y[u,g],$$

is $\mathcal{C}^2$ (Theorem 2.5); therefore, $F$ is $\mathcal{C}^2$ as well. Let $(u,g) \in L^p(Q) \times W_o^{1,p}(\Omega)$ and compute $F'(u,g)(v,h)$ for some $(v,h) \in L^p(Q) \times W_o^{1,p}(\Omega)$.

$$F'(u,g)(v,h) = \alpha(u,v)_{2,Q} + (y[u,g] - z_d, y'[u,g](v,h))_{2,Q}.$$

Let $p = p[u,g]$ be the adjoint state given by (2.5) and set $z = y'[u,g](v,h)$ (satisfying (2.4)). Then

$$F'(u,g)(v,h) = \alpha(u,v)_{2,Q} + (-\partial_t p + A^* p + f'(y[u,g])p, z)_{2,Q};$$

an integration by parts and the use of (2.4) yield

$$F'(u,g)(v,h) = (p[u,g] + \alpha u, v)_{2,Q} - (p[u,g](0), h)_{2,\Omega}. \tag{3.10}$$

A similar computation for $F''(u,g)((v,h),(v,h))$ gives

$$\begin{aligned} &F''(u,g)((v,h),(v,h)) \\ &= \alpha\|v\|^2_{2,Q} + (p'[u,g](v,h), v)_{2,Q} - (p'[u,g](v,h)(0), h)_{2,\Omega}. \end{aligned}$$

By Proposition 2.1, $q = p'[u,g](v,h)$ satisfies (2.6) and we get

$$F''(u,g)((v,h),(v,h)) = \alpha\|v\|^2_{2,Q} + (q,v)_{2,Q} - (q(0),h)_{2,\Omega},$$

that is, using the definition of $z$,

$$\begin{aligned} &F''(u,g)((v,h),(v,h)) \\ &= \alpha\|v\|^2_{2,Q} + (q, \partial_t z + Az + f'(y[u,g])z)_{2,Q} - (q(0),h)_{2,\Omega}. \end{aligned}$$

An integration by parts and (2.6) give the result. ■

PROPOSITION 3.1 *For every $h$ in $W_o^{1,p}(\Omega)$, the mapping $v \mapsto F''(u,g)((v,h),(v,h))$ is weakly lsc from $L^p(Q)$ into $\mathbb{R}$.*

Proof. We know that

$$\begin{aligned} &F''(u,g)((v,h),(v,h)) \\ &= \alpha\|v\|^2_{2,Q} + \|z\|^2_{2,Q} - \int_Q p f''(y[u,g]) z^2\, dx\, dt - 2\int_\Omega q(0) h\, dx, \end{aligned}$$

where $z = y'[u,g](v,h) \overset{def}{=} z_v + z_h$, with

$$\partial_t z_v + Az_v + f'(y\,[u,g])z_v = v \text{ in } Q,$$

and

$$\partial_t z_h + A z_h + f'(y\,[u,g]) z_h = 0 \text{ in } Q,$$
$$z_h = 0 \text{ on } \Sigma,\ z_h(0) = h \text{ in } \Omega. \tag{3.12}$$

Similarly $q = p'[u,g](v,h) = q_v + q_h$ with

$$-\partial_t q_v + A^* q_v + f'(y\,[u,g]) q_v = (1 - f''(y\,[u,g]) p\,[u,g]) z_v \text{ in } Q,$$
$$q_v = 0 \text{ on } \Sigma,\ q_v(T) = 0 \text{ in } \Omega,$$

and

$$-\partial_t q_h + A^* q_h + f'(y\,[u,g]) q_h = (1 - f''(y\,[u,g]) p\,[u,g]) z_h \text{ in } Q,$$
$$q_h = 0 \text{ on } \Sigma,\ q_h(T) = 0 \text{ in } \Omega.$$

$F''(u,g)$ may be written as

$$F''(u,g)((v,h),(v,h)) = \alpha \|v\|^2_{2,Q} + Q_u(v) + Q_{(u,g)}(v,h) + Q_g(h) \tag{3.13}$$

where

$$Q_u(v) = \|z_v\|^2_{2,Q} - \int_Q p[u,g] f''(y[u,g]) z_v^2 \, dx\, dt, \tag{3.14}$$

$$Q_{(u,g)}(v,h) = 2 \int_Q z_v z_h \, dx\, dt$$
$$-2 \int_Q p[u,g] f''(y[u,g]) z_v z_h \, dx\, dt - 2 \int_\Omega q_v(0) h \, dx, \tag{3.15}$$

$$Q_g(h) = \|z_h\|^2_{2,Q} - \int_Q p[u,g] f''(y[u,g]) z_h^2 \, dx\, dt - 2 \int_\Omega q_h(0) h \, dx. \tag{3.16}$$

We know that $p \in \mathcal{C}(\overline{Q})$; moreover, $f''(y) \in \mathcal{C}(\overline{Q})$ since $f$ is $\mathcal{C}^2$ from $\mathcal{C}(\overline{Q})$ to $\mathcal{C}(\overline{Q})$. Therefore $pf''(y) \in L^\infty(Q)$.

The mapping $v \mapsto z_v$ is continuous and linear from $L^p(Q)$ to $\mathcal{C}(\overline{Q})$ and $L^2(Q)$; so, it is weakly continuous from $L^p(Q)$ to $L^2(Q)$ and $v \mapsto Q_u(v)$ is a weakly continuous quadratic form on $L^p(Q)$. The mapping $v \mapsto Q_{(u,g)}(v,h)$ is a linear weakly continuous form on $L^p(Q)$ and $Q_g(h)$ does not depend on $v$. Therefore, $v \to F''(u,g)((v,h),(v,h))$ is weakly lsc on $L^p(Q)$. ■

Note that $F''(u,g)$ is a priori defined on $L^p(Q) \times W^{1,p}_o(\Omega)$ but may be extended to $L^2(Q) \times L^2(\Omega)$:

PROPOSITION 3.2 *For any $h \in L^2(\Omega)$, the mapping $v \mapsto F''(u,g)((v,h),(v,h))$ is well-defined on $L^2(Q)$ and is weakly lsc from $L^2(Q)$ to $\mathbb{R}$.*

Proof. We have observed that $pf''(y) = p[u,g] f''(y[u,g]) \in L^\infty(Q)$; in addition, if $h \in L^2(\Omega)$ then $z_h \in L^2(0,T; H^1_o(\Omega)) \cap \mathcal{C}(0,T; L^2(\Omega))$ (see, for example, Lions-Magenes, 1968, p. 265). Similarly, $v \in L^2(Q)$ implies $z_v \in L^2(0,T; H^1_o(\Omega)) \cap \mathcal{C}(0,T; L^2(\Omega))$. Finally, since $z \in L^2(Q)$, $q \in \mathcal{C}(0,T; L^2(\Omega))$ and $q(0) \in L^2(\Omega)$, again the same regularity result yields that $F''(u,g)((v,h),(v,h))$ is well defined on $L^2(Q) \times L^2(\Omega)$.

## 3.2. Necessary optimality conditions

We end this section by recalling (classical) first and second order optimality conditions.

### 3.2.1. First order optimality conditions

PROPOSITION 3.3 *Let be $g \in W_o^{1,p}(\Omega)$. If $u$ is a solution of $(\mathcal{P}_g)$ then*

$$(F'_u(u,g), v-u)_{2,Q} \geq 0, \ \forall v \in \mathcal{K}. \tag{3.17}$$

Proof. The first order optimality condition is

$$F'_u(u,g) + N^p_{\mathcal{K}}(u) \ni 0,$$

where $N^p_{\mathcal{K}}(u)$ is the normal cone at $u$ with respect to the $L^p$-norm (we recall the definition in the sequel); $F'_u(u,g)$ is the partial derivative of $F$ with respect to $u$ at $(u,g)$. This is equivalent to

$$\langle F'_u(u,g), v-u\rangle_{p,p'} \geq 0, \ \forall v \in \mathcal{K}$$

where $\langle \cdot,\cdot\rangle_{p,p'}$ denotes the duality pairing $L^p, L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Since $F'_u(u,g)$ $= p\,[u,g] + \alpha\, u \in L^2(Q)$, we obtain (3.17) which is equivalent to

$$(p[u,g] + \alpha u, v-u)_{2,Q} \geq 0, \ \forall v \in \mathcal{K}.$$

■

### 3.2.2. Second order optimality conditions

Before we express second order optimality conditions we introduce some notations.

