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On the inverse problems of Lyapunov theorem and Riccati equation

by

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Abstract: In this note, the inverse problem of Lyapunov theorem is reconsidered and the inverse problem of Riccati equation is introduced. Simple proofs are provided to guarantee the solution of such problems.

Keywords: linear systems theory, inverse problems.

1. Introduction

The inverse problem of the Lyapunov theorem has been proven, by a more complex mathematical proof shown in Sun (1998), but how to determine such equivalent system has not been provided in this reference. We will show, in this note, that the inverse problem of Lyapunov theorem is always true by using a simple proof. Furthermore, it will been proved that, for a given completely controllable system and any given positive definite Hermitian matrix, an equivalent systems always exists such that the given positive definite Hermitian definite matrix is actually the solution to the Riccati equation for the equivalent system.

2. Lyapunov theorem and its inverse problem

For convenience, we define some notations and abbreviations as follows:
$C^{m\times n}$:= the set of all complex m by n matrices,
$P > 0$ (res. $Q\cdot < 0$): P is a positive (res. negative) definite Hermitian matrix,
$P \geq 0$ (res. $Q\cdot \leq 0$): P is a positive (res. negative) semidefinite Hermitian matrix,
A^* := the conjugate transpose of the matrix A,
A^{-*} := the conjugate transpose of the inverse of the nonsingular matrix A,
H_c := the set of all matrices whose eigenvalues have negative real parts,
H_d := the set of all matrices whose eigenvalues have absolute values less than 1.

LEMMA 1 *Let $P \in C^{n\times n}$ and $G \in C^{n\times n}$ be positive definite Hermitian matrices. Then there exits a nonsingular matrix $T \in C^{n\times n}$ satisfying $G = T^{-*}PT^{-1}$.*

Proof. Owing to $P > 0$ and $G > 0$, there exist nonsingular matrices G_1 and P_1 such that

$$G = G_1^* G_1 \text{ and } P = P_1^* P_1.$$

Let $T = G_1^{-1}$. Then we have

$$T^{-*} P T = G_1^* P_1^{-*} P_1^* P_1 P_1^{-1} G_1 = G_1^* G_1 = G.$$

This completes our proof. ■

Now we present our main results.

THEOREM 1 *Let* $A \in C^{n \times n}$.

(a) *If* $A \in H_c$, *then, for any given* $P > 0$, *there exists a nonsingular matrix* T *such that* $(T^{-1}AT)^* P + P(T^{-1}AT) < 0$;

(b) *If* $A \in H_d$, *then, for any given* $P > 0$, *there exists a nonsingular matrix* T *such that* $(T^{-1}AT)^* P (T^{-1}AT) - P < 0$.

Proof. (a) By Lyapunov theorem from Chen (1984), there exists a matrix $G > 0$ satisfying $A^* G + GA = -I$. By Lemma 1, there exists a nonsingular matrix T satisfying $G = T^{-*} P T^{-1}$. It can be readily obtained that

$$A^* T^{-*} P T^{-1} + T^{-*} P T^{-1} A = -I$$
$$\Rightarrow (T^{-1}AT)^* P + P(T^{-1}AT) = -T^* T < 0.$$

(b) By Lyapunov theorem from Chen (1984), there exists a matrix $G > 0$ satisfying $A^* G A - G = -I$. By Lemma 1, there exists a nonsingular matrix T satisfying $G = T^{-*} P T^{-1}$. It can be readily obtained that

$$A^* T^{-*} P T^{-1} A - T^{-*} P T^{-1} = -I$$
$$\Rightarrow (T^{-1}AT)^* P (T^{-1}AT) - P = -T^* T < 0.$$

This completes our proof. ■

REMARK 1 *Theorem 1 has been proven in Sun (1998) by using a more complex mathematical proof. Only the existence of the equivalent system* $\dot{x}(t) = \widetilde{A}x(t)$ *(or* $x(k+1) = \widetilde{A}x(k)$*) can be guaranteed, but how to determine such an equivalent system has not been provided in Sun (1998).*

3. Riccati equation and its inverse problem

Consider the linear continuous system

$$\dot{x}(t) = Ax(t) + Bu(t), \; x \in \Re^n, \; u \in \Re^m, \tag{1}$$

with (A, B) being completely controllable. The performance measure to be minimized is

$$V(u(t)) = \int_{t_0}^{\infty} x^T(t) Q x(t) + u^T(t) u(t) \, dt, \tag{2}$$

with $Q \geq 0$. The optimal control is given through the following lemma.

LEMMA 2 [Barnett and Cameron (1985)] *Consider the system* (1) *with* (2). *Suppose that* (A, Q_1) *is completely observable with* $Q_1^* Q_1 = Q$. *Then, the optimal control which minimizes* (2) *is given by* $u(t) = -B^* P x(t)$, *where* P *is the unique positive Hermitian definite matrix which satisfies the Riccati equation (Bittanti, Laub and Willems, 1991)*

$$A^* P + PA - PBB^* P = -Q. \tag{3}$$

Now we present our another main result.

