Control and Cybernetics

vol. **30** (2001) No. 2

An algebraic analysis approach to 2-D discrete problems

by

Danuta Przeworska-Rolewicz[1] and Hubert Wysocki[2]

[1]Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-950 Warszawa, Poland
E-mail: rolewicz@impan.gov.pl

[2]Department of Mathematics, Polish Naval Academy
Jana Śmidowicza 69, 81-103 Gdynia, Poland
E-mail: hwys@amw.gdynia.pl

Abstract: A model of algebraic analysis for the 2-index sequences (of the type 2-D) is considered. For difference operators of the form

$$D\{x_{m,n}\} := \{x_{m+1,n+1} - a_{m,n}x_{m,n}\}$$

the right inverses and the corresponding initial operators are constructed. Having already known the initial operators, one can determine solutions of the corresponding initial value problems.

Keywords: algebraic analysis, 2-D systems, difference operator, right invertible operator, right inverse, initial operator, forward shift, backward shift.

1. Foundations of algebraic analysis

Let X be a linear space over a field $\mathbb{F}$ of scalars (of the characteristic zero). Let $L(X)$ be the set of all linear operators A whose domains $\operatorname{dom} A$ and sets of values $\operatorname{range} A = A \operatorname{dom} A$ are linear subsets of the space X. Write

$$L_0(X) := \{A \in L(X) : \operatorname{dom} A = X\}.$$

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \operatorname{dom} D$ and $DR = I$, where $I \in L_0(X)$ is the identity operator. The operator R is a *right inverse* of D. Denote by $\mathcal{R}_D$ the set of all right inverses of D. Clearly, $\mathcal{R}_D \subset L_0(X)$.

An operator $F \in L_0(X)$ is said to be an *initial operator* for D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F,\ FX = \ker D \text{ and } FR = 0,$$

where $\ker D$ denotes the kernel of the operator D.

It can be shown (cf. Theorem 2.2.1 in Przeworska-Rolewicz, 1988) that F is an initial operator for D corresponding to R if and only if

$$Fx = x - RDx \text{ for } x \in \operatorname{dom} D.$$

We therefore conclude that the family $\mathcal{R}_D = \{R_\gamma\}_{\gamma\in\Gamma}$ of right inverses of D induces a family $\mathcal{F}_D = \{F_\gamma\}_{\gamma\in\Gamma}$ of initial operators for D, where

$$F_\gamma = I - R_\gamma Dx \text{ on } \operatorname{dom} D,\ \gamma \in \Gamma.$$

EXAMPLE 1 (see Example 5.3 in Przeworska-Rolewicz, 1998) *Let $X := (s)_{\mathbb{F}}$ be the space of all sequences $x = \{x_n\}$, where $x_n \in \mathbb{F}$, $n \in \mathbb{N}$, with the usual coordinatewise addition of sequences and multiplication of sequences by scalars belonging to $\mathbb{F}$.*

Consider the forward shift

$$D\{x_n\} = \{x_{n+1}\},\ \{x_n\} \in \operatorname{dom} D = (s)_{\mathbb{F}},$$

which is right invertible. Indeed, $\ker D = \{c\delta : c \in \mathbb{F}\} \neq \{0\}$, *where* $\delta := \{\delta_1^n\}$ *and δ_1^n is the Kronecker symbol, i.e.*

$$\delta_1^n = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n \neq 1. \end{cases}$$

A right inverse of D is the backward shift*:*

$$R_1\{x_n\} = \{x_{n-1}\},\ \text{where } x_0 := 0,$$

which determines the initial operator

$$F_1\{x_n\} = x_1\delta.$$

Let $i \in \mathbb{N}$ and let

$$F_i\{x_n\} := (x_1 + \cdots + x_i)\delta,\ i > 1.$$

Observe that every operator defined by the following formula:

$$R_i x := R_1 x - F_{i-1}x,\ x \in X,\ i > 1,$$

is a right inverse of D, since $F_{i-1}X = \ker D$, $i > 1$. Moreover, F_i, $i > 1$, is an initial operator for D corresponding to R_i, $i > 1$. Indeed,

$$\begin{aligned} x - R_iDx &= x - (R_1Dx - F_{i-1}Dx) = (x - R_1Dx) + F_{i-1}Dx \\ &= F_1x + F_{i-1}Dx = x_1\delta + (x_2 + \cdots + x_i)\delta \\ &= (x_1 + \cdots + x_i)\delta = F_ix,\ i > 1, \end{aligned}$$

for every $x \in X$. Hence, to the family $\{R_i\}_{i\in\mathbb{N}} \subset \mathcal{R}_D$ of right inverses of D