Let $q \in \mathbb{N} \cap [2,p]$ and $u \in \mathcal{K}$. The cone of admissible directions at $u$ (in $L^q(Q)$) is

$$R^q_{\mathcal{K}}(u) := \{y \in L^q(Q) \mid \exists \delta > 0, \ x + \delta y \in \mathcal{K}\}, \tag{3.18}$$

the tangent cone at $u$ to $\mathcal{K}$ in $L^q$:

$$T^q_{\mathcal{K}}(u) := \{v \in L^q(Q) \mid \exists \delta > 0 \text{ such that } u + \delta v + o_q(\delta) \in \mathcal{K}\}, \tag{3.19}$$

where $o_q(\delta)$ is the remainder term in the sense of the $L^q$-norm; the normal cone at $u$ (in $\mathcal{K}$) is

$$N^q_{\mathcal{K}}(u) := \{u^* \in L^{q'}(Q) \mid \langle u^*, v-u\rangle_{q,q'} \geq 0, \ \forall v \in \mathcal{K}\}. \tag{3.20}$$

Finally, the $L^q(Q)$-critical cone at $u$ is

$$C_q(u,g) := \{v \in L^q(Q) \mid F'_u(u,g)v = 0, \ v \in T^q_{\mathcal{K}}(u)\}.$$

This last definition is quite formal. In the sequel, we shall distinguish two cases the "natural" space of data ($u$ and $g$) and the state-space $\mathcal{Y}$, while the case of $q = 2$ corresponds to the hilbertian case $L^2(Q)$, for which classical regularity conditions are easier to verify.

On the other hand, the $q$-polyhedricity of $\mathcal{K}$ will be necessary to get a second order necessary optimality condition. Let us recall the definition (Haraux, 1977, Mignot, 1976)

DEFINITION 3.1 *A convex, closed set $\mathcal{K} \subset L^q(Q)$ is $q$-polyhedric at $u \in \mathcal{K}$ in the direction $v^* \in N^q_{\mathcal{K}}(u)$ if*

$$T^q_{\mathcal{K}}(u) \cap (v^*)^{\perp} = \overline{R^q_{\mathcal{K}}(u) \cap (v^*)^{\perp}} \tag{3.21}$$

Following Bonnans-Shapiro (2000), Proposition 5.33, one can show, for example, that for $a, b \in \mathbb{R}^2$ the set

$$\mathcal{K}_{a,b} = \{u \in L^q(Q) \mid a \leq u \leq b \text{ a.e. in } Q\},$$

is $L^q$-polyhedric.

THEOREM 3.3 *Let $g \in W^{1,p}_o(\Omega)$ and $u \in \mathcal{S}_g$. If $\mathcal{K}$ is $p$-polyhedric, then*

$$\forall v \in C_p(u,g) \ F''_{u^2}(u,g)(v,v) \geq 0. \tag{3.22}$$

*($F''_{u^2}$ is the second order partial derivative of $F$ with respect to $u$.)*

Proof. See Bonnans (1998a), Theorem 2.5. ■

Let us mention, as well, that if $\mathcal{K}$ is 2-polyhedric then the following relations

$$\exists \nu > 0 \ \forall v \in C_2(u,g) \ F''_{u^2}(u,g)(v,v) \geq \nu \|v\|_2^2 \tag{3.23}$$

and

$$\forall v \in C_2(u,g) - \{0\} \ F''_{u^2}(u,g)(v,v) > 0$$

are equivalent (see Bonnans, 1998a). This is due to the fact the Hessian of $F$ is a Legendre form.

## 4. First order sensitivity results

We may now give a stability result for the "solution" of $(\mathcal{P}_g)$ which can be viewed as a (upper) continuity result for the multi-function $\mathcal{S}$ which associates the set $\mathcal{S}_g$ of solutions to $(\mathcal{P}_g)$ to a given $g$.

THEOREM 4.1 *Let $(g_k)$ be a sequence weakly converging to $\bar g$ in $W^{1,p}_o(\Omega)$ and $u_k \in \mathcal{S}_{g_k}$.*

*Then, one can extract a subsequence still denoted $(u_k)$ which converges to $\bar u \in \mathcal{S}_{\bar g}$ weakly in $L^p(Q)$ and strongly in $L^2(Q)$*

Proof. As $u_k$ is a solution to $(\mathcal{P}_{g_k})$, $u_k \in \mathcal{K}$. Since $\mathcal{K}$ is bounded in $L^p(Q)$, one can extract a subsequence of $(u_k)$ still denoted $(u_k)$ that weakly converges to $\bar{u}$ in $L^p(Q)$. Since $\mathcal{K}$ is closed and convex, $\bar{u} \in \mathcal{K}$.

We prove first that $\bar{u}$ is a solution of $(\mathcal{P}_{\bar{g}})$. By Theorem 2.3, $y_k = y[u_k, g_k]$ converges to $\bar{y} = y[\bar{u}, \bar{g}]$ strongly in $L^2(Q)$. Therefore $(\bar{u}, \bar{y})$ is feasible for $(\mathcal{P}_{\bar{g}})$.

Since $u_k$ is a solution to $(\mathcal{P}_{g_k})$, then $F(u_k, g_k) \le F(u, g_k)$, $\forall u \in \mathcal{K}$. Moreover $F$ is weakly lsc from $L^p(Q) \times W_o^{1,p}(\Omega)$ to $\mathbb{R}$ (Theorem 3.1) and we get

$$F(\bar{u}, \bar{g}) \le \liminf_k F(u_k, g_k) \le \liminf_k F(u, g_k) \ \forall u \in \mathcal{K}.$$

The weak continuity of $F$ with respect to $g$ yields

$$\liminf_k F(u, g_k) = \lim_k F(u, g_k) = F(u, \bar{g}).$$

Finally,

$$F(\bar{u}, \bar{g}) \le \liminf_k F(u_k, g_k) \le F(u, \bar{g}) \ \forall u \in \mathcal{K}. \tag{4.24}$$

Therefore, $\bar{u} \in \mathcal{S}_{\bar{g}}$. Note that with $u = \bar{u}$ relation (4.24) gives

$$\lim_k F(u_k, g_k) = F(\bar{u}, \bar{g}),$$

that is

$$\lim_k J(u_k, y_k) = J(\bar{u}, \bar{y}). \tag{4.25}$$

Since $y_k$ converges to $\bar{y}$ strongly in $L^2(Q)$, relation (4.25) implies strong convergence of $u_k$ to $\bar{u}$ in $L^2(Q)$. ∎

Now we make this result more precise by estimating the rate of convergence. This will provide a first order expansion of the optimal value function with respect to the parameter $g$. This section is devoted to **first** order sensitivity analysis and we only assume that the constraint set $\mathcal{K}$ is a convex, bounded and closed subset of $L^p(Q)$. We do not need any polyhedricity assumption for the moment.

The first result is a fundamental lemma which ascertains that the remainder term of the second order expansion of $F(u, g)$ is $o((\|u\|_2 + \|g\|_2)^2)$. This is not obvious since $F$ is $\mathcal{C}^2$ from $L^p(Q) \times W_o^{1,p}(\Omega)$ to $\mathbb{R}$ and the remainder term is a priori $o((\|v\|_p + \|g\|_{1,p})^2)$ (with test functions in $L^p(Q) \times W_o^{1,p}(\Omega)$).

