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Existence in optimal control problems of certain Fredholm integral equations

by

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Abstract: Existence of an optimal control for certain systems governed by nonlinear Fredholm integral equations that are of a Hammerstein type with respect to the control is proved under convexity of the orientor field by using a relaxed problem. Then, this convexity assumption is put away by finer analysis of the maximum principle. Illustrative examples are presented.

Keywords: optimal control, integral equation, existence, relaxation, orientor field, Young measures.

1. Problem formulation, first existence results

Existence theory in optimal control, which started from classical works by Filippov (1959) and Roxin (1962), has been successfully applied to various problems governed by ordinary differential equations, see Cesari (1983) or Mordukhovich (1977) for a survey. This theory is essentially based on convexity of the sets of admissible velocities, so-called orientor fields. Such convexity is often necessary, as demonstrated by Brunovský (1968) and, as Bittner (1994) remarked, it seemed to be the reason why no direct generalization to integral processes was available. Actually, Schmeling (1979, 1981) gave a sophisticated example for

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nonexistence of optimal solutions for a control process with Volterra equation of the type $y(t) = y_0(t) + \int_0^t f(t,\tau,y(\tau),u(\tau))\,\mathrm{d}\tau$, $u(t) \in S$, with the orientor field $f(t,\tau,r,S)$ always convex and compact, see (5.2) below, though for some rather special problems of this kind the existence can be proved, see Schmeling (1981) or Roubíček and Schmidt (1997).

Nevertheless, the Filippov–Roxin theory was extended for the nonlinear integral equations of the Fredholm (sometimes also called Urysohn) type that are of Hammerstein type with respect to control by Balder (1993), Bennati (1979), Cowles (1973) and, for less general equations, also by Zolezzi (1972). For the Volterra-type equations, even some more references exist; see Remark 2 below. Besides, the Fredholm case was also addressed by the authors, Roubíček and Schmidt (1997), but in completely different situations, via the Bauer's extremal principle.

In this paper, we want to present the Filippov–Roxin theory for this special class of Fredholm equations that are of Hammerstein type with respect to control, similarly as already done in Balder (1993), Bennati (1979), but by using an auxiliary relaxation by Roubíček (1998) similarly as proposed in Muñoz and Pedregal (2001) for problems governed by ordinary differential equations and in Roubíček (1999) for general situations. This provides a deeper insight and enables us to refine the existence results to cover also problems with nonconvex orientor field like it was done in Gabasov and Mordukhovich (1974), Ioffe and Tikhomirov (1974), Mordukhovich (1988, 1999), Muñoz and Pedregal (2001) for problems governed by ordinary differential equations.

To pursue this goal, we will treat the following isoperimetrically constrained optimal control problem:

$$\left.\begin{array}{l}
\text{Minimize } \displaystyle\int_\Omega \varphi(x,y(x),u(x))\,\mathrm{d}x \quad \text{(a cost functional)} \\
\quad \text{subject to } y(x) = y_0(x) + \displaystyle\int_\Omega K(x,\xi,y(\xi))f(\xi,y(\xi),u(\xi))\,\mathrm{d}\xi, \\
\qquad \text{(state equation)} \\
\quad \displaystyle\int_\Omega \vartheta(x,y(x),u(x))\,\mathrm{d}x \le 0, \quad \text{(state/control constraints)} \\
\quad u(x) \in S(x) \text{ for a.e. } x \in \Omega, \quad \text{(control constraints)} \\
\quad y \in L^q(\Omega;\mathbb{R}^n),\ u \in L^p(\Omega;\mathbb{R}^m),
\end{array}\right\} \qquad (\mathsf{P})$$

where Ω is a subset in a Euclidean space with a finite Lebesgue measure and the functions $K : \Omega\times\Omega\times\mathbb{R}^n \to \mathbb{R}^{n\times l}$, $f : \Omega\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^l$, $\varphi : \Omega\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$, $\vartheta : \Omega\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^k$ with $\mathbb{R}^k$ ordered by a closed convex cone (to give a sense to "$\vartheta \le 0$"), $y_0 : \Omega \to \mathbb{R}^n$ and the multivalued mapping $S : \Omega \rightrightarrows \mathbb{R}^m$ will be subjected to the following basic data qualification:

φ, K, f, ϑ are Carathéodory mappings, i.e.

measurable in $x, \xi \in \Omega$ and continuous in the other variables, (1.1a)

$$|\varphi(x,r_1,s)-\varphi(x,r_2,s)|$$
$$\le \left(a_{q/(q-1)}(x)+b|r_1|^{q-1}+b|r_2|^{q-1}+c|s|^{p(q-1)/q}\right)|r_1-r_2|, \tag{1.1c}$$
$$|f(x,r,s)|\le a_\gamma(x)+c|r|^{q/\gamma}+c|s|^{p/\gamma}, \tag{1.1d}$$
$$|f(x,r_1,s)-f(x,r_2,s)|$$
$$\le \left(a_{q\gamma/(q-\gamma)}(x)+b|r_1|^{(q-\gamma)/\gamma}+b|r_2|^{(q-\gamma)/\gamma}+c|s|^{p(q-\gamma)/(q\gamma)}\right)|r_1-r_2|, \tag{1.1e}$$
$$|K(x,\xi,r)|\le a^x_{\gamma/(\gamma-1)}(\xi)+c^x|r|^{q(\gamma-1)/\gamma}, \tag{1.1f}$$
$$|[Kf](x,\xi,r,s)|\le \sum_{j=1}^{J} a_{q,j}(x)a_{1,j}(\xi)+a_q(x)(|r|^{q-\varepsilon}+|s|^{p-\varepsilon}), \tag{1.1g}$$
$$|[Kf](x,\xi,r_1,s)-[Kf](x,\xi,r_2,s)|\le \ell(x,\xi)|r_1-r_2|, \tag{1.1h}$$
$$|\vartheta(x,r,s)|\le a_1(x)+c(|r|^q+|s|^{p-\varepsilon}), \tag{1.1i}$$
$$|\vartheta(x,r_1,s)-\vartheta(x,r_2,s)|$$
$$\le \left(a_{q/(q-1)}(x)+b|r_1|^{q-1}+b|r_2|^{q-1}+c|s|^{p(q-1)/q}\right)|r_1-r_2|, \tag{1.1j}$$
$$S:\Omega\rightrightarrows\mathbb{R}^m \text{ is measurable and } S(x) \text{ is closed for a.a. } x\in\Omega, \tag{1.1k}$$
$$y_0\in L^q(\Omega;\mathbb{R}^n), \tag{1.1 l}$$