2. Formulation of the problem

The set $(s)_{\mathbb{F}}$ of all sequences $x = \{x_{m,n}\}$, where $x_{m,n} \in \mathbb{F}$, $m, n \in \mathbb{N}$, with the coordinatewise addition and multiplication by scalars

$$x + y := \{x_{m,n} + y_{m,n}\},\ \lambda x := \{\lambda x_{m,n}\},$$

where $x = \{x_{m,n}\} \in (s_2)_{\mathbb{F}}$, $y = \{y_{m,n}\} \in (s_2)_{\mathbb{F}}$, $\lambda \in \mathbb{F}$, is a linear space over the field $\mathbb{F}$. Moreover, if for $x, y \in (s_2)_{\mathbb{F}}$ we define the coordinatewise multiplication

$$xy := \{x_{m,n} y_{m,n}\},$$

then $(s_2)_{\mathbb{F}}$ is a commutative algebra over $\mathbb{F}$ with the unit $e = \{e_{m,n}\}$, where $e_{m,n} = 1$ for all $m, n \in \mathbb{N}$.

Suppose therefore that $X := (s)_{\mathbb{F}}$ is an algebra with the structure operations defined as above. Suppose, moreover, that there are given sequences $\alpha, \beta \in (s)_{\mathbb{F}}$ and $a \in (s)_{\mathbb{F}}$ with the property

$$a_{m,n} \neq 0 \text{ for every } m, n \in \mathbb{N}.$$

The second author of the present paper posed in the paper Wysocki (2002) a problem, which can be presented in the algebraic analysis approach as follows:

Determine for the operator

$$D\{x_{m,n}\} := \{x_{m+1,n+1} - a_{m,n} x_{m,n}\} \tag{1}$$

a right inverse $R \in \mathcal{R}_D$ and the corresponding initial operator F induced by the following conditions:

$$x_{m,n_0} = \alpha_m,\ x_{m_0,n} = \beta_n,\ m, n \in \mathbb{N}, \tag{2}$$

where $a_{m_0} = \beta_{n_0}$ and m_0, n_0 are fixed positive integers such that $m_0 \neq n_0$.[1]

3. Solution of the problem

We shall solve the problem posed in Section 2 in the case when instead of Conditions (2) the following conditions are imposed:

$$x_{m,1} = \alpha_m^1,\ x_{m,2} = \alpha_m^2, \ldots,\ x_{m,n_0} = \alpha_m^{n_0},\ m \in \mathbb{N} \tag{3}$$

and

$$x_{1,n} = \beta_n^1,\ x_{2,n} = \beta_n^2, \ldots,\ x_{m_0,n} = \beta_n^{m_0},\ n \in \mathbb{N}, \tag{4}$$

where given sequences

$$\alpha^j = \{\alpha_m^j\} \in (s)_{\mathbb{F}},\ j \in \overline{1, n_0}\ [2],$$
$$\beta^i = \{\beta_n^i\} \in (s)_{\mathbb{F}},\ i \in \overline{1, m_0},$$

[1]The case $m_0 = n_0$ has been considered in Wysocki (2002).

This implies

$$y_{m,n} = \begin{cases} z_{m,n} \sum_{k=1}^{m} \frac{x_{k,n-m+k}}{z_{k,n-m+k}} & \text{for } m \le n, \\ z_{m,n} \sum_{l=1}^{n} \frac{x_{m-n+l,l}}{z_{m-n+l,l}} & \text{for } m > n, \end{cases} \qquad m, n \in \mathbb{N}. \tag{11}$$

Finally, we get

$$(I - R_1 R_2 A)^{-1}\{x_{m,n}\} = \{y_{m,n}\},$$

where the sequence $\{y_{m,n}\}$ is given by Formulae (11).