LEMMA 4.1 *Let $g, h \in W_o^{1,p}(\Omega)$ and $u, v \in L^p(Q)$. Let $r(u, v, g, h)$ be the remainder term of the second-order expansion of the cost functional $F$ at $(u, g)$ in the direction $(v, h)$, that is*

$$F(u + v, g + h)$$

$$= F(u, g) + F'(u, g)(v, h) + \frac{1}{2} F''(u, g)(v, h)^2 + r(u, v, g, h)$$

*If $v \to 0$ strongly in $L^2(\Omega)$ and $v$ remains bounded in $L^p(Q)$, if $h \rightharpoonup 0$ weakly in $W_o^{1,p}(\Omega)$, then*

$$\frac{|r(u,v,g,h)|}{(\|v\|_{2,Q} + \|h\|_{2,\Omega})^2} \to 0.$$

Proof. As $u$ and $g$ are fixed, we drop the dependence with respect to $u$ and $g$ in what follows and set $r(v,h) := r(u,v,g,h)$. As $F$ is $C^2$ in $L^p(Q) \times W_o^{1,p}(\Omega)$, we may write a Taylor-expansion with an integral remainder term: for any $(v,h) \in L^p(Q) \times W_o^{1,p}(\Omega)$

$$F(u+v, g+h) = F(u,g) + F'(u,g)(v,h) + \frac{1}{2}F''(u,g)(v,h)^2 + r(v,h),$$

where

$$r(v,h) = \frac{1}{2}\int_0^1 (1-s)F''(u+sv, g+sh)(v,h)^2\, ds - \frac{1}{2}F''(u,g)(v,h)^2,$$

that is

$$r(v,h) = \frac{1}{2}\int_0^1 (1-s)[F''(u+sv, g+sh) - F''(u,g)](v,h)^2\, ds.$$

Set $z = y'[u,g](v,h)$ (resp. $z_s = y'[u+sv, g+sh](v,h)$), the solution to the linearized state equation corresponding to $y := y[u,g]$ (resp. $y_s := y[u+sv, g+sh]$):

$$\begin{aligned}
&\partial_t z + Az + f'(y)z = v \text{ in } Q,\ z = 0 \text{ on } \Sigma,\ z(0) = h \text{ in } \Omega,\\
&\partial_t z_s + Az_s + f'(y_s)z_s = v \text{ in } Q,\ z_s = 0 \text{ on } \Sigma,\ z_s(0) = h \text{ in } \Omega.
\end{aligned}$$

Similarly $q = p'[u,g](v,h)$ (resp. $q_s = p'[u+sv, g+sh](v,h)$) is the solution to the linearized adjoint state equation corresponding to $p := p[u,g]$ (resp. $p_s := p[u+sv, g+sh]$):

$$\begin{aligned}
&-\partial_t q + A^*q + f'(y)q = (1 - pf''(y))z \text{ in } Q,\ q = 0 \text{ on } \Sigma,\ q(T) = 0 \text{ in } \Omega,\\
&-\partial_t q_s + A^*q_s + f'(y_s)q_s = (1 - p_s f''(y_s))z_s \text{ in } Q,\\
&q_s = 0 \text{ on } \Sigma,\ q_s(T) = 0 \text{ in } \Omega.
\end{aligned}$$

The expression for $F''$ is given by (3.9) so that

$$\begin{aligned}
r(v,h) = &\int_0^1 \int_Q (1-s)[(1 - p_s f''(y_s))z_s^2 - (1 - pf''(y))z^2]\, dx\, dt\, ds\\
&-2\int_0^1 \int_\Omega (1-s)(q_s(0) - q(0))h\, dx\, ds.
\end{aligned}$$

Therefore $|r(v,h)| \le |r_1| + |r_2| + |r_3|$, with

$$r_1 = \int_0^1 \int_Q (1-s)(1 - pf''(y))(z_s^2 - z^2)\, dx\, dt\, ds, \tag{4.26}$$

$$r_2 = \int_0^1 \int_Q (1-s)(pf''(y) - p_s f''(y_s)) z_s^2 \, dx \, dt \, ds, \tag{4.27}$$

$$r_3 = 2\int_0^1 \int_\Omega (1-s)(q_s(0) - q(0)) h \, dx \, ds. \tag{4.28}$$

$$|r_1| \le \frac{1}{2} \sup_{s\in[0,1]} \left| \int_Q (1 - pf''(y))(z_s^2 - z^2) \, dx \, dt \right|$$

$$|r_1| \le \frac{1}{2} \sup_{s\in[0,1]} \|1 - pf''(y)\|_\infty \|z_s^2 - z^2\|_{1,Q},$$

$$|r_1| \le C \sup_{s\in[0,1]} \|z_s^2 - z^2\|_{1,Q}.$$

The mapping $s \to p[u+sv, g+sh]f''(y[u+sv, g+sh])$ is uniformly continuous from $[0,1]$ to $\mathcal{C}(\overline{Q})$. Here, we use the fact that $(v,h)$ weakly converges (up to a subsequence) in $L^p(Q) \times W_o^{1,p}(\Omega)$ and Corollary 2.1. So, for any $\|v\|_2$ small enough, $\|v\|_p$ bounded and $h \rightharpoonup 0$ in $W_o^{1,p}(\Omega)$, we get $|r_2| \le o(\|z_s^2\|_{2,Q})$. Finally

$$|r(v,h)| \le C \sup_{s\in[0,1]} \|z_s^2 - z^2\|_{1,Q} + o(\|z_s^2\|_{2,Q})$$
$$+ \sup_{s\in[0,1]} \|q_s(0) - q(0)\|_{2,\Omega} \|h\|_{2,\Omega}.$$

Note that $z - z_s$ is the solution of the following linear equation

$$\begin{cases} (z-z_s)' + A(z-z_s) + f'(y)(z-z_s) = (f'(y_s) - f'(y)) z_s \text{ in } Q \\ z - z_s = 0 \text{ on } \Sigma, \ (z-z_s)(0) = 0 \text{ in } \Omega; \end{cases}$$

therefore (see for example Dautray-Lions, 1984) we obtain for any $s \in [0,1]$

$$\|z - z_s\|_{2,Q} \le \|f'(y[u+sv, g+sh]) - f'(y[u,g])\|_{2,Q} \|z_s\|_{2,Q} \le o(\|z_s\|_{2,Q}).$$

Similarly we have

$$\|q(0) - q_s(0)\|_{2,\Omega} \le \|q - q_s\|_{L^\infty(0,T;L^2(\Omega))}$$
$$\le o(\|q_s\|_{2,Q}) + o(\|z - z_s\|_{2,Q}) + o(\|z_s\|_{2,Q}).$$

Using

$$\|z_s\|_2^2 \le C(\|v\|_{2,Q}^2 + \|h\|_{2,\Omega}^2) \text{ and } \|z^2 - z_s^2\|_{1,Q} \le \|z - z_s\|_{2,Q} \|z + z_s\|_{2,Q},$$

we finally obtain

$$\|z^2 - z_s^2\|_{1,Q} \le o(\|z_s\|_2^2) \le o(\|v\|_{2,Q}^2 + \|h\|_{2,\Omega}^2)$$
$$\text{and } \|q(0) - q_s(0)\|_{2,\Omega} \le o(\|z_s\|_{2,Q}).$$

This implies

$$|r(v,h)| \le o(\|v\|_{2,Q}^2 + \|h\|_{2,\Omega}^2) + o(\|v\|_{2,Q}\|h\|_{2,\Omega} + \|h\|_{2,\Omega}^2)$$
$$= o([\|v\|_{2,Q} + \|h\|_{2,\Omega}]^2). \qquad \blacksquare$$

COROLLARY 4.1 *Let $g \in W_o^{1,p}(\Omega)$, $u, v_k \in L^p(Q)$ and let $r(u,g,v_k)$ be the remainder term of the second order expansion of the partial function $F(\cdot, g)$ at $(u,g)$ in the direction $v_k$:*

$$F(u+v_k, g) = F(u,g) + F'_u(u,g)v_k + \frac{1}{2}F''_{u^2}(u,g)v_k^2 + r(u,g,v_k).$$

*If $v_k \to 0$ strongly in $L^2(Q)$ and weakly in $L^p(Q)$, then* $\dfrac{|r(u,g,v_k)|}{\|v_k\|^2_{2,Q}} \to 0$.