where $n,m,l,k,J\in\mathbb{N}$ and $p\in[1,+\infty)$, $q,\gamma\in(1,+\infty)$, $q\ge\gamma$, $b,c\in\mathbb{R}$, $\ell\in L^q(\Omega;L^{q/(q-1)}(\Omega))$, $\varepsilon>0$, and $a_\alpha\in L^\alpha(\Omega)$, i.e. the lower indices in a's indicate integrability. The assumptions (1.1a,b,g,i) ensure existence of all three integrals appearing in (P) and together with (1.1c–f,h) make possible to do a correct relaxation by using Roubíček (1998) and also to rewrite the relaxed problems in terms of Young measures.

We will confine ourselves to the case when, for every admissible control u, the existence and uniqueness of the solution $y=y(u)$ to the involved integral equation is ensured by the Banach contraction principle, which can be guaranteed by the assumption

$$\|\ell\|_{L^q(\Omega;L^{q/(q-1)}(\Omega))}<1 \tag{1.2}$$

with ℓ referring to (1.1h), though other principles which are standard in integral-equation theory can be used, too. Of course, we need also a feasibility and coercivity of the whole problem (P). For simplicity, we can assume

$$\exists(u,y)\in L^p(\Omega;\mathbb{R}^m)\times L^q(\Omega;\mathbb{R}^n):\int_\Omega\vartheta(x,y(x),u(x))\,\mathrm{d}x\le 0,\ u(x)\in S(x),$$
$$y(x)=y_0(x)+\int_\Omega K(x,\xi,y(\xi))f(\xi,y(\xi),u(\xi))\,\mathrm{d}\xi \text{ a.e.}, \tag{1.3a}$$
$$\varphi(x,r,s)\ge\varepsilon|s|^p \text{ for some } \varepsilon>0. \tag{1.3b}$$

It turns out that the relevant orientor field has the form

$$Q(x,r):=\{(q_0,q_1,q_2)\in\mathbb{R}^{1+l+k};$$

for $x \in \Omega$ and $r \in \mathbb{R}^n$, see also Balder (1993), Bennati (1979), Zolezzi (1972). From the assumptions (1.1a) and (1.3b) it follows that $Q(x,r)$ is closed for a.a. x and for all r. A solution u to (P) will be called stable if any sequence of controls converging to some optimal control is minimizing asymptotically feasible, i.e.

$$\lim_{k\to\infty} \|u_k - u\|_{L^p(\Omega;\mathbb{R}^n)} = 0 \Rightarrow \lim_{k\to\infty} \left| \int_\Omega \vartheta(x, y_k(x), u_k(x))\,\mathrm{d}x \right| = 0$$
$$\& \limsup_{k\to\infty} \int_\Omega \varphi(x, y_k(x), u_k(x))\,\mathrm{d}x \le \inf(\mathsf{P}), \tag{1.5}$$

where y_k is a response to the control u_k, i.e. the solution to $y_k(x) = y_0(x) + \int_\Omega K(x,\xi,y_k(\xi)) f(\xi, y_k(\xi), u_k(\xi))\,\mathrm{d}\xi$.

Now we can formulate the generalization of the Filippov and Roxin theorem, which follows directly from Propositions 1–3 given further on:

THEOREM 1 *Let (1.1)–(1.3) hold and let*

$$Q(x,r) \text{ be convex} \tag{1.6}$$

for a.a. $x \in \Omega$ and all $r \in \mathbb{R}^n$. Then (P) *has an optimal solution. Moreover, any solution to* (P) *is stable in the sense (1.5).*

REMARK 1 *The above statement is similar to the existence theorem by Balder (1984, 1993) and Bennati (1979) where, however, no isoperimetric inequalities constraints are considered (in fact, Balder, 1984 and Bennati, 1979 consider the state constraint of the type $y(x) \in A$ not involving explicitly the control) and, because of weaker assumptions used in Balder (1984, 1993) and Bennati (1979), no stability is obtained. Yet, having not guaranteed that a perturbed control is ϵ-optimal in a suitable sense seems to have a little practical usage, although the existence of a (possibly unstable) optimal control is theoretically interesting.*

Let us also mention that Cesari's Q-type property, i.e. $Q(x,r) = \bigcap_{\delta>0} \mathrm{cl\,co} \bigcup_{\|r-\tilde{r}\|\le\delta} Q(x,\tilde{r})$, is valid because of (1.6) and continuity of the data with respect to r-variable as assumed, in particular, in (1.1a).

