Silent-noisy duel with two kinds of weapon

by

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Abstract: In this paper the duel is solved in which Players I and II have two kinds of weapon, the first one with a single bullet, which Player can use when he wants, and the second one that can be used only when the distance between players is zero. Player I hears the shot of Player II, Player II does not hear the shot of Player I.

Keywords:

1. Introduction

At the beginning let us consider a special case of the above game. This special case will be denoted by $(1, e; 1)$. It is defined as follows:

Two Players I and II fight a duel. They move toward each other. They have one bullet each and this fact is known to both of them. It is also known that Player I has the second weapon which he can use when the distance between players is zero. The second weapon destroys opponent with probability p, $0 \leq p \leq 1$. The duel considered here is silent-noisy: at each moment t, $0 \leq t < 1$, Player II does not know whether or not his opponent has fired, whereas Player I has this information about Player II.

At the beginning of the duel players are at the distance 1 from each other. Let $P(s)$ be the probability of succeeding (destroying the opponent) by the first weapon when the distance between players is $1 - s$. The function $P(s)$ will be called accuracy function. It is the same for both players. It is assumed that the function $P(s)$ is increasing and continuous in $[0, 1]$, has continuous first derivatives in $(0, 1)$, and $P(s) = 0$ for $s \leq 0$, $P(1) = 1$. The time is taken to be equal to s.

Player I gains 1 if only be succeeds, gains -1 if only his opponent succeeds and gains 0 in the remaining cases. The duel is a zero-sum game.

The silent duel with two kinds of weapon is considered in Trybuła (1999).

2. The expected gain

Denote by $K(s, t)$ the expected gain of Player I if he fires the first weapon at the

$K(s,t)$ is called the payoff function. We have

$$K(s,t) = \begin{cases} P(s) - (1 - P(s))P(t) \\ \quad + p(1 - P(s))(1 - P(t)) & \text{if } s < t < 1, \\ p(1 - P(t))^2 & \text{if } s = t, \\ 1 - 2P(t) & \text{if } t < s, \\ P(s) - (1 - p)(1 - P(s)) & \text{if } s < t = 1. \end{cases} \tag{1}$$

Suppose that Player II has fired at the moment t and did not succeed. Then, the best what Player I can do, if he has a bullet, is to wait until the moment $s = 1$ and succeed with probability 1. Such behaviour of Player I is assumed for the function $K(s,t)$ given by (1).

In Sections 2–4 we shall suppose that $0 \le p < 1$. For $p = 1$ the function $K(s,t)$ is the same as for the corresponding silent duel and the optimal strategies follow from the paper of Trybuła (1999).

These strategies are identical for the silent and the above duels since the strategy pair (s,t) yields different outcomes in the two duels only if Player II fires first $(t < s)$ and misses. If so, Player I responds at time s in the silent duel, but holds off until time 1 in the silent-noisy duel. But even this difference in behaviour affects the outcome only if $p < 1$, because if $p = 1$ Player II wins with probability $P(t)$ and Player I wins otherwise.

Denote by ξ_a the mixed strategy of Player I in the game $(1, e; 1)$, in which he fires the first weapon at a random moment s distributed according to the density $f_1(s)$ in the interval $[a, 1]$. Denote by $K(\xi_a, t)$ the expected gain of Player I if he applies the strategy ξ_a and if Player II fires at a moment t, $0 \le t \le 1$. We have, when $a \le t < 1$,

$$K(\xi_a, t) = \int_a^1 K(s,t)f_1(s)\, ds$$
$$= \int_a^t (P(s) - (1 - P(s))P(t) + p(1 - P(s))(1 - P(t)))f_1(s)\, ds$$
$$+ \int_t^1 (1 - 2P(t))f_1(s)\, ds.$$

We shall look for a strategy ξ_a of Player I, for which

$$K(\xi_a, t) \equiv \text{const.} \stackrel{\text{def.}}{=} v_1$$

for $a \le t < 1$.