COROLLARY 4.2 *Let $g, g_k \in W_o^{1,p}(\Omega)$, $u \in L^p(Q)$ and let $r(u,g,g_k)$ be the remainder term of the second order expansion of the partial function $F(u,\cdot)$ at $(u,g)$ in the direction $g_k$:*

$$F(u, g+g_k) = F(u,g) + F'_g(u,g)g_k + \frac{1}{2}F''_{g^2}(u,g)g_k^2 + r(u,g,g_k).$$

*If $g_k \rightharpoonup 0$ weakly in $W_o^{1,p}(\Omega)$, then* $\dfrac{|r(u,g,g_k)|}{\|g_k\|^2_{2,\Omega}} \to 0$.

THEOREM 4.2 *Let $(g_k)$ be a sequence convergent to $\bar{g}$ weakly in $W_o^{1,p}(\Omega)$, and $u_k \in \mathcal{S}_{g_k}$. Assume that $(\bar{u} = u_{\bar{g}}, \bar{g})$ satisfies condition (3.23); then one can extract a subsequence still denoted $(u_k)$ such that*

$$\|u_k - \bar{u}\|_2 = O(\|g_k - \bar{g}\|_2). \tag{4.29}$$

Proof. Let us assume

$$\frac{\|u_k - \bar{u}\|_2}{\|g_k - \bar{g}\|_2} \to \infty, \tag{4.30}$$

and set $\alpha_k = \dfrac{\|g_k - \bar{g}\|_2}{\|u_k - \bar{u}\|_2}$ $(\to 0)$. We are going to exhibit a critical direction at $(\bar{u}, \bar{g})$ which doest not verify (3.23). Setting

$$v_k = \frac{u_k - \bar{u}}{\|u_k - \bar{u}\|_2},$$

we get $\|v_k\|_{2,Q} = 1$ and we may extract a subsequence still denoted $(v_k)$ converging to $\bar{v}$ weakly in $L^2(Q)$. Since $v_k \in T^2_{\mathcal{K}}(\bar{u})$ (which is a closed, convex subset of $L^2(Q)$), then $\bar{v} \in T^2_{\mathcal{K}}(\bar{u})$.

We prove now that $F'_u(\bar{u},\bar{g})\bar{v} = 0$. Relation (3.17) gives $F'_u(\bar{u},\bar{g})v_k \geq 0$, for all $k$ and the second order expansion of $F$ at $(u_k, g_k)$ (together with Lemma 4.1) yields

$$F(u_k, g_k) = F(\bar{u},\bar{g}) + F'(\bar{u},\bar{g})(\|u_k - \bar{u}\|_2 v_k, g_k - \bar{g})$$
$$+ \frac{1}{2}F''(\bar{u},\bar{g})((\|u_k - \bar{u}\|_2 v_k, g_k - \bar{g}), (\|u_k - \bar{u}\|_2 v_k, g_k - \bar{g}))$$

Since

$$F'(\bar{u},\bar{g})(\|u_k-\bar{u}\|_2\, v_k, g_k-\bar{g}) = \|u_k-\bar{u}\|_2\, F'_u(\bar{u},\bar{g})\, v_k + F'_g(\bar{u},\bar{g})\,(g_k-\bar{g}),$$

we deduce

$$\begin{aligned}
&F'_u(\bar{u},\bar{g})v_k = \frac{F(u_k,g_k)-F(\bar{u},\bar{g})}{\|u_k-\bar{u}\|_2} - F'_g(\bar{u},\bar{g})\frac{g_k-\bar{g}}{\|u_k-\bar{u}\|_2}\\
&-\frac{\|u_k-\bar{u}\|_2}{2}F''(\bar{u},\bar{g})\left(\left(v_k,\frac{g_k-\bar{g}}{\|u_k-\bar{u}\|_2}\right),\left(v_k,\frac{g_k-\bar{g}}{\|u_k-\bar{u}\|_2}\right)\right)\\
&+o(\|u_k-\bar{u}\|_2(1+\alpha_k)^2).
\end{aligned}\tag{4.32}$$

By setting $h_k = \dfrac{g_k-\bar{g}}{\|u_k-\bar{u}\|_2}$ ($\|h_k\|_2=\alpha_k\to 0$), we get

$$\begin{aligned}
&F'_u(\bar{u},\bar{g})v_k = \frac{F(u_k,g_k)-F(\bar{u},\bar{g})}{\|u_k-\bar{u}\|_2} - F'_g(\bar{u},\bar{g})h_k\\
&-\frac{\|u_k-\bar{u}\|_2}{2}F''(\bar{u},\bar{g})(v_k,h_k)^2 + o(\|u_k-\bar{u}\|_2).
\end{aligned}$$

Since $u_k$ is a solution to $(\mathcal{P}_{g_k})$, we have $F(u_k,g_k)\le F(\bar{u},g_k)$ and

$$\begin{aligned}
&0\le F'_u(\bar{u},\bar{g})v_k \le \frac{F(\bar{u},g_k)-F(\bar{u},\bar{g})}{\|u_k-\bar{u}\|_2} - F'_g(\bar{u},\bar{g})h_k\\
&-\frac{\|u_k-\bar{u}\|_2}{2}F''(\bar{u},\bar{g})(v_k,h_k)^2 + o(\|u_k-\bar{u}\|_2).
\end{aligned}\tag{4.33}$$

By Corollary 4.2, we observe that

$$\frac{F(\bar{u},g_k)-F(\bar{u},\bar{g})}{\|u_k-\bar{u}\|_2} = \alpha_k\frac{F(\bar{u},g_k)-F(\bar{u},\bar{g})}{\|g_k-\bar{g}\|_2}\to 0.$$

Similarly, the continuity of $F'_g(\bar{u},\bar{g})$ and (4.30) yield

$$F'_g(\bar{u},\bar{g})h_k\to 0.$$

At last, $F''(\bar{u},\bar{g})$ is lsc and quadratic, so $-F''$ is usc and the limit of the corresponding term is zero as well. Therefore the passage to the limit in (4.33) proves that $\bar{v}$ is a critical direction ($\bar{v}\in\mathcal{C}_2(\bar{u})$).