REMARK 2 *This convexity imposed on $Q(x,r)$ is trivially satisfied if $\varphi(x,r,\cdot)$ is convex, $f(x,r,\cdot)$ is linear, $\vartheta(x,r,\cdot)$ is convex (with respect to the ordering of $\mathbb{R}^k$), and $S(x)$ is convex for a.a. $x \in \Omega$ and all $r \in \mathbb{R}^n$. Then, however, the standard direct method based on weak compactness can be applied. In such linear/convex case, other classes of integral equations can also be handled, see Pełczewski (1989), Yusifov and Karagezov (1990), etc.*

REMARK 3 (Volterra integral equations) *The special case (of the Volterra type) of our integral equations has been already investigated by Angel (1976), Balder (1993), Carlson (1990) and Yeh (1978); they used $\Omega := [0,T]$, $l := 2$, $K(t,\tau,y)$*

constraints, and ensured existence by assuming convexity of an orientor field like (1.4). For some special Volterra case see also Schmeling (1981).

Moreover, for the even more special case of $n = l$ and $K(t, \tau, y) =$ identity matrix for $t \geq \tau$ otherwise $K = 0$, we can cover basically the original results by Filippov (1959) and Roxin (1962) but here with isoperimetrical state constraints; in this case the existence of a unique response of the controlled system is ensured without (1.2).

REMARK 4 (Semilinear elliptic equations) *For the special case of Ω being a smooth domain, $K \equiv K(x, \xi)$ the Green function of the Laplace operator $-\Delta$ on Ω, and y_0 a harmonic function, i.e. $\Delta y_0 = 0$ on Ω, our integral equation represents the Dirichlet boundary value problem for the semilinear equation*

$$-\Delta y(x) = f(x, y(x), u(x)) \text{ for a.a. } x \in \Omega, \tag{1.7a}$$

$$y(x) = y_0(x) \text{ for a.a. } x \in \partial\Omega \text{ (the boundary of } \Omega), \tag{1.7b}$$

as pointed out already by Pachpatte (1987), see also Stuart (1974) for the Neumann boundary conditions. Our assumption (1.2) then bounds Lipschitz continuity of $f(x, \cdot, s)$ in correlation with the integrability of Green's function K depending on the dimension of Ω, and Theorem 1 gives then existence of an optimal control of (1.7) under integral state/control constraints. For such sort of results we refer also to Papageorgiou (1991a, 1991b) and a one-dimensional illustrative Example 5.3 further on.

2. Auxiliary relaxed problem

In this section we will construct and analyze a continuously extended (so-called relaxed) problem. Essentially, we can use directly the results from Roubíček (1998) modified to our special case or, if $K(x, \xi, \cdot)$ is constant like in Remark 3, we can even directly use Roubíček (1997), Section 4.6. We first construct a suitable convex locally (sequentially) compact envelope of the Lebesgue space $L^p(\Omega; \mathbb{R}^m)$ of controls. To do this, we take a suitable linear space of Carathéodory integrands containing all possible nonlinearities, e.g. one can consider

$$\begin{aligned} H := \operatorname{span}\{&\varphi \circ y + a \cdot (f \circ y) + \vartheta_j \circ y + h_S; \\ &a \in L^{\gamma/(\gamma-1)}(\Omega; \mathbb{R}^l),\ y \in L^q(\Omega; \mathbb{R}^n),\ j = 1, \ldots, k\}, \end{aligned} \tag{2.1}$$

where $f \circ y : \Omega \times \mathbb{R}^m \to \mathbb{R}^l$ is defined by $[f \circ y](x, s) := f(x, y(x), s)$ and similarly also $\varphi \circ y : \Omega \times \mathbb{R}^m \to \mathbb{R}$ and $\vartheta \circ y : \Omega \times \mathbb{R}^m \to \mathbb{R}^k$, and moreover

$$h_S(x, s) := \max(1, \min_{\sigma \in S(x)} |\sigma - s|). \tag{2.2}$$

Note that h_S is a Carathéodory function because S is measurable, see (1.1k). In view of (1.1b,d,i), it is natural to equip H with which is a norm, see Roubíček (1997), Example 3.4.13. Since $\gamma > 1$ and $q < +\infty$, both $L^{\gamma/(\gamma-1)}(\Omega)$ and $L^q(\Omega)$ are separable, and thus also H is separable if equipped with the norm (2.3). For a ranging $C(\Omega;\mathbb{R}^l)$, this separability was shown in Roubíček (1999), Lemma 1. Here however, instead of the obvious inequality $\|a\cdot h\|_H \le \|a\|_{L^\infty(\Omega)}\|h\|_H$, we must use $\|a\cdot h\|_H \le \|a\|^{\gamma/(\gamma-1)}_{L^{\gamma/(\gamma-1)}(\Omega)} + \||h|^\gamma\|_H$, which allows us to estimate

$$\begin{aligned}
&\|a\cdot(f\circ y) - \tilde a\cdot(f\circ\tilde y)\|_H \le \delta_1\||f\circ y|^\gamma\|_H + C_{\delta_1}\|a-\tilde a\|^{\gamma/(\gamma-1)}_{L^{\gamma/(\gamma-1)}(\Omega;\mathbb{R}^l)}\\
&+\delta_2\|\tilde a\|^{\gamma/(\gamma-1)}_{L^{\gamma/(\gamma-1)}(\Omega;\mathbb{R}^l)} + C_{\delta_2}\||(f\circ y)-(f\circ\tilde y)|^\gamma\|_H\\
&=: I_1 + I_2 + I_3 + I_4
\end{aligned}\tag{2.4}$$

with $\delta_1,\delta_2 > 0$ arbitrarily small and $C_{\delta_1}, C_{\delta_2}$ depending on δ's. Then, for a given $\varepsilon > 0$, $a \in L^{\gamma/(\gamma-1)}(\Omega;\mathbb{R}^l)$ and $y \in L^q(\Omega;\mathbb{R}^n)$, one can take δ_1 small enough so that $I_1 \le \varepsilon/4$, then take $\tilde a$ from a (chosen fixed) dense countable subset close enough to a so that $I_2 \le \varepsilon/4$, further take δ_2 small enough so that $I_3 \le \varepsilon/4$, and finally $\tilde y$ from a (chosen fixed) dense countable subset close enough to y so that $I_4 \le \varepsilon/4$ by using (1.1e). This proves the separability of the set $\{a\cdot(f\circ y);\ a\in L^{\gamma/(\gamma-1)}(\Omega;\mathbb{R}^l),\ y\in L^q(\Omega;\mathbb{R}^n)\}$. From this, the separability of the whole space H follows easily by using also (1.1c,j).