5. Algebraic description of 2-D systems in control theory

The generalized 2-D model considered in control theory is the state-space model with deviating arguments, Kaczorek (1993),

$$\begin{aligned} &\boldsymbol{E}_{m+1,n+1}\boldsymbol{x}_{m+1,n+1} = \boldsymbol{A}^0_{m,n}\boldsymbol{x}_{m,n} + \boldsymbol{A}^1_{m+1,n}\boldsymbol{x}_{m+1,n} + \boldsymbol{A}^2_{m,n+1}\boldsymbol{x}_{m,n+1} \\ &+ \boldsymbol{B}^0_{m,n}\boldsymbol{x}_{m-k,n-l} + \boldsymbol{B}^1_{m+1,n}\boldsymbol{x}_{m-k+1,n-l} + \boldsymbol{B}^2_{m,n+1}\boldsymbol{x}_{m-k,n-l+1} \\ &+ \boldsymbol{C}^0_{m,n}\boldsymbol{u}_{m,n} + \boldsymbol{C}^1_{m+1,n}\boldsymbol{u}_{m+1,n} + \boldsymbol{C}^2_{m,n+1}\boldsymbol{u}_{m,n+1}, \end{aligned} \tag{12}$$

$$\boldsymbol{y}_{m,n} = \boldsymbol{G}_{m,n}\boldsymbol{x}_{m,n} + \boldsymbol{H}_{m,n}\boldsymbol{u}_{m,n}, \tag{13}$$

where $\boldsymbol{x}_{m,n} \in \mathbb{R}^p$ is the local state vector at the point $(m,n) \in \mathbb{N} \times \mathbb{N}$, $\boldsymbol{u}_{m,n} \in \mathbb{R}^q$ is the input vector, $\boldsymbol{y}_{m,n} \in \mathbb{R}^r$ is the output vector and the variable coefficients $\boldsymbol{E}_{m,n}, \boldsymbol{A}^0_{m,n}, \boldsymbol{A}^1_{m,n}, \boldsymbol{A}^2_{m,n}, \boldsymbol{B}^0_{m,n}, \boldsymbol{B}^1_{m,n}, \boldsymbol{B}^2_{m,n}, \boldsymbol{C}^0_{m,n}, \boldsymbol{C}^1_{m,n}, \boldsymbol{C}^2_{m,n}, \boldsymbol{G}_{m,n}, \boldsymbol{H}_{m,n}$ for each $(m,n) \in \mathbb{N} \times \mathbb{N}$ are the real matrices of appropriate dimensions.

Upon admitting

$$\begin{aligned} &\boldsymbol{x} := [x_\mu],\ x_\mu \in (s_2)_{\mathbb{R}},\ \mu \in \overline{1,p}, \\ &\boldsymbol{u} := [u_\nu],\ u_\nu \in (s_2)_{\mathbb{R}},\ \nu \in \overline{1,q} \end{aligned}$$

we get $\boldsymbol{x} = \{\boldsymbol{x}_{m,n}\}$, $\boldsymbol{u} = \{\boldsymbol{u}_{m,n}\}$.

Using the operators considered in this paper, the state equation (12) is transformed into the vector-matrix 'partial integro-differential equation'

$$\begin{aligned} &D_1 D_2(E\boldsymbol{x}) = A_0\boldsymbol{x} + D_1(A_1\boldsymbol{x}) + D_2(A_2\boldsymbol{x}) \\ &+ B_0 R_1^k R_2^l \boldsymbol{x} + D_1(B_1 R_1^k R_2^l \boldsymbol{x}) \\ &+ D_2(B_2 R_1^k R_2^l \boldsymbol{x}) + C_0\boldsymbol{u} + D_1(C_1\boldsymbol{u}) + D_2(C_2\boldsymbol{u}), \end{aligned} \tag{14}$$

where operators $D_i, R_i, E, A_j, B_j, C_j$ are defined in the following way

$$D_i\boldsymbol{x} := [D_i x_\mu],\ R_i\boldsymbol{x} := [R_i x_\mu],\ i = 1,2,\ E\boldsymbol{x} := \boldsymbol{E}_{m,n}\boldsymbol{x}_{m,n},$$

In the case, when the coefficients of the equation (12) are real-valued constant matrices, from (14) we obtain the first and the second Fornasini–Marchesini model, respectively (see Fornasini and Marchesini, 1978; Antoniou and Emmons, 2000)

$$\begin{aligned} D_1 D_2 \boldsymbol{x} &= A_1 D_1 \boldsymbol{x} + A_2 D_2 \boldsymbol{x} + C_0 \boldsymbol{u}, \\ D_1 D_2 \boldsymbol{x} &= A_1 D_1 \boldsymbol{x} + A_2 D_2 \boldsymbol{x} + C_1 D_1 \boldsymbol{u} + C_2 D_2 \boldsymbol{u} \end{aligned}$$

and the generalized linear model (see Kaczorek, 1985; Dzieliński, 1993)

$$D_1 D_2 \boldsymbol{x} = A_0 \boldsymbol{x} + A_1 D_1 \boldsymbol{x} + A_2 D_2 \boldsymbol{x} + C_0 \boldsymbol{u} + C_1 D_1 \boldsymbol{u} + C_2 D_2 \boldsymbol{u}. \tag{15}$$