It remains to prove that $\bar{v}$ cannot satisfy $F''_{u^2}(\bar{u},\bar{g})(\bar{v},\bar{v})\ge\nu\|\bar{v}\|_2^2$. By Corollary 4.2 if $u_g\in\mathcal{S}_g$, then

$$\begin{aligned}
&F(u_g,g)\le F(\bar{u},g) = F(\bar{u},\bar{g}) + F'_g(\bar{u},\bar{g})(g-\bar{g})\\
&+\frac{1}{2}F''_{g^2}(\bar{u},\bar{g})(g-\bar{g},g-\bar{g}) + o(\|g-\bar{g}\|_2^2).
\end{aligned}\tag{4.34}$$

In particular, for $g=g_k$, $u_g=u_k$

$$F(u_k,g_k)\le F(\bar{u},g_k) = F(\bar{u},\bar{g}) + F'_g(\bar{u},\bar{g})(g_k-\bar{g})$$

On the other hand,

$$F(u_k, g_k) = F(\bar{u}, \bar{g}) + F'(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g})$$
$$+ \frac{1}{2} F''(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g})^2 + o((\|u_k - \bar{u}\|_2 + \|g_k - \bar{g}\|_2)^2). \tag{4.36}$$

The optimality condition (3.17) implies

$$F'(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g}) \geq F'_g(\bar{u}, \bar{g})(g_k - \bar{g}).$$

Combining (4.35) and (4.36), we get

$$F''(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g})^2$$
$$\leq F''_{g^2}(\bar{u}, \bar{g})(g_k - \bar{g})^2 + o((\|u_k - \bar{u}\|_2 + \|g_k - \bar{g}\|_2)^2),$$
$$F''_{u^2}(\bar{u}, \bar{g})(u_k - \bar{u})^2 + 2F''_{ug}(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g})$$
$$\leq o((\|u_k - \bar{u}\|_2 + \|g_k - \bar{g}\|_2)^2).$$

In addition,

$$|F''_{ug}(\bar{u}, \bar{g})(u_k - \bar{u}, g_k - \bar{g})| \leq \|F''_{ug}(\bar{u}, \bar{g})\| \|u_k - \bar{u}\|_2 \|g_k - \bar{g}\|_2,$$

so that

$$F''_{u^2}(\bar{u}, \bar{g})(u_k - \bar{u}, u_k - \bar{u}) \leq M\alpha_k \|u_k - \bar{u}\|_2^2 + o(\|u_k - \bar{u}\|_2^2(1 + \alpha_k))^2$$

where $M > 0$ is a constant independent of $k$. As $\alpha_k$ converges to 0 we finally obtain

$$F''_{u^2}(\bar{u}, \bar{g})(u_k - \bar{u}, u_k - \bar{u}) \leq o(\|u_k - \bar{u}\|_2^2). \tag{4.37}$$

Then we pass to the inf limit and use the weak lower semicontinuity of $F''_{u^2}(\bar{u}, \bar{g})$; this gives

$$F''_{u^2}(\bar{u}, \bar{g})(\bar{v}, \bar{v}) \leq \liminf F''_{u^2}(\bar{u}, \bar{g})(v_k, v_k)$$
$$\leq \limsup F''_{u^2}(\bar{u}, \bar{g})(v_k, v_k) \leq 0. \tag{4.38}$$

The proof is achieved as soon as we have proved that $\bar{v} \neq 0$. By Theorem 3.2 we have

$$F''_{u^2}(\bar{u}, \bar{g})(v, v)$$
$$= \alpha \|v\|_{2,Q}^2 + \int_Q (1 - p[\bar{u}, \bar{g}](x, t) f''(y[\bar{u}, \bar{g}])) z_v^2(x, t)\, dx\, dt. \tag{4.39}$$

Let us set

$$Q_{(\bar{u}, \bar{g})}(v) = \int_Q (1 - p[\bar{u}, \bar{g}] f''(y[\bar{u}, \bar{g}])) z_v^2(x, t)\, dx\, dt$$

where $z_v$ is given by (3.11). By the weak continuity of $Q_{(\bar u,\bar g)}$ we have

$$Q_{(\bar u,\bar g)}(\bar v) = \lim Q_{(\bar u,\bar g)}(v_k) \le \liminf Q_{(\bar u,\bar g)}(v_k)$$
$$= \liminf F''_{u^2}(\bar u,\bar g)(v_k,v_k) - \alpha\|v_k\|^2_{2,Q} \le -\alpha < 0.$$

Therefore $\bar v$ cannot be zero: we have found a nonzero critical direction which does not satisfy (4.38). ∎

Let us define now the "optimal value" function for problem $(\mathcal{P}_g)$:

$$\mathcal{V}(g) = F(u_g, g) = J(u_g, y[u_g, g]) \text{ for every } u_g \in \mathcal{S}_g. \tag{4.40}$$

The previous study shows that $\mathcal{V}$ is weakly continuous at any $\bar g \in W^{1,p}_o(\Omega)$. This is a "zero-order" result. Now, we look for a higher order representation of $\mathcal{V}$ in a neighborhood of $\bar g$ and we have to perform a second order analysis.

## 5. Second order sensitivity analysis

### 5.1. A linear problem associated to $(\mathcal{P}_g)$

It is known that under certain regularity assumptions the solution of a nonlinear control problem is also the solution of the linearized problem (see Zowe and Kurcyusz, 1979). We enounce here a similar result

THEOREM 5.1 *Let be $g_k = \bar g + t_k h_k$, where $t_k > 0$, $t_k \to 0^+$ and $\|h_k\|_{1,p} = 1$; then the solution $\bar u$ to $(\mathcal{P}_{\bar g})$ (given by Theorem 4.1) is also a solution to*

$$\min \quad \{F'_g(u,\bar g)h \mid u \in \mathcal{S}_{\bar g}\}. \tag*{$(\mathcal{P}^{\ell}_{\bar g,h})$.}$$

*where $h \in W^{1,p}_o(\Omega)$ is a weak cluster point of the sequence $(h_k)$.*

Proof. Since $\|h_k\|_{1,p} = 1$, one may extract a subsequence still denoted $(h_k)$ which converges to some $h$ weakly in $W^{1,p}_o(\Omega)$ (and strongly in $\mathcal{C}(\overline{\Omega})$). Let $\bar u \in \mathcal{S}_{\bar g}$ be the solution of $(\mathcal{P}_{\bar g})$ given by Theorem 4.1. We prove that $\bar u$ is solution to $(\mathcal{P}^{\ell}_{\bar g,h})$ as well, that is

$$\forall u \in \mathcal{S}_{\bar g} \; F'_g(\bar u,\bar g)\,h \le F'_g(u,\bar g)\,h.$$

Let us choose $u \in \mathcal{S}_{\bar g}$ $(\mathcal{V}(\bar g) = F(u,\bar g))$ and let $u_k$ be a solution of $(\mathcal{P}_{g_k})$:

$$\mathcal{V}(g_k) \le F(u, g_k)$$
$$= F(u,\bar g) + t_k F'_g(u,\bar g)h_k + \frac{t_k^2}{2} F''_{g^2}(u,\bar g)(h_k,h_k) + o(t_k^2\|h_k\|^2_{1,p}),$$
$$\frac{\mathcal{V}(g_k) - \mathcal{V}(\bar g)}{t_k} \le F'_g(u,\bar g)h_k + \frac{t_k}{2} F''_{g^2}(u,\bar g)(h_k,h_k) + o(t_k),$$

and

$$\limsup \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar g)}{t_k} \le F'_g(u,\bar g)h, \quad \forall u \in \mathcal{S}_{\bar g}.$$

Setting

$$\mathcal{V}'_+(\bar{g};h) = \limsup_k \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g})}{t_k},$$

we have just proved

$$\mathcal{V}'_+(\bar{g};h) \leq \inf\{F'_g(u,\bar{g})h \mid u \in \mathcal{S}(\mathcal{P}_{\bar{g}})\}. \tag{5.41}$$

On the other hand

$$\mathcal{V}(g_k) = F(u_k, g_k) = F(u_k, \bar{g}) + t_k F'_g(u_k, \bar{g})h_k + o(t_k).$$