Furthermore, we define a (norm,weak*)-continuous (possibly not injective) embedding $i : L^p(\Omega;\mathbb{R}^m) \to H^*$ by

$$\langle i(u), h\rangle := \int_\Omega h(x,u(x))\,\mathrm{d}x. \tag{2.5}$$

Of course, if h is valued in $\mathbb{R}^n$ (or in $\mathbb{R}^k$), then $\langle i(u),h\rangle$ is defined by (2.5) component-wise. Let us abbreviate also $K^x(\xi,r) := K(x,\xi,r)$. Then if $K^x f : \Omega\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ and $(K^x f)\circ y$ defined by $[(K^x f)\circ y]_i(\xi,s) := \sum_{j=1}^l K_{ij}(x,\xi,y(\xi))f_j(\xi,y(\xi),s)$ belongs to H^n due to (2.1) with (1.1f). We can rewrite the original problem (P) in the equivalent form

$$\left.\begin{array}{l}
\text{Minimize } \langle i(u),\varphi\circ y\rangle\\
\quad\text{subject to } y(x) = y_0(x) + \langle i(u),(K^x f)\circ y\rangle \text{ for a.a. } x\in\Omega,\\
\quad\langle i(u),\vartheta\circ y\rangle \le 0,\ \langle i(u),h_S\rangle = 0,\\
\quad y\in L^q(\Omega;\mathbb{R}^n)),\ u\in L^p(\Omega;\mathbb{R}^m).
\end{array}\right\}\tag{P$'$}$$

Note that, as $S(x)$ is closed (see (1.1k)), we have $h_S(x,s) > 0$ for $s\in\mathbb{R}^m\setminus S(x)$ while $h_S(x,s) = 0$ for $s\in S(x)$, and therefore $\langle i(u),h_S\rangle = 0$ is indeed equivalent to $u(x)\in S(x)$ for a.a. $x\in\Omega$.

Furthermore, we define the set of the so-called generalized Young functionals by $Y^p_H(\Omega;\mathbb{R}^m) := \text{w*-cl}\, i(L^p(\Omega;\mathbb{R}^m))$. It is known from Roubíček (1997) that, as a consequence of (2.1) with (1.3), $Y^p_H(\Omega;\mathbb{R}^m)$ is a convex locally (sequentially) us remind that $\varphi \circ y \in H$, $(K^x f) \circ y \in H^n$ (for a.a. $x \in \Omega$), $\vartheta \circ y \in H^k$, and $h_S \in H$ because of the choice (2.1) and (1.1f). Then, in view of (P′), we can define the relaxed problem as follows:

$$\left.\begin{array}{l} \text{Minimize } \langle \eta, \varphi \circ y \rangle \\ \quad \text{subject to } y(x) = y_0(x) + \langle \eta, (K^x f) \circ y \rangle \text{ for a.a. } x \in \Omega, \\ \quad \langle \eta, \vartheta \circ y \rangle \le 0, \ \langle \eta, h_S \rangle = 0, \\ \quad y \in L^q(\Omega; \mathbb{R}^n)), \ \eta \in Y_H^p(\Omega; \mathbb{R}^m). \end{array}\right\} \qquad (\mathsf{RP})$$

PROPOSITION 1 *Let (1.1)–(1.3) hold. Then the relaxed problem* (RP) *always has a solution* $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$. *Moreover,* $\min(\mathsf{RP}) \le \inf(\mathsf{P})$.

Proof. It just follows from Roubíček (1998), Proposition 4.1, together with Remark 3.2. Note that it can be explicitly seen from Roubíček (1998), Remark 3.3 that Roubíček (1998), Condition (10) is indeed fulfilled. ■

Let $L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(\mathbb{R}^m))$ denote the space of weakly measurable essentially bounded functions on Ω with values in the space of Borel measures on $\mathbb{R}^m$, and $\mathrm{rca}_1^+(\mathbb{R}^m)$ the set of all probability measures on $\mathbb{R}^m$. Further, let us define the set of the so-called L^p-Young measures as

$$\begin{aligned} \mathcal{Y}^p(\Omega; \mathbb{R}^m) := \Big\{ &(x \mapsto \nu_x) \in L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(\mathbb{R}^m)); \\ &\nu_x \in \mathrm{rca}_1^+(\mathbb{R}^m) \text{ for a.a. } x \in \Omega \text{ and } \int_\Omega \int_{\mathbb{R}^m} |s|^p \, \nu_x(\mathrm{d}s) \, \mathrm{d}x < +\infty \Big\}. \end{aligned} \qquad (2.6)$$

The natural imbedding $L^p(\Omega; \mathbb{R}^m) \to \mathcal{Y}^p(\Omega; \mathbb{R}^m)$ is then defined by $u \mapsto \nu$ with $\nu_x = \delta_{u(x)}$, where δ_s denotes the Dirac measure supported at $s \in \mathbb{R}^m$.