Suppose additionally in this Section that the function $P(t)$ has continuous second derivatives in $(0,1)$. Under this condition we obtain

$$\frac{\partial K(\xi_a, t)}{\partial t} = (P^2(t) + 2P(t) - 1 + p(1 - P(t))^2)f_1(t)$$
$$- (1 + p)P'(t)\int_a^t (1 - P(s))f_1(s)\, ds - 2P'(t)\int_t^1 f_1(s)\, ds = 0 \tag{2}$$

$$\frac{\partial^2 K(\xi_a, t)}{\partial t^2} = (P^2(t) + 2P(t) - 1 + p(1 - P(t))^2) f_1'(t)$$
$$+ 2(P(t) + 1 - p(1 - P(t))) P'(t)) f_1(t)$$
$$- (1 + p) P''(t) \int_a^t (1 - P(s)) f_1(s) \, ds - 2P''(t) \int_t^1 f_1(s) \, ds$$
$$- (1 + p)(1 - P(t)) P'(t) f_1(t) + 2P'(t) f_1(t) \equiv 0. \tag{3}$$

By eliminating the integrals from the equations (2) and (3) we get

$$((1 + p)P^2(t) + 2(1 - p)P(t) - (1 - p)) f_1'(t)$$
$$+ 3((1 + p)P(t) + 1 - p) P'(t) f_1(t)$$
$$- \frac{P''(t)}{P'(t)}((1 + p)P^2(t) + 2(1 - p)P(t) - (1 - p)) P'(t) f_1(t) = 0.$$

Solving above differential equation we obtain

$$f_1(t) = \frac{CP'(t)}{\left(P^2(t) + \frac{2(1 - p)}{1 + p} P(t) - \frac{1 - p}{1 + p}\right)^{3/2}}, \tag{4}$$

where C is a constant.

Let ξ_a be a (mixed) "strategy" of Player II in the game $(1, e; 1)$, in which he fires at a random moment t according to the density $f_2(t)$ in the interval $[a, 1]$ and according to the probability q, $0 \leq q < 1$ at the "moment" $1-$, where $K(s, 1-) \overset{\text{def.}}{=} \lim_{\varepsilon \to 0+} K(s, 1 - \varepsilon)$. We obtain for $s \in [a, 1)$

$$K(s, \eta_a) = \int_a^1 K(s, t) f_2(t) \, dt + q K(s, 1-)$$
$$= \int_a^s (1 - 2P(t)) f_2(t) \, dt$$
$$+ \int_a^1 (P(s) - (1 - P(s))P(t) + p(1 - P(s))(1 - P(t))) f_2(t) \, dt$$
$$+ q(2P(s) - 1).$$

We look for a "strategy" η_a of Player II, for which

$$K(s, \eta_a) \equiv \text{const.} \overset{\text{def.}}{=} v_2$$

for $a \leq t < 1$.

Acting in the same manner as for $K(\xi_a, t)$ we obtain

$$f_2(s) = \frac{DP'(s)}{\left(P^2(s) + \frac{2(1 - p)}{1 + p} P(s) - \frac{1 - p}{1 + p}\right)^{3/2}}, \tag{5}$$

Let us put

$$r = \frac{1-p}{1+p}, \quad 0 \le p < 1. \tag{6}$$

Then we have $0 < r \le 1$ and

$$f_1(t) = \frac{CP'(t)}{(P^2(t) + 2rP(t) - r)^{3/2}}, \quad f_2(s) = \frac{DP'(s)}{(P^2(s) + 2rP(s) - r)^{3/2}}. \tag{7}$$

3. Determination of parameters of equalizing strategies

One can see that for $P(t) > P_1 \overset{\text{def.}}{=} -r + \sqrt{r(r+1)}$ we have $P^2(t) + 2rP(t) - r > 0$ and

$$\int \frac{P'(t)\,dt}{(P^2(t) + 2rP(t) - r)^{3/2}} = -\frac{1}{1+r} \frac{P(t)/r + 1}{(P^2(t) + 2rP(t) - r)^{1/2}} + K, \tag{8}$$

$$\int \frac{P(t)P'(t)\,dt}{(P^2(t) + 2rP(t) - r)^{3/2}} = \frac{1}{1+r} \frac{P(t) - 1}{(P^2(t) + 2rP(t) - r)^{1/2}} + K \tag{9}$$

where K is a constant.

