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Dynamic flows with supply and demand

by

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Abstract: We are given a network $G = (N, A, h, c)$ with node set N, arc set A, time function h, capacity function c, and P the set of periods, s the source and s' the sink of the network G. Associated with s, there is a non-negative real number $q(t)$ called the supply of source s at time t, and with s' — a nonnegative real number $q'(t)$ called the demand of sink s' at time t, $t \in P$. The objective is to determine the existence of a dynamic flow in G for p periods, so that the demands at sink s' can be fulfilled from the supplies at the source s. A numerical example is presented.

Keywords: dynamic flows, networks, graph algorithms.

1. Dynamic flows

Let $G = (N, A)$ be a connected digraph with $N = \{s, \ldots, x, \ldots, s'\}$, $|N| = n$ the node set and $A = \{(x, y)/x, y \in N\}$, $|A| = m \leq n(n-1)$ the arc set. Let $\mathbb{R}$ be the set of real numbers and $P = \{0, 1, \ldots, p\}$ the set of periods. Let us state the time function $h : A \to \mathbb{R}$ and the capacity function $c : A \times P \to \mathbb{R}$, where $h(x, y)$ represents the arc transit time and $c(x, y; t)$ the arc capacity at time t for $(x, y) \in A$, $t \in P$.

Let us designate by s the source and by s' the sink of the dynamic network $G = (N, A, h, c)$.

The dynamic flows problem for p time periods may be formulated as follows. Let us determine the function $f : A \times P \to \mathbb{R}$, which should satisfy the following relations:

$$\sum_{t=0}^{p} \Big(\sum_{y} f(s, y; t) - \sum_{y} f(y, s; t') \Big) = v(P), \tag{1a}$$

$$\sum_{y} f(x, y; t) - \sum_{y} f(y, x; t') = 0, \ x \neq s, s', \ t \in P, \tag{1b}$$

$$\sum_{t=0}^{p} \Big(\sum_{y} f(s', y; t) - \sum_{y} f(y, s'; t') \Big) = -v(P), \tag{1c}$$

where $t' = t - h(y,x)$, $v(P) = \sum_{t=0}^{p} v(t)$, $v(t)$ is the flow value at time t and $f(x,y;t)$ is the amount that leaves x along (x,y) at time t, consequently, arriving at y at $t + h(x,y)$.

If $f(x,y;t)$ and $v(P)$ satisfy (1) and (2), we call f a dynamic flow from s to s' for p periods with value $v(P)$. If $v(P)$ is also maximal, then f is a maximal dynamic flow.

Ford and Fulkerson (1962) have shown that a dynamic flow for p time periods in the dynamic network $G = (N, A, h, c)$ can be represented as a static flow in the static network $G(p) = (N(p), A(p), c)$, where

$$N(p) = \{x(t) \mid x \in N,\ t \in P\}$$
$$A(p) = \{(x(t), y(t')) \mid (x,y) \in A;\ t, t' \in P,\ t' = t + h(x,y)\}$$
$$c(x(t), y(t')) = c(x,y;t),\ (x,y) \in A;\ t, t' \in P.$$

In Ciurea (1979, 1995) it is shown that a dynamic flow for p time periods in the dynamic network $G = (N, A, h, c)$ is equivalent to a dynamic flow for the same time periods in the dynamic network $G'(p) = (N'(p), A'(p), h', c')$ and can be represented as a static flow in the static network $G^*(p) = (N^*(p), A^*(p), c^*)$.

The networks $G'(p)$ and $G^*(p)$ may be constructed as follows. Let $d(s,x)$ be the length of the shortest route from the source s to the node x and $d(x,s')$ — the length of the shortest route from node x to the sink s', with respect to $h(x,y)$. Computation of $d(s,x), d(x,s')$ for all $x \in N$ is performed by means of the usual shortest path algorithms and is not discussed in this paper. Let us consider:

$$P(x) = \{t \mid t \in P,\ d(s,x) \le t \le p - d(x,s')\},\ x \in N,$$
$$P(x,y) = \{t \mid t \in P,\ d(s,x) \le t \le p - (h(x,y) + d(y,s'))\},\ (x,y) \in A.$$

The network $G'(p) = (N'(p), A'(p), h', c')$ may be constructed as follows:

$$N'(p) = \{x \mid x \in N,\ P(x) \ne \emptyset\},$$
$$A'(p) = \{(x,y) \mid (x,y) \in A,\ P(x,y) \ne \emptyset\}$$
and h', c' are the restrictions to $A'(p)$.