We will also need a relaxed problem written in terms of L^p-Young measures, which looks as follows:

$$\left.\begin{array}{l} \text{Minimize } \displaystyle\int_\Omega \int_{\mathbb{R}^m} \varphi(x, y(x), s) \, \nu_x(\mathrm{d}s) \, \mathrm{d}x \\ \quad \text{subject to } y(x) = y_0(x) \\ \quad + \displaystyle\int_\Omega K(x, \xi, y(\xi)) \Big(\int_{\mathbb{R}^m} f(\xi, y(\xi), s) \, \nu_\xi(\mathrm{d}s) \Big) \mathrm{d}\xi, \\ \quad \displaystyle\int_\Omega \int_{\mathbb{R}^m} \vartheta(x, y(x), s) \, \nu_x(\mathrm{d}s) \, \mathrm{d}x \le 0, \\ \quad \mathrm{supp}(\nu_x) \subset S(x) \text{ for a.e. } x \in \Omega, \\ \quad y \in L^q(\Omega; \mathbb{R}^n), \ \nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m). \end{array}\right\} \qquad (\mathsf{RP}')$$

We call $\eta \in Y_H^p(\Omega; \mathbb{R}^m)$ p-nonconcentrating if it is attainable by a sequence $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$ (i.e. $i(u_k) \to \eta$ weakly* in H^*) such that $\{|u_k|^p; \, k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$. Since H is separable, any such η possesses at least one L^p-Young measure representation $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$ in the sense that for all $h \in H$:

and, conversely, any $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ determines by the formula (2.7) some p-nonconcentrating $\eta \in Y_H^p(\Omega;\mathbb{R}^m)$, see Roubíček (1997), Proposition 3.4.15.

PROPOSITION 2 *Under the assumptions of Proposition 1, any solution $\eta \in Y_H^p(\Omega;\mathbb{R}^m)$ of* (RP) *is p-nonconcentrating and any L^p-Young-measure representation $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ of such η solves* (RP$'$). *Moreover*, $\min(\mathsf{RP}') = \min(\mathsf{RP})$.

Proof. The p-nonconcentration of any η solving (RP) follows from Roubíček (1998), Propositions 4.2 with 4.1. Thus we do not change the set of minimizers by restricting (RP) only to p-nonconcentrating functionals in $Y_H^p(\Omega;\mathbb{R}^m)$.

By using L^p-Young measure representation of these functionals, we can equivalently write this modified relaxed problem in the form:

$$\left.\begin{array}{l}
\text{Minimize } \displaystyle\int_\Omega\int_{\mathbb{R}^m}\varphi(x,y(x),s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x \\
\quad\text{subject to } y(x)=y_0(x) \\
\quad+\displaystyle\int_\Omega\int_{\mathbb{R}^m}[K^x f](\xi,y(\xi),s)\,\nu_\xi(\mathrm{d}s)\,\mathrm{d}\xi \text{ for a.a. } x\in\Omega, \\
\quad\displaystyle\int_\Omega\int_{\mathbb{R}^m}\vartheta(x,y(x),s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x\le 0, \\
\quad\displaystyle\int_\Omega\int_{\mathbb{R}^m}h_S(x,s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x=0, \\
\quad y\in L^q(\Omega;\mathbb{R}^n),\ \nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m).
\end{array}\right\}\qquad (\mathsf{RP}'')$$

In particular, $\min(\mathsf{RP}'') = \min(\mathsf{RP})$.

As $K(x,\xi,y(\xi))$ is simply a constant in terms of s for a.a. $\xi\in\Omega$, we get

$$\begin{aligned}
&\int_\Omega\int_{\mathbb{R}^m}[K^x f](\xi,y(\xi),s)\,\nu_\xi(\mathrm{d}s)\,\mathrm{d}\xi \\
&=\int_\Omega\int_{\mathbb{R}^m}K^x(\xi,y(\xi))f(\xi,y(\xi),s)\,\nu_\xi(\mathrm{d}s)\,\mathrm{d}\xi \\
&=\int_\Omega K(x,\xi,y(\xi))\Big(\int_{\mathbb{R}^m}f(\xi,y(\xi),s)\,\nu_\xi(\mathrm{d}s)\Big)\,\mathrm{d}\xi.
\end{aligned}$$

Also, $\int_\Omega\int_{\mathbb{R}^m}h_S(x,s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x=0$ is equivalent to the condition $\mathrm{supp}(\nu_x)\subset S(x)$ (for a.e. $x\in\Omega$) because it holds $h_S(x,s)>0$ for $s\in\mathbb{R}^m\setminus S(x)$ while $h_S(x,s)=0$ for $s\in S(x)$.

Altogether, we can see that (RP$''$) is equivalent to (RP$'$). ∎

3. Proof of Theorem 1

Now we are ready to give a quite simple proof of Theorem 1, following Roubíček (1999), Lemma 2. Let us remark that the optimal Young-measure solution and the measurable-selection technique has already been used in Muñoz and but in a bit different arrangement relying on the maximum principle. Inspired by Muñoz and Pedregal (2001), it will be more suitable for the proof and especially for further modification in Section 4 to reformulate the convexity condition (1.6) into the form

$$\mathrm{co}[\varphi \times f \times \vartheta](x, r, S(x)) \subset Q(x, r). \tag{3.1}$$

Indeed, (3.1) implies (1.6) because, taking $q^1, q^2 \in Q(x,r)$, one has $s^1, s^2 \in S(x)$ such that $q_1^i \geq \varphi(x,r,s^i)$, $q_2^i = f(x,r,s^i)$, and $q_3^i \geq \varphi(x,r,s^i)$ for $i = 1, 2$, and then (3.1) guarantees existence of $s^3 \in S(x)$ such that $\sum_{i=1,2} \frac{1}{2}(\varphi(x,r,s^i),$ $f(x,r,s^i), \varphi(x,r,s^i)) \in Q$, which eventually results in $\sum_{i=1,2} \frac{1}{2} q^i \in Q(x,r)$. Conversely, (1.6) implies (3.1) because always $\mathrm{co}[\varphi \times f \times \vartheta](x,r,S(x)) \subset \mathrm{co}Q(x,r)$.