Recall that $\mathcal{V}(\bar{g}) = F(\bar{u}, \bar{g}) \leq F(u_k, \bar{g})$ and

$$\frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g})}{t_k} \geq F'_g(u_k, \bar{g})h_k + o(1);$$

setting

$$\mathcal{V}'_-(\bar{g};h) = \liminf_k \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g})}{t_k},$$

we get

$$\mathcal{V}'_-(\bar{g};h) \geq \liminf_k F'_g(u_k, \bar{g})h_k. \tag{5.42}$$

We have seen (Theorem 4.1) that $u_k$ converges to $\bar{u}$ weakly in $L^p(Q)$ and strongly in $L^2(Q)$. Moreover by (3.8), $F'_g(u_k, \bar{g})h_k = -(p[u_k, \bar{g}](0), h_k)_{2,\Omega}$. Since $\bar{g}$ is fixed, Theorem 2.3 yields that $y[u_k, \bar{g}]$ converges to $y[\bar{u}, \bar{g}]$ strongly in $L^\infty(\overline{Q})$. Therefore $p[u_k, \bar{g}]$ converges to $p[\bar{u}, \bar{g}]$ strongly in $\mathcal{C}(\overline{Q})$ and $p[u_k, \bar{g}](0)$ converges to $p[\bar{u}, \bar{g}](0)$ strongly in $L^\infty(\Omega)$. Finally

$$F'_g(u_k, \bar{g})h_k \to F'_g(\bar{u}, \bar{g})h \text{ strongly in } L^2(\Omega)$$

and with (5.41) and (5.42), we obtain

$$F'_g(\bar{u}, \bar{g})h \leq \mathcal{V}'_-(\bar{g};h) \leq \mathcal{V}'_+(\bar{g};h) \leq F'_g(\bar{u}, \bar{g})h,$$

that is

$$\mathcal{V}'(\bar{g})h = \lim_{k \to +\infty} \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g})}{t_k} = F'_g(\bar{u}, \bar{g})h. \qquad \blacksquare$$

REMARK 5.1 *Theorem 5.1 remains valid if we choose a sequence $g_k$ strongly convergent to some $\bar{g}$ in $W^{1,p}_o(\Omega)$. Indeed, one chooses*

$$t_k = \|\bar{g} - g_k\|_{1,p} \ (\to 0) \text{ and } h_k = \frac{\bar{g} - g_k}{t_k}.$$

*Note that the weak convergence of any sequence $g_k$ is not sufficient since we do not know how $t_k = \|\bar{g} - g_k\|_{1,p}$ behaves.*

*Moreover, Theorem 5.1 is not valid if the sequence $h_k$ is strongly convergent to $h$ in $L^2(\Omega)$ without any further assumption; indeed, we cannot use Theo-*

### 5.2. A quadratic auxiliary problem associated to $(\mathcal{P}_g)$

Since we want to perform a second order analysis, it is natural to introduce a "quadratic approximation of the original problem" $(\mathcal{P}_g)$. We may also remark that it is the basic idea of SQP methods. Let us consider the following quadratic problem which corresponds to the formal second order expansion of the functional $F$ at $(u,g)$, for a fixed $h \in W_o^{1,p}(\Omega)$:

$$\begin{cases} \min \quad F''(u,g)((v,h),(v,h)) \\ v \in C_2(u,g) \end{cases} \qquad (\mathcal{Q}_{u,g,h})$$

as it has been previously set in Bonnans (1998b). Remark that

$$F''(u,g)((v,h),(v,h)) = \alpha\|v\|_{2,Q}^2 + Q_u(v) + Q_{(u,g)}(v,h) + Q_g(h)$$

where $Q_u$, $Q_{(u,g)}$ and $Q_g$ are given, respectively, by (3.14), (3.15) and (3.16). The direction $h$ is fixed so that the minimum is to be taken with respect to the variable $v$; therefore, the objective function of $(\mathcal{Q}_{u,g,h})$ turns to be

$$\Psi[u,g,h](v) = \alpha\|v\|_{2,Q}^2 + Q_u(v) + Q_{(u,g)}(v,h).$$

THEOREM 5.2 *Assume that the weak second order condition (3.23) holds at $(u,g) \in L^p(Q) \times W_o^{1,p}(\Omega)$; then, for any $h \in L^2(\Omega)$ problem $(\mathcal{Q}_{u,g,h})$ has at least one solution.*

Proof. We have seen (Theorem 3.2) that for every fixed $h \in L^2(\Omega)$, the mapping $v \mapsto F''(u,g)((v,h),(v,h))$ is weakly lsc. As $\mathcal{C}_2(u,g)$ is convex and $L^2(Q)$-closed, it is sufficient to prove that $v \mapsto \Psi[u,g,h](v)$ is coercive. We have

$$\Psi[u,g,h](v) = \alpha\|v\|_{2,Q}^2 + Q_u(v) + Q_{(u,g)}(v,h).$$

where

$$Q_u(v) = -\int_Q (1 - p[u,g]f''(y[u,g]))z_v^2\,dx\,dt, \text{ and}$$

$$Q_{(u,g)}(v,h) = 2\int_Q (1 - p[u,g]f''(y[u,g]))z_v z_h\,dx\,dt - 2\int_\Omega q_v(0)h\,dx.$$

Assumption (3.23) at $(u,g)$ yields

$$\alpha\|v\|_{2,Q}^2 + Q_u(v) = F''_{u^2}(u,g)(v,v) \geq \nu\|v\|_{2,Q}^2 \text{ with } \nu > 0.$$

We use the Cauchy-Schwartz inequality to estimate $Q_{(u,g)}(v,h)$:

$$\int_Q z_v z_h\,dx\,dt \geq -\|z_v\|_{2,Q}\|z_h\|_{2,Q}.$$

The mapping $v \mapsto z_v$ is linear, continuous from $L^2(Q)$ to $L^2(Q)$: $\exists c_1 > 0$ such that

that is

$$\begin{aligned}
&\int_Q p[u,g]f''(y[u,g])z_v z_h\,dx\,dt\\
&\geq -\|p[u,g]f''(y[u,g])\|_\infty \int_Q z_v z_h\,dx\,dt,\\
&\geq -\|p[u,g]f''(y[u,g])\|_\infty \|z_v\|_{2,Q}\|z_h\|_{2,Q},\\
&\geq -c_1\|p[u,g]f''(y[u,g])\|_\infty \|v\|_{2,Q}\|z_h\|_{2,Q}.
\end{aligned}$$

Similarly

$$-\int_Q q_v(0)h\,dx \geq -c_2\|v\|_{2,Q}\|h\|_{2,Q}.$$

Finally

$$\begin{aligned}
&Q_{(u,g)}(v,h) \geq \|v\|_{2,Q}(-c_1\|z_h\|_{2,Q}\\
&-c_1\|p[u,g]f''(y[u,g])\|_\infty\|z_h\|_{2,Q} - c_2\|h\|_{2,Q}),
\end{aligned} \tag{5.43}$$

that is

$$\Psi[u,g,h](v) \geq \|v\|_{2,Q}[\nu\|v\|_{2,Q} - C(h,u,g)],$$

where $C(h,u,g)$ is a constant depending only on $h$, $u$ and $g$. ■

Let us call $\mathsf{V}(\mathcal{Q}_{u,g,h})$ the optimal value function for $(\mathcal{Q}_{u,g,h})$:

$$\mathsf{V}(\mathcal{Q}_{u,g,h}) = \min(\mathcal{Q}_{u,g,h}) = \min\{F''(u,g)((v,h),(v,h)) \mid v \in C_2(u,g)\}$$

### 5.3. Use of polyhedricity

We would like to get a $L^2$-expansion of the optimal value function $\mathcal{V}$ because the (weak) second order sufficient coercivity condition is satisfied only in $L^2$-norm. Unfortunately, we have seen that this function (via function $F$) is differentiable only if the state function belongs to $\mathcal{C}(\overline{Q})$ (that is why we consider a control function in $L^p(Q)$ and perturbed initial data in $W_o^{1,p}(\Omega)$). We will use the coercivity condition in $L^2$-norm. There is a gap between the two norms: this is the two-norm discrepancy phenomenon. The tool that will help us to solve the problem (to overcome the difficulty connected with the gap) is the polyhedricity of the control constraints set $\mathcal{K}$. Of course, the polyhedricity is also useful without the norm discrepancy (see Haraux, 1977, Mignot, 1976: it may be useful to control variational inequalities for example).