Taking into account the formulae (7), (8) and (9) we obtain, after reduction,

$$
\begin{aligned}
K(\xi_a, t) &= \int_a^t \Bigg(P(s) - (1 - P(s))P(t) \\
&\quad + \frac{1-r}{1+r}(1 - P(s))(1 - P(t)) \Bigg) f_1(s)\,ds + \int_t^1 (1 - 2P(t))f_1(s)\,ds \\
&= \frac{2C}{r(1+r)} \left[-\frac{P(a)}{(P^2(a) + 2rP(a) - r)^{1/2}} + (1+r)^{1/2} \right] P(t) \\
&\quad + \frac{C}{r(1+r)} \left[\frac{r + (1 - 2r)P(a)}{(P^2(a) + 2rP(a) - r)^{1/2}} - (1+r)^{1/2} \right] \equiv \text{const.} = v_1
\end{aligned}
$$

if

$$\frac{P(a)}{(P^2(a) + 2rP(a) - r)^{1/2}} - (1+r)^{1/2} = 0 \tag{10}$$

and for this a

$$v_1 = \frac{C}{r(1+r)} \left[\frac{r + (1 - 2r)P(a)}{(P^2(a) + 2rP(a) - r)^{1/2}} - (1+r)^{1/2} \right]. \tag{11}$$

$$K(s, \eta_a) = \int_a^s (1 - 2P(t)) f_2(t)\, dt$$

$$+ \int_s^1 \left(P(s) - (1 - P(s))P(t) + \frac{1-r}{1+r}(1 - P(s))(1 - P(t)) \right) f_2(t)\, dt$$

$$+ q(2P(s) - 1)$$

$$= 2\left(-\frac{D}{(1+r)^{3/2}} + q \right) P(s)$$

$$+ \frac{D}{1+r} \left[\frac{(1/r + 2)P(a) - 1}{(P^2(a) + 2rP(a) - r)^{1/2}} - \frac{1-r}{r(1+r)^{1/2}} \right] - q \equiv \text{const.} = v_2 \quad (12)$$

if

$$\frac{D}{(1+r)^{3/2}} - q = 0 \tag{13}$$

and, in this case,

$$v_2 = \frac{D}{r(1+r)} \left[\frac{(1+2r)P(a) - r}{(P^2(a) + 2rP(a) - r)^{1/2}} - \frac{1-r}{(1+r)^{1/2}} \right] - q. \tag{14}$$

Moreover, we have

$$\int_a^1 f_1(s)\, ds = C \int_a^1 \frac{P'(s)\, ds}{(P^2(s) + 2rP(s) - r)^{3/2}}$$

$$= \frac{C}{r(1+r)} \left[\frac{P(a) + r}{(P^2(a) + 2rP(a) - r)^{1/2}} - (1+r)^{1/2} \right] = 1 \tag{15}$$

and

$$\int_a^1 f_2(t)\, dt = D \int_a^1 \frac{P'(t)\, dt}{(P^2(t) + 2rP(t) - r)^{3/2}}$$

$$= \frac{D}{r(1+r)} \left[\frac{P(a) + r}{(P^2(a) + 2rP(a) - r)^{1/2}} - (1+r)^{1/2} \right] = 1 - q. \tag{16}$$

By solving the system of equations given by (10)–(11), (13)–(16) with respect to a, C, D, q, v_1, v_2 we get

$$P(a) = -(1 + r) + \sqrt{(1+r)(2+r)}, \tag{17}$$

$$C = (1 + r)(\sqrt{(2+r)} - \sqrt{1+r}), \tag{18}$$

$$D = \frac{(1+r)^{3/2}}{2 + r + \sqrt{(1+r)(2+r)}}, \tag{19}$$

$$q = \frac{1}{2 + r + \sqrt{(1+r)(2+r)}}, \tag{20}$$

For $P(a)$ given by (17) we obtain

$$P(a) - P_1 = \frac{3 + 2r - 2\sqrt{(1+r)(2+r)}}{\sqrt{(1+r)(2+r)} + \sqrt{r(1+r)} - 1}$$

$$\geq \frac{1}{(5 + 2\sqrt{6})(\sqrt{6} + \sqrt{2} - 1)} > 0 \tag{22}$$

for each $0 < r \leq 1$.

4. Proof of optimality

Let η_1^ε be a strategy according to which Player II fires with density $f_2(t)$ in the interval $[a, 1 - \varepsilon_0]$ and with probability $\bar{q} = 1 - \int_a^{1-\varepsilon_0} f_2(t)\, dt$ at the moment $1 - \varepsilon_0$ where $f_1(s), f_2(t)$ are given by (7) and ε_0 is a positive small number. The strategy η_a^ε is an approximation of the "strategy" η_a. We shall prove that the game $(1, e; 1)$ has a value and for constants a, C, D, q given by (17)–(20) the strategy ξ_a is maximin and the strategy η_a^ε is ε-minimax.