PROPOSITION 3 *Assume (1.1)–(1.3) hold. Let ν be the solution to* (RP′) *and let also (1.6) hold for a.a. $x \in \Omega$ and all $r \in \mathbb{R}^n$. Then there is $u \in L^p(\Omega;\mathbb{R}^m)$ such that*

$$\begin{aligned} u(x) \in U(x) := \Big\{ s \in S(x);\ &\varphi(x,y(x),s) \leq \int_{\mathbb{R}^m} \varphi(x,y(x),\sigma)\,\nu_x(\mathrm{d}\sigma), \\ f(x,y(x),s) &= \int_{\mathbb{R}^m} f(x,y(x),\sigma)\,\nu_x(\mathrm{d}\sigma), \\ \vartheta(x,y(x),s) &\leq \int_{\mathbb{R}^m} \vartheta(x,y(x),\sigma)\,\nu_x(\mathrm{d}\sigma) \Big\}, \quad x \in \Omega, \end{aligned} \tag{3.2}$$

and any such u is an optimal control for (P).

Proof. For a.a. $x \in \Omega$, ν_x can be approximated weakly* by a sequence $\{\nu_x^j\}_{j\in\mathbb{N}}$ of convex combinations of Dirac measures supported on $\mathrm{supp}(\nu_x)$. Then $\int_{\mathbb{R}^m} [\varphi \times f \times \vartheta](x,y(x),s)\,\nu_x^j(\mathrm{d}s) \in \mathrm{co}[\varphi \times f \times \vartheta](x,y(x),\mathrm{supp}(\nu_x))$, so that by passing to the limit and using (1.6) in the form (3.1), one gets

$$\begin{aligned} &\int_{\mathbb{R}^m} [\varphi \times f \times \vartheta](x,y(x),s)\,\nu_x(\mathrm{d}s) = \lim_{j\to\infty} \int_{\mathbb{R}^m} [\varphi \times f \times \vartheta](x,y(x),s)\,\nu_x^j(\mathrm{d}s) \\ &\in \mathrm{cl\,co}[\varphi \times f \times \vartheta](x,y(x),\mathrm{supp}(\nu_x)) \\ &\subset \mathrm{cl\,co}[\varphi \times f \times \vartheta](x,y(x),S(x)) \subset Q(x,y(x)); \end{aligned} \tag{3.3}$$

this limit passage is indeed correct at each $x \in \Omega$ for which $\int_{\mathbb{R}^m} |s|^p\,\nu_x(\mathrm{d}s)$ is finite, see Roubíček (1999), Lemma 2 for details.

This enables us to show that the set $U(x)$ defined by (3.2) is nonempty. Indeed, because of definition (1.4), for any $(q_0, q_1, q_2) \in Q(x,y(x))$ there is $s \in S(x)$ such that $q_0 \geq \varphi(x,y(x),s)$, $q_1 = f(x,y(x),s)$ and $q_2 \geq \vartheta(x,y(x),s)$. Hence, for the particular choice the inclusion (3.3) implies that $q_0 \geq \varphi(x, y(x), s)$, $q_1 = f(x, y(x), s)$ and $q_2 \geq \vartheta(x, y(x), s)$ for some $s \in S(x)$.

Moreover, by Aubin and Frankowska (1990) the multivalued mapping $U : \Omega \rightrightarrows \mathbb{R}^m$ defined by (3.2) is measurable because S is measurable, ν is weakly measurable, and φ, f and ϑ are Carathéodory mappings, see, again, Roubíček (1999), Lemma 2 for details.

Obviously, $U(x)$ is closed for a.a. $x \in \Omega$ because $S(x)$ is closed and $\varphi(x, r, \cdot)$, $f(x, r, \cdot)$ and $\vartheta(x, r, \cdot)$ are continuous, see (1.1k) and (1.1a), respectively. Then, by Aubin and Frankowska (1990), Theorem 8.1.4, the multivalued mapping U possesses a measurable selection $u(x) \in U(x)$.

In particular, $u(x) \in S(x)$. Moreover, in view of (3.2) with (3.4),

$$\begin{aligned}&\int_\Omega \vartheta(x, y(x), u(x))\,\mathrm{d}x \\ &\leq \int_\Omega q_2(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^m} \vartheta(x, y(x), s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x \leq 0 \end{aligned} \tag{3.5}$$

and also

$$f(x, y(x), u(x)) = q_1(x) = \int_{\mathbb{R}^m} f(x, y(x), s)\,\nu_x(\mathrm{d}s) \tag{3.6}$$

for a.a. $x \in \Omega$, so that

$$\begin{aligned}y(x) &= y_0(x) + \int_\Omega K(x, \xi, y(\xi)) \Big(\int_{\mathbb{R}^m} f(\xi, y(\xi), s)\,\nu_\xi(\mathrm{d}s) \Big)\,\mathrm{d}\xi \\ &= \int_\Omega K(x, \xi, y(\xi)) q_1(\xi)\,\mathrm{d}\xi = \int_\Omega K(x, \xi, y(\xi)) f(\xi, y(\xi), u(\xi))\,\mathrm{d}\xi. \end{aligned} \tag{3.7}$$

By using also Propositions 1 and 2, we get

$$\begin{aligned}&\int_\Omega \varphi(x, y(x), u(x))\,\mathrm{d}x \leq \int_\Omega q_0(x)\,\mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^m} \phi(x, y(x), s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x = \min(\mathsf{RP}') = \min(\mathsf{RP}) \leq \inf(\mathsf{P}). \end{aligned} \tag{3.8}$$

In particular, (3.8) and the coercivity (1.3b) together with (1.3a) and (1.1b) imply that

$$\varepsilon \int_\Omega |u(x)|^p\,\mathrm{d}x \leq \int_\Omega \varphi(x, y(x), u(x))\,\mathrm{d}x \leq \inf(\mathsf{P}) < +\infty \tag{3.9}$$

so that $u \in L^p(\Omega; \mathbb{R}^m)$.