THEOREM 2 *Let* $A \in C^{n\times n}$ *and* $B \in C^{n\times m}$. *If* (A, B) *is completely controllable, then, for any given positive definite Hermitian matrix* $P \in C^{n\times n}$, *there exist nonsingular matrices* $T \in C^{n\times n}$ *and* $Q_1 \in C^{n\times n}$ *such that*

(i) $(T^{-1}AT)^* P + P(T^{-1}AT) - P(T^{-1}B)(T^{-1}B)^* P = -Q_1^* Q_1 < 0$;
(ii) $(T^{-1}AT, T^{-1}B)$ *is completely controllable;*
(iii) $(T^{-1}AT, Q_1)$ *is completely observable.*

Proof. By Lemma 2, there exists a matrix $G > 0$ satisfying $A^* G + GA - GBB^* G = -I$. By Lemma 1, there exists a nonsingular matrix T satisfying $G = T^{-*} P T^{-1}$. It can be readily obtained that

$$\begin{aligned}
&A^* T^{-*} P T^{-1} + T^{-*} P T^{-1} A - T^{-*} P T^{-1} BB^* T^{-*} P T^{-1} = -I \\
&\Rightarrow T^* [A^* T^{-*} P T^{-1} + T^{-*} P T^{-1} A \\
&- T^{-*} P T^{-1} BB^* T^{-*} P T^{-1}] T = -T^* T < 0 \\
&\Rightarrow (T^{-1}AT)^* P + P(T^{-1}AT) - P(T^{-1}B)(T^{-1}B)^* P = -Q_1^* Q_1 < 0,
\end{aligned}$$

if we define $Q_1 = T$. Clearly, it can be deduced that $(T^{-1}AT, T^{-1}B)$ is completely controllable and $(T^{-1}AT, Q_1)$ is completely observable in view of the PBH rank test from Chen (1984). This completes our proof. ■

Consider the linear discrete system

$$x(k+1) = Ax(k) + Bu(k), \; x \in \Re^n, \; B \in \Re^m, \tag{4}$$

with (A, B) being completely controllable. The performance measure to be minimized is

$$V(u(k)) = \sum_{k=0}^{\infty} [x^*(k) Q x(k) + u^*(k) u(k)], \tag{5}$$

with $Q \geq 0$. The optimal control is given as the following lemma.

LEMMA 3 [Barnett and Cameron (1985)] *Consider the system* (4) *with* (5). *Suppose that* (A, Q_1) *is completely observable with* $Q_1^* Q_1 = Q$. *Then the optimal control which minimizes* (5) *is given by* $u(k) = -(I + B^* PB)^{-1} B^* PAx(k)$, *where* P *is the unique positive definite Hermitian matrix which satisfies the Riccati equation*

$$A^* PA - P - [B^* PA]^* [I + B^* PB]^{-1} [B^* PA] = -Q.$$

We will now present the last of our main results.

THEOREM 3 *Let $A \in C^{n\times n}$ and $B \in C^{n\times m}$. If (A, B) is completely controllable, then, for any given positive definite Hermitian matrix $P \in C^{n\times n}$, there exist nonsingular matrices $T \in C^{n\times n}$ and $Q \in C^{n\times n}$ such that*

(i) $(T^{-1}AT)^*P(T^{-1}AT) - P$
$- [B^*T^{-*}PT^{-1}AT]^*[I - B^*T^{-*}PT^{-1}B][B^*T^{-*}PT^{-1}AT] = -Q_1^*Q_1 < 0;$

(ii) $(T^{-1}AT, T^{-1}B)$ *is completely controllable;*

(iii) $(T^{-1}AT), Q_1)$ *is completely observable.*

Proof. By Lemma 3, there exists a matrix $G > 0$ satisfying

$$A^*GA - G - (A^*GB)[I + B^*GB]^{-1}(B^*GA) = -I.$$

By Lemma 1, there exists a nonsingular matrix T satisfying $G = T^{-*}PT^{-1}$. It can be readily obtained that

$$\begin{aligned}
&A^*T^{-*}PT^{-1}A - T^{-*}PT^{-1} \\
&- (A^*T^{-*}PT^{-1})[I + B^*T^{-*}PT^{-1}B]^{-1}(B^*T^{-*}PT^{-1}A) = -I \\
&\Rightarrow T^*\{A^*T^{-*}PT^{-1}A - T^{-*}PT^{-1} \\
&- (A^*T^{-*}PT^{-1}B)[I + B^*T^{-*}PT^{-1}B]^{-1}(B^*T^{-*}PT^{-1}A)\}T = -T^*T < 0 \\
&\Rightarrow (T^{-1}AT)^*P(T^{-1}AT) - P \\
&- [B^*T^{-*}PT^{-1}AT]^*[I + B^*T^{-*}PT^{-1}B]^{-1}[B^*T^{-*}PT^{-1}AT] \\
&= -Q_1^*Q_1 < 0,
\end{aligned}$$

if we define $Q_1 = T$. Clearly, it can be deduced that $(T^{-1}AT, T^{-1}B)$ is completely controllable and $(T^{-1}AT, Q_1)$ is completely observable in view of the PBH rank test from Chen (1984). This completes our proof. ■

4. Conclusions

In this note, the inverse problem of Lyapunov theorem has been reconsidered and the inverse problem of Riccati equation has been introduced. Simple proofs have been provided to guarantee the solution of such problems.

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