EXAMPLE 3 *Consider the scalar system described by the following state equation*

$$x_{m+1,n+1} = x_{m,n} + (m+1)(n+1), \; m, n \in \mathbb{N}. \tag{16}$$

It corresponds a particular case of the model (15) and it can be presented in the form

$$D_1 D_2 x = x + u$$

or

$$\widetilde{D} x = u, \tag{17}$$

where $x = \{x_{m,n}\}$, $u = \{(m+1)(n+1)\}$.

We determine the solution of the equation (17) with the condition

$$F^{(m_0,n_0)} x = 0, \tag{18}$$

for a fixed $m_0, n_0 \in \mathbb{N} \setminus \{1\}$.

The condition (18) will be satisfied if

$$\begin{aligned} & x_{1,n} + x_{2,n} + \cdots + x_{m_0-1,n} = 0, \; x_{m_0,n} = 0, \; n \in \mathbb{N}, \\ & x_{m,1} + x_{m,2} + \cdots + x_{m,n_0-1} = 0, \; x_{m,n_0} = 0, \; m \in \mathbb{N}. \end{aligned}$$

The solution of the initial value problem (17), (18) is given by the formula

$$x = R^{(m_0,n_0)} u,$$

i.e.

$$\begin{aligned} x &= (I - R_1 R_2)^{-1} R_1^{(m_0)} R_2^{(n_0)} u \\ &= (I - R_1 R_2)^{-1} (R_1 - F_1^{(m_0-1)})(R_2 - F_2^{(n_0-1)}) u \\ &= (I - R_1 R_2)^{-1} (R_1 R_2 - R_1 F_2^{(n_0-1)} - R_2 F_1^{(m_0-1)} + F_1^{(m_0-1)} F_2^{(n_0-1)}) \end{aligned}$$

Applying the form of the operators $R_1, R_2, F_1^{(m_0)}, F_2^{(n_0)}, (I-R_1R_2)^{-1}$, *we finally obtain*

$$x_{m,n} = \begin{cases} \dfrac{m(m+1)(3n-m+1)}{6} \\ \quad - \dfrac{(n-m+1)(m_0-1)(m_0+2)}{2} & \text{for } m < n, \\ \dfrac{m(m+1)(2m+1)}{6} \\ \quad - \dfrac{(m_0-1)(m_0+2)}{2} - \dfrac{(n_0-1)(n_0+2)}{2} \\ \quad + \dfrac{(m_0-1)(m_0+2)(n_0-1)(n_0+2)}{4} & \text{for } m = n, \\ \dfrac{n(n+1)(3m-n+1)}{6} \\ \quad - \dfrac{(m-n+1)(n_0-1)(n_0+2)}{2} & \text{for } m > n, \end{cases}$$

where $m, n \in \mathbb{N}$.

References

ANTONIOU, G.E. and EMMONS, K. (2000) Fornasini–Marchesini 2-D state space models: Transfer function computation via the DFT. *Int. J. Appl. Math. and Comp. Sci.*, **10**, 2, 321–331.

DZIELIŃSKI, A. (1993) Optimal control of two-dimensional systems. *Archives of Control Sciences*, **2**, 3–4, 185–199.

FORNASINI, E. and MARCHESINI, G. (1978) Doubly indexed dynamical systems: State space models and structural properties. *Math. Syst. Theory*, **12**, 1, 59–72.

KACZOREK, T. (1985) Two-dimensional linear systems. *Lecture Notes in Control and Information Sciences*, **68**, Springer-Verlag, Berlin.

KACZOREK, T. (1993) Generalized multidimensional linear systems with deviating arguments. *Archives of Control Sciences*, **2**, 3–4, 165–183.

PRZEWORSKA-ROLEWICZ, D. (1988) *Algebraic Analysis*. D. Reidel & PWN-Polish Scientific Publishers, Dordrecht & Warszawa.

PRZEWORSKA-ROLEWICZ, D. (1998) *Logarithms and Antilogarithms. An Algebraic Analysis Approach.* With Appendix by Z. Binderman. Kluwer Academic Publishers, Dordrecht.

WYSOCKI, H. (2002) Some 2-*D* discrete models of the Bittner Operational Calculus. *Acta Math. Hungar.* (in print).