Altogether, (3.5), (3.7), and (3.9) show that the pair (u, y) is admissible

REMARK 5 (Zero relaxation gap) *Under the assumptions (1.1)–(1.3) and (1.6), Proposition 3 implies that the first term in (3.8) equals to* $\inf(\mathsf{P})$, *so that (3.8) gives* $\min(\mathsf{RP}) = \inf(\mathsf{P}) = \min(\mathsf{P})$, *i.e. there is no relaxation gap. Let us emphasize that this is a nontrivial fact for problems involving state constraints.*

Proof of Theorem 1. By Proposition 1, we get some η solving (RP). Then Proposition 2 yields its Young-measure representation ν solving (RP'). Finally, using this ν, Proposition 3 gives a solution u to (P). The stability (1.5) of any solution to (P) is a consequence of zero relaxation gap (see Remark 5) and the correctness (in mathematical sense) of the relaxed problem, see Roubíček (1997), Sect. 4.1. ∎

4. Refined existence results

The method used above allows easily to refine the existence results on condition that one has some a priori information about the support of the Young measure representing some optimal relaxed control. Such information can be obtained sometimes by analyzing the maximum principle. In the context of optimal control of ordinary differential equations, we refer to Gabasov and Mordukhovich (1974), Ioffe and Tikhomirov (1974, Sect. 9.2.2, Proposition 1 with Theorem 3], Mordukhovich (1988), Sect. 20 or Mordukhovich (1999), and also to Muñoz and Pedregal (2001).

To use the standard maximum principle, we must still assume certain smoothness of the data with respect to the state variable to ensure the relaxed problem to be smooth. Namely, we assume that, for $q \geq 2$, the data K, f, φ, and ϑ satisfy, beside (1.1)–(1.2), also

$$\varphi_r', K_r', f_r', \vartheta_r' \text{ are Carathéodory mappings,} \tag{4.1a}$$

$$\begin{aligned}&|\varphi_r'(x, r_1, s) - \varphi_r'(x, r_2, s)| \\ &\leq (a_{q/(q-2)}(x) + b|r_1|^{q-2} + b|r_2|^{q-2} + c|s|^{p(q-2)/q})|r_1 - r_2|,\end{aligned} \tag{4.1b}$$

$$|[Kf]_r'(x, \xi, r_1, s) - [Kf]_r'(x, \xi, r_2, s)| \leq \ell_1(x, \xi)|r_1 - r_2|, \tag{4.1c}$$

$$\begin{aligned}&|\vartheta_r'(x, r_1, s) - \vartheta_r'(x, r_2, s)| \\ &\leq (a_{q/(q-2)}(x) + b|r_1|^{q-2} + b|r_2|^{q-2} + c|s|^{p(q-2)/q})|r_1 - r_2|\end{aligned} \tag{4.1d}$$

with $\ell_1 \in L^q(\Omega; L^{q/(q-2)}(\Omega))$; here φ_r' denotes the differential of $f(x, \cdot, s)$, K_r' is the differential of $K(x, \xi, \cdot)$, etc., and the remaining notation is as in (1.1). Moreover, (1.1f) is to be strengthened a bit:

$$|K(x, \xi, r)| \leq a(x, \xi) + c|r|^{q(\gamma-1)/\gamma} \tag{4.1e}$$

for some $a^* \in L^{(\gamma-1)/\gamma}(\Omega; L^q(\Omega))$ with $a^*(x, \xi) = a(\xi, x)$; equivalently, this requires $\int_\Omega (\int_\Omega |a(\xi, x)|^q \, d\xi)^{\gamma/(q(\gamma-1))} \, dx < +\infty$.

PROPOSITION 4 *Let the assumptions (1.1)–(1.3) and (4.1) hold, and let the* *and $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ be a Young-measure representation of η in the sense (2.7). Then, for some multipliers $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ not vanishing simultaneously, the following maximum principle holds*

$$\int_{\mathbb{R}^m} \mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,s)\,\nu_x(\mathrm{d}s) = \max_{s\in S(x)} \mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,s) \text{ for a.a. } x \in \Omega, \quad (4.2)$$

where the Hamiltonian $\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}$ is defined by

$$\begin{aligned}\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,s) &:= \int_\Omega \lambda(\xi)K(\xi,x,y(x))\,\mathrm{d}\xi\, f(x,y(x),s)\\ &- \lambda_0\varphi(x,y(x),s) - \lambda_1^{\mathrm{T}}\vartheta(x,y(x),s) \qquad (4.3)\end{aligned}$$

and $\lambda \in L^{q/(q-1)}(\Omega;\mathbb{R}^n)$ solves the adjoint linear integral equation

$$\begin{aligned}\lambda(x) &= \int_\Omega\int_{\mathbb{R}^m} [Kf]'_r(\xi,x,y(x),s)^{\mathrm{T}}\lambda(\xi)\,\nu_x(\mathrm{d}s)\,\mathrm{d}\xi\\ &- \lambda_0\int_{\mathbb{R}^m}\varphi'_r(x,y(x),s)\,\nu_x(\mathrm{d}s) - \lambda_1^{\mathrm{T}}\int_{\mathbb{R}^m}\vartheta'_r(x,y(x),s)\,\nu_x(\mathrm{d}s) \qquad (4.4)\end{aligned}$$

for a.a. $x \in \Omega$ with $(\cdot)^{\mathrm{T}}$ denoting the transposition. Moreover, the following transversality condition holds:

$$\lambda_1^{\mathrm{T}}\int_\Omega\int_{\mathbb{R}^m}\vartheta(x,y(x),s)\,\nu_x(\mathrm{d}s)\,\mathrm{d}x = 0. \qquad (4.5)$$