THEOREM 5.3 *Consider a sequence $(g_k) \in W_o^{1,p}(\Omega)$ weakly convergent to $\bar{g}$.*

*(i) If $(\bar{u}, \bar{g})$ satisfies (3.23) then $\mathcal{V}$ admits the following second-order expansion:*

$$\begin{aligned}
&\mathcal{V}(g_k) = \mathcal{V}(\bar{g}) + F'_g(\bar{u}, \bar{g})(g_k - \bar{g}) \\
&+ \frac{\|g_k - \bar{g}\|_2^2}{2} \mathsf{V}(\mathcal{Q}_{\bar{u},\bar{g},h}) + o(\|g_k - \bar{g}\|_2^2),
\end{aligned} \tag{5.44}$$

*where $h$ is a weak cluster point of $\dfrac{g_k - \bar{g}}{\|g_k - \bar{g}\|_2}$ in $L^2(\Omega)$.*

*(ii) If $v$ is a weak cluster point of $\dfrac{u_k - \bar{u}}{\|g_k - \bar{g}\|_2}$ in $L^2(Q)$, then $v$ is a strong cluster point of $\dfrac{u_k - \bar{u}}{\|g_k - \bar{g}\|_2}$ in $L^2(Q)$ and it is a solution to $(\mathcal{Q}_{\bar{u},\bar{g},h})$.*

Proof. (i) Theorem (4.2) yields $\|u_k - \bar{u}\|_2 = O(\|g_k - \bar{g}\|_2)$. Therefore the sequence

$$v_k := \frac{u_k - \bar{u}}{\|g_k - \bar{g}\|_2} = \frac{u_k - \bar{u}}{t_k}$$

is bounded in $L^2(Q)$ and there exists $v \in L^2(Q)$ such that a subsequence of $(v_k)$ converges to $v$ weakly in $L^2(Q)$.

**Let us show that $v$ is a critical direction.** The proof is similar to that given in Theorem 4.2: we consider now the $L^2$-norm instead of the $W^{1,p}$ norm. We set

$$h_k = \frac{g_k - \bar{g}}{\|g_k - \bar{g}\|_2} \text{ and } t_k = \|g_k - \bar{g}\|_2.$$

The second order expansion of $F$ and Lemma 4.1 gives

$$\begin{aligned}
&F(u_k, g_k) - F(\bar{u}, \bar{g}) = t_k F'(\bar{u}, \bar{g})(v_k, h_k) \\
&+ \frac{t_k^2}{2} F''(\bar{u}, \bar{g})(v_k, h_k)^2 + o(t_k^2[\|v_k\|_2 + \|h_k\|_2]^2).
\end{aligned}$$

Furthermore, by Theorem 4.2, $\|v_k\|_2 = O(\|h_k\|_2)$, so that the remainder term in the previous expression is $o(t_k^2\|h_k\|_2^2) = o(t_k^2)$. We obtain

$$\begin{aligned}
&F(u_k, g_k) = F(\bar{u}, \bar{g}) + t_k F'(\bar{u}, \bar{g})(v_k, h_k) \\
&+ \frac{t_k^2}{2} F''(\bar{u}, \bar{g})((v_k, h_k), (v_k, h_k)) + o(t_k^2). \\
&0 \leq F'_u(\bar{u}, \bar{g}) v_k = \frac{F(u_k, g_k) - F(\bar{u}, \bar{g})}{t_k} - F'_g(\bar{u}, \bar{g})\, h_k \\
&- \frac{t_k}{2} F''(\bar{u}, \bar{g})((v_k, h_k), (v_k, h_k)) + o(t_k).
\end{aligned}$$

Since $F(u_k, g_k) \leq F(\bar{u}, g_k)$, we get

$$0 \leq F'_u(\bar{u}, \bar{g}) v_k \leq \frac{F(\bar{u}, g_k) - F(\bar{u}, \bar{g})}{t_k} - F'_g(\bar{u}, \bar{g}) h_k$$

$t_k$

Passing to the limit yields that $v$ is a critical direction: indeed, using Corollary 4.2 gives

$$\frac{F(\bar{u}, g_k) - F(\bar{u}, \bar{g})}{t_k} = F_g'(\bar{u}, \bar{g}) h_k + o(t_k).$$

On the other hand, by (3.9)

$$F''(\bar{u}, \bar{g})((v_k, h_k), (v_k, h_k)) = \alpha \|v_k\|_{2,Q}^2 + \|y'[\bar{u}, \bar{g}](v_k, h_k)\|_{2,Q}^2 \\ - \int_Q p[\bar{u}, \bar{g}] f''(y[\bar{u}, \bar{g}]) y'[\bar{u}, \bar{g}](v_k, h_k)^2 \, dx \, dt - 2 \int_\Omega p'[\bar{u}, \bar{g}](v_k, h_k)(0) h_k \, dx;$$

this term remains bounded as $k \to +\infty$ and we finally obtain $F_u'(\bar{u}, \bar{g}) v = 0$.

Now, we give lower and upper estimates of

$$\frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g}) - t_k F_g'(\bar{u}, \bar{g}) h_k}{t_k^2},$$

and we start with a **lower estimate**: from (5.45) we have

$$F(u_k, g_k) - F(\bar{u}, \bar{g}) - t_k F_g'(\bar{u}, \bar{g}) \, h_k \geq \frac{t_k^2}{2} F''(\bar{u}, \bar{g})(v_k, h_k)^2 + o(t_k^2).$$

Passing to the inf-limit and using the weak lower semicontinuity of $F_{u^2}''(\bar{u}, \bar{g})(v_k, v_k)$ we get

$$\liminf_k \frac{F(u_k, g_k) - F(\bar{u}, \bar{g}) - t_k F_g'(\bar{u}, \bar{g}) h_k}{t_k^2} \geq \frac{1}{2} F''(\bar{u}, \bar{g})(v, h)^2.$$

Since $v \in C_2(\bar{u}, \bar{g})$, we obtain

$$\liminf_k \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g}) - F_g'(\bar{u}, \bar{g})(g_k - \bar{g})}{t_k^2} \geq \frac{1}{2} \mathsf{V}(\mathcal{Q}_{\bar{u}, \bar{g}, h}). \tag{5.46}$$

**Upper estimate**: let $u_k$ be a solution to $(\mathcal{P}_{g_k})$:

$$\mathcal{V}(g_k) = F(u_k, g_k) \leq F(u, \bar{g}_k), \ \forall u \in \mathcal{K}.$$

Let $w \in R_{\mathcal{K}}^2(\bar{u}) \cap F_u'(\bar{u}, \bar{g})^\perp$ and $u := \bar{u} + t_k w$; then

$$\mathcal{V}(g_k) \leq F(\bar{u} + t_k w, g_k).$$

Again, we perform a second order expansion of $F$

$$F(\bar{u} + t_k w, g_k) = F(\bar{u}, \bar{g}) + t_k F'(\bar{u}, \bar{g})(w, h_k) + \frac{t_k^2}{2} F''(\bar{u}, \bar{g})(w, h_k)^2 + o(t_k^2).$$

Since $F_u'(\bar{u}, \bar{g}) w = 0$, for every $w \in C_2(\bar{u}, \bar{g})$, we obtain

$$F(\bar{u} + t_k w, g_k) = F(\bar{u}, \bar{g}) + t_k F_g'(\bar{u}, \bar{g}) h_k + \frac{t_k^2}{2} F''(\bar{u}, \bar{g})(w, h_k)^2 + o(t_k^2),$$

$$\frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g}) - t_k F_g'(\bar{u}, \bar{g}) h_k}{t_k^2} \leq \frac{1}{2} F''(\bar{u}, \bar{g})(w, h_k)^2 + o(1).$$

and

$$\limsup_k \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g}) - t_k F'_g(\bar{u},\bar{g})h_k}{t_k^2} \le \frac{1}{2} F''(\bar{u},\bar{g})(w,h)^2. \tag{5.47}$$

**We use here the 2-polyhedricity of $\mathcal{K}$:** the previous inequality is a priori valid for any $w \in R^2_{\mathcal{K}}(\bar{u}) \cap F'_u(\bar{u},\bar{g})^\perp$; therefore it is also valid for any $w \in T^2_{\mathcal{K}}(\bar{u}) \cap F'_u(\bar{u},\bar{g})^\perp$, that is, at least for any $w \in C_2(\bar{u},\bar{g})$. Consequently,

$$\begin{aligned}&\limsup_k \frac{\mathcal{V}(g_k) - \mathcal{V}(\bar{g}) - t_k F'_g(\bar{u},\bar{g})h_k}{t_k^2} \\ &\le \frac{1}{2} \min_{w \in C_2(\bar{u},\bar{g})} F''(\bar{u},\bar{g})(w,h)^2 = \frac{1}{2} \mathsf{V}(\mathcal{Q}_{\bar{u},\bar{g},h}).\end{aligned} \tag{5.48}$$

This completes the proof of (i).