Suppose that Player II fires at a moment t, $a \leq t < 1$. From Section 2 and 3 it follows that

$$K(\xi_a, t) = v.$$

Suppose that Player II fires at a moment t, $0 \leq t < a$. We have

$$K(\xi_a, t) = \int_a^1 (1 - 2P(t))f_1(s)\, ds > \int_a^1 (1 - 2P(a))f_1(s)\, ds = v.$$

Suppose that Player II fires at the moment $t = 1$. In this case

$$K(\xi_a, t) = \int_a^1 (P(s) - (1 - p)(1 - P(s)))f_1(s)\, ds$$

$$\geq \int_a^1 (P(s) - (1 - P(s)))f_1(s)\, ds = \lim_{t \to 1-} K(\xi_a, t) = v.$$

On the other hand, suppose that Player I fires at the moment s where $a \leq s < 1 - \varepsilon_0$. In this case

$$K(s, \eta_1^\varepsilon) \leq v + \varepsilon_1$$

for each $\varepsilon_1 > \bar{\varepsilon} > 0$. The constant $\bar{\varepsilon}$ is chosen with respect to ε_0 and tends to 0 if $\varepsilon_0 \to 0$.

Suppose that Player I fires at s, $0 \leq s < a$. We have

$$K(s, \eta_1^\varepsilon) = \int_a^{1-\varepsilon_0} (P(s) - (1 - P(s))P(t) + p(1 - P(s))(1 - P(t)))f_2(t)\, dt$$

$$\leq \int_a^1 (P(s) - (1 - P(s))P(t)$$
$$+ p(1 - P(s))(1 - P(t)))f_2(t)\, dt + q(2P(s) - 1) + \varepsilon_2$$
$$= \int_a^1 (-P(t) + p(1 - P(t))$$
$$+ P(s)(1 + P(t) - p(1 - P(t))))f_2(t)\, dt + q(2P(s) - 1) + \varepsilon_2$$
$$< \int_a^1 (-P(t) + p(1 - P(t))$$
$$+ P(a)(1 + P(t) - p(1 - P(t))))f_2(t)\, dt + q(2P(s) - 1) + \varepsilon_2$$
$$= v + \varepsilon_2$$

for each $\varepsilon_2 > \bar{\varepsilon} > 0$.

Suppose that Player I fires at the moment $1 - \varepsilon_0$. We obtain

$$K(1 - \varepsilon_0, \eta_a^\varepsilon) \leq \int_a^1 (1 - 2P(t))f_2(t)\, dt + \varepsilon_3$$
$$< \int_a^1 (1 - 2P(t))f_2(t)\, dt + q + \varepsilon_3 \stackrel{(12)}{=} v + \varepsilon_3 \tag{23}$$

for each $\varepsilon_3 > \bar{\varepsilon} > 0$.

Finally, suppose that Player I fires at $1 - \varepsilon'$, where $\varepsilon_0 > \varepsilon' \geq 0$. We obtain

$$K(1 - \varepsilon', \eta_a^\varepsilon) \leq \int_a^1 (1 - 2P(t))f_2(t)\, dt - q + \varepsilon_4 \stackrel{(23)}{<} v + \varepsilon_4$$

for each $\varepsilon_4 > \bar{\varepsilon} > 0$.

Assume that $\varepsilon = \max(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. From the above inequalities and the fact that $\bar{\varepsilon} \to 0$ when $\varepsilon_0 \to 0$ it follows that the game $(1, e; 1)$ has a value and for a, C, D, q given by (17)–(20) the strategy ξ_a is maximin and the strategy η_a^ε is ε-minimax.

5. Generalization of the duel $(1, e; 1)$

Suppose now that both players have two kinds of weapon, the first with one bullet which a player can use when he wants, and the second, which he can use only when the distance between players is zero. All other suppositions are the same as in the duel solved in Sections 2–4. Let the probabilities of success using the second weapon by Players I and II be p_1, p_2, respectively. Now if both players use only the second weapon the expected gain of Player I is

Then, the expected gain of Player I when both can use two kinds of weapon is

$$
K_0(s,t) = \begin{cases}
P(s) - (1-P(s))P(t) & \\
\quad + p(1-P(s))(1-P(t)) & \text{if } s < t < 1, \\
p(1-P(s))^2 & \text{if } s = t, \\
1 - 2P(t) - \varepsilon(t) & \text{if } t < s, \\
P(s) - (1-p_1)(1-P(s)) & \text{if } s < t = 1
\end{cases}
\tag{25}
$$

where now p can be negative.