The network $G^*(p) = (N^*(p), A^*(p), c^*)$ is constructed from network $G'(p)$ in the following manner:

$$N^*(p) = \{x(t) \mid x \in N'(p),\ t \in P(x)\},$$
$$A^*(p) = \{(x(t), y(t')) \mid (x,y) \in A'(p),\ t \in P(x,y),\ t' = t + h(x,y)\},$$
$$c^*(x(t), y(t')) = c(x,y;t),\ (x,y) \in A'(p),\ t \in P(x,y).$$

The network $G'(p)$ is, in general, a partial subnetwork of G and the network

2. Dynamic flows with supply and demand

Associate with s a nonnegative real number $q(t)$ called the supply of source s at time t, and with s' — a nonnegative real number $q'(t)$ called the demand of sink s' at time t, $t \in P$.

The objective is to determine the existence of a dynamic flow in G so that the demands at the sink can be fulfilled from the supplies at the source s, satisfying the constraints

$$\sum_y f(s,y;t) - \sum_y f(y,s;t') \le q(t), t \in P, \tag{3a}$$

$$\sum_y f(x,y;t) - \sum_y f(y,x;t') = 0,\ x \neq s, s',\ t \in P, \tag{3b}$$

$$\sum_y f(y,s';t') - \sum_y f(s',y;t) \ge q'(t);\ t \in P, \tag{3c}$$

$$0 \le f(x,y;t) \le c(x,y;t),\ (x,y) \in A,\ t \in P, \tag{4}$$

where $t' = t - h(y,x)$.

If such a solution exists, we say that the constraints (3), (4) are feasible. Otherwise, they are infeasible.

Let us consider:

$$\begin{aligned}
&X(p) \subset N(p),\ \overline{X}(p) = N(p) \setminus X(p),\\
&S(p) = \{s(t) \mid t \in P\},\ \bar{S}(p) = \{s'(t) \mid t \in P\},\\
&\widetilde{S}(p) = \{x(t) \mid x(t) \neq s(t), s'(t),\ t \in P\}\\
&T(p) = \{t \mid s(t) \in S(p) \cap \overline{X}(p)\},\ \overline{T}(p) = \{t \mid s'(t) \in \bar{S}(p) \cap \overline{X}(p)\},\\
&\widetilde{T}(p) = \{t \mid x(t) \in \widetilde{S}(p) \cap \overline{X}(p)\},\ R(p) = T(p) \cup \overline{T}(p) \cup \widetilde{T}(p)\\
&M(p) = \{(x(t), y(t')) \mid x(t) \in X(p),\ y(t') \in \overline{X}(p),\ (x,y) \in A\}.
\end{aligned}$$

The supply-demand theorem in network $G(p)$ is stated as follows:

THEOREM 1 *The constraints* (3), (4) *are feasible if and only if*

$$\sum_{\overline{T}(p)} q'(t) - \sum_{T(p)} q(t) \le \sum_{M(p)} c(x(t), y(t')) \tag{5}$$

holds for every subset $X(p) \subset N(p)$.

Proof. The constraints (3), (4) in G are equivalent in $G(p)$ to constraints:

$$\sum_{y(t')} f(s(t), y(t')) - \sum_{y(t')} f(y(t'), s(t)) \le q(t), t \in P, \tag{6a}$$

$$\sum f(x(t), y(t')) - \sum f(y(t'), x(t)) = 0,\ x(t) \neq s(t), s(t'),\ t \in P, \tag{6b}$$

$$\sum_{y(t')} f(y(t'), s'(t)) - \sum_{y(t')} f(s'(t), y(t')) \geq q'(t);\ t \in P, \tag{6c}$$

$$0 \leq f(x(t), y(t')) \leq c(x(t), y(t')),\ (x(t), y(t')) \in A(p). \tag{7}$$

Necessity. Assume that there is a flow f in $G(p)$ satisfying the constraints (6), (7). Then, by summing constraints (6) over $x(t) \in \overline{X}(p)$ we obtain