**Let us demonstrate (ii).** Relation (3.9) of Theorem 3.2 yields

$$\|v_k\|_2^2 = \frac{1}{\alpha}\{F''(\bar{u},\bar{g})(v_k,h_k)^2 - \mathsf{Q}_{\bar{u},\bar{g}}(v_k,h_k)\},$$

where

$$\begin{aligned}&\mathsf{Q}_{\bar{u},\bar{g}}(v_k,h_k) = \|z_k\|^2_{2,Q} - \int_Q p[\bar{u},\bar{g}] f''(y[\bar{u},\bar{g}]) z_k^2(x,t)\,dx\,dt \\ &- 2\int_\Omega q(x,0) h_k(x,0)\,dx,\end{aligned}$$

and $z_k = y'[\bar{u},\bar{g}](v_k,h_k)$. We have seen that $\mathsf{Q}_{\bar{u},\bar{g}}$ is weakly continuous. In addition, point (i) gives

$$\lim_k F''(\bar{u},\bar{g})(v_k,h_k) = F''(\bar{u},\bar{g})(v,h) = \mathsf{V}(\mathcal{Q}_{\bar{u},\bar{g},h}).$$

Therefore $v_k$ strongly converges to $v$ in $L^2(Q)$. ■

REMARK 5.2 *Theorem 5.3 is of course valid if we choose a $W^{1,p}_o(\Omega)$ expansion $h$ of $g$ around $\bar{g}$. Indeed, we are in the $L^2$-frame and the $W^{1,p}_o(\Omega)$ weak convergence is sufficient to ensure the strong one in $L^2(\Omega)$. This theorem is stronger than Theorem 5.1. Though $\mathcal{V}$ is not differentiable in $L^2(\Omega)$, we are able to give an expansion for tests functions in $L^2(\Omega)$.*

**Acknowledgment.** We would like to thank the anonymous referees for all their suggestions that helped us to improve this paper.

## References

AHMED, N.U. and XIANG, X. (1997) Nonlinear uncertain systems and necessary conditions of optimality. *SIAM Journal on Control and Optimization*, ARADA, N., BERGOUNIOUX, M. and RAYMOND, J.P. (2000) Minimax controls for uncertain distributed parabolic systems. *SIAM J. Control and Optimization*, **38**, 5, 1481–1500.

ARADA, N. and RAYMOND, J.P. (1998) Dirichlet boundary control of semilinear parabolic equations. Part 1: Problems with no state constraints. UMR CNRS 5640, *Report n° 98-05.*

ARADA, N. and RAYMOND, J.P. (1999) Optimality Conditions for State-Constrained Dirichlet Boundary Control Problems. *J. Optimization Theory and Applications*, **102**, n° 1, 51–68.

BERGOUNIOUX, M. and TRÖLTZSCH, F. (1996) Optimality conditions and generalized Bang-Bang principle for a state-constrained semilinear parabolic problem. *Numerical Functional Analysis and Optimization*, **17**, n° 5 & 6, 517–537.

BERGOUNIOUX, M. and ZIDANI, H. (1999) Pontryagin Maximum Principle For Optimal Control of Variational Inequalities. *SIAM Journal on Control and Optimization*, **37**, n° 4, 1273–1290.

BONNANS, J.F. (1998a) Second order analysis for control constrained optimal control problems of semilinear elliptic systems. *Applied Mathematics and Optimization*, **38**, 303–325.

BONNANS, J.F. (1998b) Extended Quadratic Tangent Optimization Problems. *Lecture Notes in Pure and Applied Mathematics*, 3 vol. 195, Mathematical Programming with Data Perturbations, A.V. Fiaccon, ed., 31–45.

BONNANS, J.F. and COMINETTI, R. (1996a) Perturbed optimization in Banach spaces I: a general theory based on a weak directional constraint qualification. *SIAM Journal on Control and Optimization*, **34**, n° 4, 1151–1171.

BONNANS, J.F. COMINETTI, R. (1996b) Perturbed optimization in Banach spaces II: a general theory based on a strong directional constraint qualification. *SIAM Journal on Control and Optimization*, **34**, n° 4, 1172–1189.

BONNANS, J.F. and SHAPIRO, A. (2000) *Perturbed Analysis of Optimization Problems.* Springer-Verlag, New York.

DAUTRAY, R. and LIONS, J.L. (1984) *Analyse mathématique et calcul numérique pour les sciences et les techniques. Evolution: semi-groupe, variationnel.* Tome 8, Masson.

HARAUX, A. (1997) How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *Journal Mathematical Society of Japan*, **29**, n° 4.

LIONS, J.L. (1969) *Quelques méthodes de résolution de problèmes aux limites non linéaires.* Dunod, Paris.

LIONS, J.L. and MAGENES, E. (1968) *Problèmes aux limites non homogènes et applications.* Dunod, Paris.

MALANOWSKI, K. (1995) Stability and Sensitivity of Solutions to Nonlinear Optimal Control problems. *Applied Mathematics and Optimization*, **32**, MALANOWSKI, K. MAURER, H. (1996) Sensitivity Analysis for Parametric Control problems with Control-State Constraints. *Computational Optimization and Applications*, **5**, 253–283.

MALANOWSKI, K. and TRÖLTZSCH, F. (1999) Lipschitz stability of solutions to parametric optimal control for parabolic equations. *Journal of Analysis and its Applications (ZAA)*, **18**, 2, 469–489.

MAURER, H. and PESCH, H.J. (1994) Solution differentiability for nonlinear parametric control problems. *SIAM Journal on Control and Optimization*, **32**, n° 6, 1542–1554.

MERABET, N. (2000) Analyse de sensibilité de problèmes de contrôle optimal gouvernés par une équation semilinéaire parabolique. *PhD Thesis*, Université d'Orléans.

MIGNOT, F. (1976) Contrôle dans les inéquations variationnelles elliptiques. *Journal of Functional Analysis*, **22**, 130–185.

RAYMOND, J.P. and ZIDANI, H. (1998) Pontryagin's Principles for State-Constrained Control Problems Governed by Semilinear Parabolic Equations with unbounded Controls. *SIAM Journal on Control and Optimization*, **36**, n° 6, 1853–1879.

SOKOLOWSKI, J. (1987) Sensitivity analysis of control constrained optimal problem for distributed parameter systems. *Siam Journal on Control and Optimization*, **25**, n° 6, 1542–1556.

TRÖLTZSCH, F. (1996) A stability theorem for linear-quadratic parabolic control problems. *Control of partial differential equations and applications.* Lecture Notes in Pure and Applied Mathematics, **174**, E. Casas, ed., Marcel Dekker, New York 1996, 287–296.

TRÖLTZSCH, F. (1997) Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations. Report 97-12, TU-Chemnitz, to appear in *Dynamics of Continuous, Discrete and Impulsive Systems.*

ZOWE, J. and KURCYUSZ, S. (1979) Regularity and stability for the mathematical programming problem in Banach spaces. *Applied Mathematics and Optimization*, **5**, 49–62.