When determining the function $K_0(s,t)$ it is now assumed for $p_2 > 0$ that if Player II has fired and Player I has yet the bullet, he waits until a time near 1 (but lower than 1) and succeeds with probability close to 1. Then $\varepsilon(t) > 0$ when $p_2 > 0$ and $\varepsilon(t)$ is small.

The case of $p_2 = 0$ was solved in Sections 2–4.

Let $K(s,t) = 1 - 2P(t)$ when $t < s \leq 1$, $K(s,t) = K_0(s,t)$ in the other cases. Let $-1 < p < 1$. It is easy to see from Sections 2–3 that also now, when p can be negative, for the gain function $K(s,t)$, the strategy ξ_a of Player I and the "strategy" η_a of Player II have absolute continuous components with densities $f_1(s), f_2(t)$ given, as well, by (7), where $r = (1-p)/(1+p)$ and the "strategy" η_a has discrete component q in point 1.

The constants $P(a), C, D, q, v$ will be also given by formulae (17)–(21), but now, differently than in (6), $0 < r < \infty$.

The number P_1, defined at the beginning of Section 3, and $P(a)$, given by (17), can be presented in the form

$$
P_1 = \frac{1}{1 + \sqrt{1 + 1/r}}, \quad P(a) = \frac{1}{1 + \sqrt{1 + 1/(1+r)}}.
\tag{26}
$$

It follows from the above that $P(a) > P_1$ and that the densities $f_1(s), f_2(t)$ given by the formulae (7) exist. It is also easy to see that $P(a)$ is an increasing function of r and that $P(a) \to 1/2$ when $r \to \infty$.

Let $-1 < p < 1$ and let ξ_a and η_a^ε be the strategies defined at the beginning of Section 4. We shall prove the optimality of the strategy ξ_a and ε-optimality of the strategy η_a^ε for the constants a, C, D, q given by (17)–(20), and the payoff function $K(s,t)$ obtained from (25) by putting $\varepsilon(t) = 0$.

Suppose that Player II fires at the moment t, $a \leq t < 1$. From Sections 2 and 3 it follows that

$$
K(\xi_a, t) = v
$$

where v is given by (21).

Suppose that Player II fires at the moment t, $0 \leq t < a$. We obtain

$$
K(\xi_a, t) = \int^1 (1 - 2P(t)) f_a(s)\, ds > \int^1 (1 - 2P(s)) f_a(s)\, ds = v
$$

Suppose that Player II fires at the moment $t = 1$. In this case

$$K(\xi_a, t) = \int_a^1 (P(s) - (1 - p_1)(1 - P(s)))f_1(s)\, ds$$

$$\geq \int_a^1 (P(s) - (1 - P(s)))f_1(s)\, ds = \lim_{t \to 1-} K(\xi_a, t) = v.$$

Similarly as in Section 4 we can prove also that

$$K(s, \eta_1^\varepsilon) \leq v + \varepsilon$$

for each $0 \leq s \leq 1$ and for each $\varepsilon > 0$.

Let us notice that these parts of proofs in the cases $0 \leq p < 1$ and $-1 < p < 1$ do not differ much from each other.

Let as in Section 4

$$\bar{q} = 1 - \int_a^{1-\varepsilon_0} f_2(t)\, dt.$$

Because $K_0(s, t)$ differs from $K(s, t)$ by no more that $\varepsilon = \max \varepsilon(t)$ and $|K(s, t)| \leq 1$ then it follows from the above inequalities that for the constants a, C, D, q given by (17)–(20) and for $\bar{q}$ defined above the strategy ξ_a is an ε-maximin strategy of Player I and η_a^ε is an ε-minimax strategy of Player II. The game has a value.

When, moreover, $p_2 = 0$, the strategy ξ_a is maximin as it follows from Sections 2–4, since in this case $\varepsilon(t) = 0$ in (25).

When $p_1 = p_2 = 0$ the game considered is a duel with one kind of weapon treated in Karlin's book (1959). In this case both players have optimal strategies. The solution is

$$C^{-1} f_1(s) = D^{-1} f_2(s) = \frac{P'(s)}{(P^2(s) + 2P(s) - 1)^{3/2}}$$

for $P(s) \in [\sqrt{6} - 2, 1]$, $f_1(s) = f_2(s) = 0$ in the other cases,

$$C = 2(\sqrt{3} - \sqrt{2}), \quad D = \frac{2\sqrt{2}}{3}(3 - \sqrt{6}), \quad q = \frac{1}{2}(3 - \sqrt{6}).$$

The value of the game is $5 - 2\sqrt{6}$.