$$\begin{aligned}
&\sum_{\overline{T}(p)} \sum_{y(t')} f(y(t'), s'(t)) + \sum_{T(p)} \sum_{y(t')} f(y(t'), s'(t)) \\
&+ \sum_{\widetilde{T}(p)} \sum_{y(t')} f(y(t'), x(t)) - \sum_{\overline{T}(p)} \sum_{y(t')} f(s'(t), s(t')) \\
&- \sum_{T(p)} \sum_{y(t')} f(s(t), y(t')) - \sum_{\widetilde{T}(p)} \sum_{y(t')} f(x(t), y(t')) \\
&\geq \sum_{\overline{T}(p)} q'(t) - \sum_{T(p)} q(t),
\end{aligned}$$

or, equivalently,

$$\sum_{R(p)} \sum_{y(t')} f(y(t'), x(t)) - \sum_{R(p)} \sum_{y(t')} f(x(t), y(t')) \geq \sum_{\overline{T}(p)} q'(t) - \sum_{T(p)} q(t).$$

Rewriting this constraint as

$$\sum_{N(p)} \sum_{\overline{X}(p)} f(x(t), y(t')) - \sum_{N(p)} \sum_{\overline{X}(p)} f(y(t'), x(t)) \geq \sum_{\overline{T}(p)} q'(t) - \sum_{T(p)} q(t),$$

where $x(t) \in N(p)$, $y(t') \in \overline{X}(p)$, and using $N(p) = X(p) \cup \overline{X}(p)$ gives

$$\sum_{X(p)} \sum_{\overline{X}(p)} f(x(t), y(t')) - \sum_{X(p)} \sum_{\overline{X}(p)} f(y(t'), x(t)) \geq \sum_{\overline{T}(p)} q'(t) - \sum_{T(p)} q(t),$$

where $x(t) \in X(p)$, $y(t') \in \overline{X}(p)$.

Since f satisfies (7), summing over $x(t) \in X(p)$, $y(t') \in \overline{X}(p)$ results in

$$\sum_{X(p)} \sum_{\overline{X}(p)} f(x(t), y(t')) - \sum_{X(p)} \sum_{\overline{X}(p)} f(y(t'), x(t)) \leq \sum_{X(p)} \sum_{\overline{X}(p)} c(x(t), y(t'))$$

or, equivalently, in

$$\sum_{X(p)} \sum_{\overline{X}(p)} f(x(t), y(t')) - \sum_{X(p)} \sum_{\overline{X}(p)} f(y(t'), x(t)) \leq \sum_{M(p)} c(x(t), y(t')).$$

We obtain that the constraint (6) must hold for every subset $X(p) \subset N(p)$.

Sufficiency. To prove sufficiency, we construct a new network $G_1(p) =$

sink s'_1, and the arc set $E(p) = E(s_1) \cup E(s'_1)$, where $E(s_1) = \{(s_1, s(t)) \mid t \in P\}$, $E(s'_1) = \{(s'(t), s'_1) \mid t \in P\}$.

The capacity function c_1 on $A_1(p)$ is defined by

$$c_1(x(t), y(t')) = c(x(t), y(t')),\ (x(t), y(t')) = A(p),$$
$$c_1(s_1, s(t)) = q(t),\ t \in P,$$
$$c_1(s'(t), s'_1) = q'(t),\ t \in P.$$

We now show that the inequality (6) holds for every subset $X(p) \subset N(p)$ if and only if it holds for the cut $E(s'_1)$ that is a minimum $s_1 - s'_1$ cut in $G_1(p)$. Define $X(p) = X_1(p) \setminus \{s_1\}$, $\overline{X}(p) = X_1(p) \setminus \{s'_1\}$. Consider the expression

$$\begin{aligned}
&\sum_{X_1(p)} \sum_{\overline{X}_1(p)} c_1(x(t), y(t')) - \sum_{E(s'_1)} c_1(s'(t), s'_1) \\
&= \sum_{X(p)} c_1(x(t), s'_1) + \sum_{X(p)} \sum_{\overline{X}(p)} c_1(x(t), y(t')) \\
&+ \sum_{\overline{X}(p)} c_1(s_1, y(t')) - \sum_{E(s'_1)} c_1(s'(t), s'_1) \\
&= \sum_{M(p)} c(x(t), y(t')) + \sum_{T(p)} q(t) - \sum_{\overline{T}(p)} q'(t).
\end{aligned}$$

By assumption, (5) holds for every subset $X(p) \subset N(p)$. Thus, the inequality

$$\sum_{X_1(p)} \sum_{\overline{X}_1(p)} c_1(x(t), y(t')) - \sum_{E(s'_1)} c_1(s_1(t), s'_1) \geq 0 \tag{8}$$

holds for all $s_1 - s'_1$ cuts $(X_1(p), \overline{X}_1(p))$ of $G_1(p)$, showing that $E(s'_1)$ is a minimum $s_1 - s'_1$ cut in $G_1(p)$. Our conclusion is that (6) holds for all $X(p) \subset N(p)$ if and only if (8) holds for all $s_1 - s'_1$ cuts in $G_1(p)$.

Since $E(s'_1)$ is a minimum $s_1 - s'_1$ cut in $G_1(p)$, there exists a flow f_1 from s_1 to s'_1 in $G_1(p)$ that saturates all the arcs of $E(s'_1)$.

Define $f(x(t), y(t')) = f_1(x(t), y(t'))$, $(x(t), y(t')) \in A(p)$.

Clearly, f satisfies (6b) and (7). To see that it also satisfies (6a) and (6c), we consider for all $s(t), s(t')$, $t \in P$ the equations

$$f_1(s_1, s(t)) = \sum_{y(t')} f_1(s(t), y(t')) - \sum_{y(t')} f_1(y(t'), s(t))$$
$$f_1(s'(t), s'_1) = \sum_{y(t')} f_1(y(t'), s'(t')) - \sum_{y(t')} f_1(s'(t), y(t')).$$

Since by construction $q(t) \geq f_1(s_1, s(t))$, $q'(t) = f_1(s'(t), s'_1)$ the inequalities

In the network $G^*(p) = (N^*(p), A^*(p), c^*)$ the sets $X^*(p)$, $\overline{X}^*(p)$, and so on, are defined analogously to $X(p)$, $\overline{X}(p)$, and so on, from the network $G(p) = (N(p), A(p), c)$.

The supply-demand theorem in network $G^*(p)$ is stated as follows:

THEOREM 2 *The constraints* (3), (4) *are feasible if and only if*

$$\sum_{\overline{T}^*(p)} q'(t) - \sum_{T^*(p)} q(t) \leq \sum_{M^*(p)} c^*(x(t), y(t')) \tag{9}$$

holds for every subset $X^*(p) \subset N^*(p)$.

Proof. The result derives directly from Theorem 1 and the fact that a static flow in the static network $G(p) = (N(p), A(p), c)$ is equivalent to a static flow in the static network $G^*(p) = (N^*(p), A^*(p), c^*)$. ■

We will consider in network $G = (N, A, h, c)$

$$Q(x) \subset P(x),\ \overline{Q}(x) = P(x) \setminus Q(x),\ x \in N,\ \text{and}$$
$$M(t) = \{(x, y; t) \mid x \in N,\ t \in Q(x)\}.$$

The supply-demand theorem in network G is stated as follows:

THEOREM 3 *The constraints* (3), (4) *are feasible if and only if*

$$\sum_{\overline{Q}(s')} q'(t) - \sum_{Q(s)} q(t) \leq \sum_{M(t)} c(x, y; t) \tag{10}$$

holds for all subsets $Q(x) \subset P(x)$, $x \in N$.

Proof. This result derives directly from Theorem 2 and the fact that a dynamic flow for p time periods in the dynamic network $G = (N, A, h, c)$ can be represented as a static flow in the static network $G^*(p) = (N^*(p), A^*(p), c^*)$.

In practice, if we are interested in ascertaining the feasibility of a given supply-demand network $G = (N, A, h, c)$, the most efficient way to do this is to use the maximum flow algorithm to solve the equivalent maximum flow problem in the extended network $G_1^*(p) = (N_1^*(p), A_1^*(p), c_1^*)$. We construct the network $G_1^*(p)$ from network $G^*(p)$ by adjoining a fictitious source s_1, sink s_1', and the arc set

$$E^*(p) = E^*(s_1) \cup e^*(s_1'),$$
$$E^*(s_1) = \{(s_1, s(t)) \mid t \in P(s)\}$$
$$E^*(s_1') = \{(s'(t), s_1') \mid t \in P(s')\}.$$

Thus, we have:

$$N_1^*(p) = N^*(p) \cup \{s_1, s_1'\}$$

The capacity function $A_1^*(p)$ on is defined by

$$c_1^*(s_1, s(t)) = q(t),\ t \in P(s)$$
$$c_1^*(x(t), y(t')) = c^*(x(t), y(t')),\ (x(t), y(t')) \in A^*(p)$$
$$c_1^*(s'(t), s_1') = q'(t),\ t \in P(s').$$

Since $E^*(s_1')$ is a minimum $s_1 - s_1'$ cut in G_1^*, there exists a flow f_1 from s_1 to s_1' in G_1^* which saturates all the arcs of $E^*(s_1')$.