When $p = 0$, the game can be reduced to this duel.

When $p = 1$, that is - when $p_1 = 1$, $p_2 = 0$, the function $K(s, t)$ is of the form

$$K(s, t) = K_0(s, t) = \begin{cases} P(s) - (1 - P(s))P(t) \\ \quad + (1 - P(s))(1 - P(t)) & \text{if } s < t < 1, \\ (1 - P(s))^2 & \text{if } s = t, \\ 1 - 2P(t) & \text{if } s > t, \end{cases} \tag{27}$$

Now $\varepsilon(t) = 0$, because $p_2 = 0$, and the optimal behaviour of Player I after the shot of Player II is to fire at the point 1, not before this point. We have $r = 0$ and

$$f_1(s) = \frac{CP'(s)}{P^3(s)}, \quad f_2(s) = \frac{DP'(s)}{P^3(s)}. \tag{28}$$

The constants $P(a), C, D, q, v$ are also expressed by the formulae (17)–(21), which can be shown by direct computation.

The proof of the optimality of the strategies ξ_a and η_a^ε is omitted since it is the same as that in Section 4. In this case the strategy ξ_a is maximin and η_a^ε is ε-minimax.

Let, at the end, $p = -1$ or, $p_1 = 0$, $p_2 = 1$. Now

$$K_0(s,t) = \begin{cases} 2P(s) - 1 & \text{if } s < t, \\ -(1 - P(s))^2 & \text{if } s = t, \\ 1 - 2P(t) - \varepsilon(t) & \text{if } t < s. \end{cases} \tag{29}$$

Let, as before, $K(s,t) = K_0(s,t)$ for $\varepsilon(t) = 0$. Now the function $K(s,t)$ is the same as for noisy duel with one kind of weapon (with the exception of the case when $s = t$). From Karlin's book (1959) it follows that the optimal, strategy of Player II for the payoff function $K(s,t)$ is to fire at the moment s_0 such that

$$2P(s_0) - 1 = 1 - 2P(s_0), \tag{30}$$

that is, when $P(s_0) = 1/2$, and ε-optimal strategy of Player I (when he did not hear the shot of Player II before) is to fire with the constant density $f_1(s)$ in the interval $(s_0, s_0 + \varepsilon)$, In this case $v = 0$ and since $\varepsilon(t) > 0$ then the strategy of Player II remains optimal for the payoff function $K_0(s,t)$.

Then, we have considered all the cases.

Let us notice that for all p_1, p_2, except for the case of $p_1 = 0$, $p_2 = 1$, the value of the game is

$$v = 3 + 2r - 2\sqrt{(1+r)(2+r)} = \frac{1}{3 + 2r + 2\sqrt{(1+r)(2+r)}} \tag{31}$$

and because for these cases $0 \le r < \infty$, then $v > 0$. When $p_1 = 0$, $p_2 = 1$, we have obtained $v = 0$. Then, $v \ge 0$ always, even when Player II has two kinds of weapon and Player I only one!

The considered duel can be a little more generalized by assuming that when using only the second weapon four results are possible with probabilities: q_1 — that Player I wins only, q_2 — that Player II wins only, q_3 — that both win, q_4 — that both lose (that both are destroyed). Here, q_1, q_2, q_3, q_4 are any numbers satisfying the conditions $0 \le q_i \le 1$, $q_1 + q_2 + q_3 + q_4 = 1$.

The considered duel can be treated in other manner. If Player I has no bullet and II keeps the bullet at the end-time of the game, then one can consider two

a) I is destroyed alone with probability 1 (i.e. the second weapon is useless since II keeps a bullet),

or

b) I is destroyed alone with probability $\bar{p}_1 = 1 - p_1 \leq 1$ (i.e. the second weapon is useful even if II has a bullet).

The author follows the case b). But the case a) can be also considered.

Mixed duels are considered in Karlin (1959), Kimeldorf (1983), Styszyński (1974).

For other duels see Cegielski (1986 A, B), Davydov (1978), Fox and Kimeldorf (1969), Karlin (1959), Radzik and Orłowski (1985A, B), Restrepo (1957), Teraoka (1979), Trybuła (1992, 1993–1995, 1999).

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