We illustrate the above results by the following example.

3. Example

Consider the network $G = (N, A, h, c)$ of Fig. 1, where node s is the source, s' is the sink, and x, y are the intermediate nodes. The arc transit times are given as the first members and the capacities of the arc are given as the second members of the pairs of numbers written adjacent to the arcs of Fig. 1.

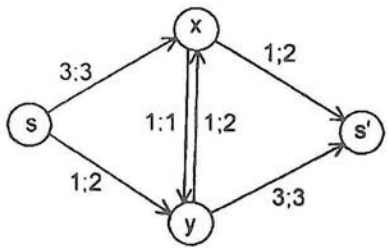
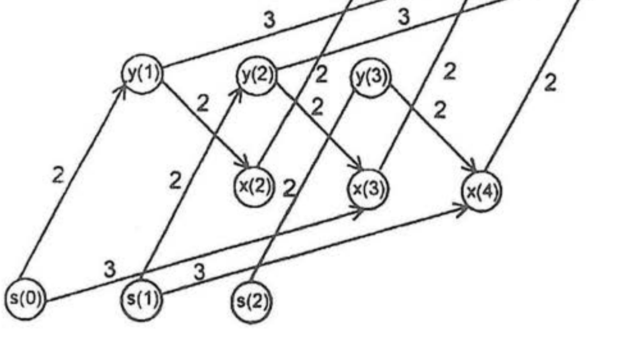


Fig. 1

For $p = 5$, the network $G^*(5)$ is shown in Fig. 2. The supplies and demands are given by $q(0) = 4$, $q(1) = 4$, $q(2) = 1$, $q'(3) = 2$, $q'(4) = 2$, $q'(5) = 4$.



A maximum flow f_1^* in the extended network $G_1^*(5)$ is presented in Fig. 3, in which the first numbers adjacent to the arcs denote their capacities and the second numbers denote their flow values. Since the maximum flow f_1^* saturates

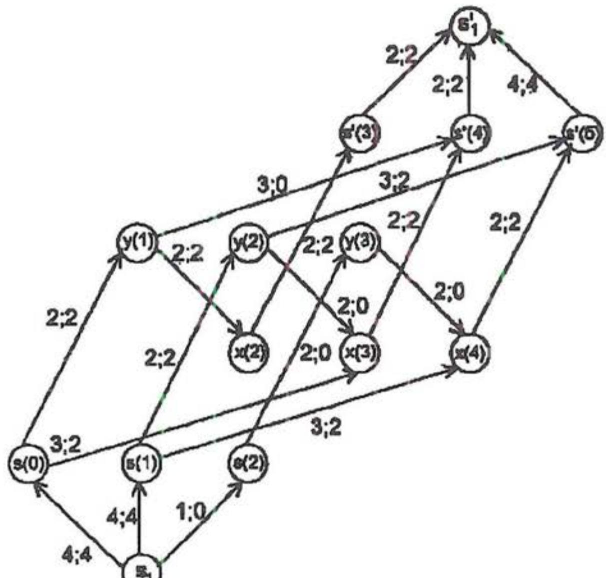


Fig. 3

all the arcs of $E^*(s_1')$, the problem is feasible. If $q'(3) = 3$, then a maximum flow in the extended network $G_1^*(5)$ is as shown in Fig. 3, and $f_1^*(s'(3), s_1') = 2 < 3 = c_1^*(s'(3), s_1')$, so that the problem is infeasible. Alternatively, the cut $(X^*(5), \overline{X}^*(5))$, where

$$X^*(5) = \{s(1), s(2), y(3), x(4)\},\ \overline{X}^*(5) = N^*(5) - X^*(5)$$

violates condition (6):

$$T^*(5) = \{0\},\ \overline{T}^*(5) = \{3, 4, 5\},$$

$$\sum_{\overline{T}^*(5)} q'(t) - \sum_{T^*(5)} q(t) = 3 + 2 + 4 - 4 = 5,$$

$$\sum_{M^*(5)} c^*(x(t), y(t')) = c^*(s(1), y(2)) + c^*(x(4), s'(5)) = 2 + 2 = 4,$$

confirming that the problem is infeasible.

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