Proof. Let us enlarge H from (2.1) so that also

$$(\varphi'_r \circ y)\tilde{y} \in H,\ (\vartheta'_r \circ y)\tilde{y} \in H^k,\ ([Kf]'_r \circ y)\tilde{y} \in H^n$$

for all $y, \tilde{y} \in L^q(\Omega;\mathbb{R}^n)$. Moreover, (4.1e) ensures that $x \mapsto \int_\Omega \lambda(\xi)K(\xi,x,y(x))\,\mathrm{d}\xi$ belongs to $L^{\gamma/(\gamma-1)}(\Omega;\mathbb{R}^l)$ for any $\lambda \in L^{q/(q-1)}(\Omega;\mathbb{R}^n)$ so that also $\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y} \in H$ due to (2.1). In such a way, we get a refined relaxation scheme yielding a smooth relaxed problem (RP). Moreover, we can assume H separable; it just suffices to modify the arguments of continuity (2.4) and to use (4.1a–d) and the separability of $L^q(\Omega;\mathbb{R}^n)$. Then, our assertion just follows as a special case from Roubíček (1998), Proposition 5.1 with Remark 3.2. ∎

Now we can state a rather strong existence principle which does not require the orientor field Q to be convex. The usage of such result for nonconvex Q is, however, not straightforward because it requires a certain information about at least one solution of the relaxed problem and the corresponding adjoint state. Yet, in particular cases a relevant information can sometimes be isolated, see Section 5 below.

THEOREM 2. *Let the assumptions (1.1)–(1.3) and (4.1) hold, and let the cone* *with corresponding λ's from Proposition 4, and let the following condition hold for a.a. $x \in \Omega$:*

$$\mathrm{co}[\varphi \times f \times \vartheta](x, y(x), M(x)) \subset Q(x, y(x)), \tag{4.6}$$

where $M(x) \subset S(x)$ is an estimate of the set of maximizers of the Hamiltonian $\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}$, i.e.

$$M(x) \supset \{s \in S(x);\ \mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,s) = \max_{\tilde{s}\in S(x)} \mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,\tilde{s})\}. \tag{4.7}$$

Then (P) *has at least one solution and, moreover, every solution to* (P) *is stable in the sense (1.5).*

Proof. Let ν denote some Young-measure representation of the optimal relaxed control η in question. By Proposition 4, for a.a. $x \in \Omega$, ν_x must be supported on $M(x)$. Then, Proposition 3 gives a solution u to (P) if used in a refined way, namely (3.3) can simply exploit $M(x)$ and (4.6) instead of $S(x)$ and (3.1), respectively. The stability (1.5) is again the consequence of zero relaxation gap (by arguments as in Remark 5) and of the correctness of the relaxed problem. ■

REMARK 6 *There are two extreme situations. First, the maximum principle does not give any specific information, see Section 5.1 below, or the particular problem often is so complicated that one is unable to extract such information; then we can say that $M(x) = S(x)$ only, and (4.6) coincides with (3.1), i.e. with the Filippov–Roxin-type condition (1.6). Second, sometimes it may happen that one can a priori guarantee, by a specific analysis, that the Hamiltonian $\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,\cdot)$ is maximized only at a single point for a.a. $x \in \Omega$, see Remark 7 below, i.e. the set on the right-hand side of (4.7) is a singleton for a.a. $x \in \Omega$, and by choosing $M(x)$ equal to this set we make the condition (4.6) trivially satisfied.*

5. Examples

In this last section we present three concrete, rather simple, illustrative examples.

5.1. Schmeling's example of nonexistence in Fredholm equations revisited

To present the example by Schmeling (1979, 1981) in our context, we must modify it slightly. For example, we consider the following problem:

$$\left.\begin{array}{l} \text{Minimize } \displaystyle\int_0^T (2y(t) - t^2)^2\,\mathrm{d}t \\ \quad \text{subject to } y(t) = \displaystyle\int_0^t [(T-\tau)u(\tau) + (t-\tau)u(\tau)^2]\,\mathrm{d}\tau, \\ u(t) \in [-1, 1] \text{ for a.a. } t \in [0, T], \end{array}\right\} \tag{P_1}$$

As usual for Volterra equations, we put $\Omega := (0,T)$ and write t and τ instead of x and ξ, respectively.

For reader's convenience, let us remind the slightly modified arguments from Schmeling (1981) to show the nonexistence of solutions of (P_1): Taking a fast oscillating sequence $\{u_k : (0,T) \to \{1,-1\}\}_{k\in\mathbb{N}}$ converging to 0 weakly in $L^p(0,T)$, we get a sequence of corresponding states $\{y_k\}_{k\in\mathbb{N}}$ converging to $y(t) = \frac{1}{2}t^2$ in $C(0,T)$, which shows $\inf(\mathsf{P}_1) = 0$. Therefore, if there exists a solution (u,y) to (P_1), then inevitably $y(t) = \frac{1}{2}t^2$ would hold. By differentiating the Volterra equation involved in (P_1), one would get

$$\begin{aligned} t &= (T-t)u(t) - tu(t)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t tu(\tau)^2 \,\mathrm{d}\tau \\ &= (T-t)u(t) + \int_0^t u(\tau)^2 \,\mathrm{d}\tau \end{aligned} \tag{5.1}$$

for a.a. $t \in [0,T]$. Note that, since u is bounded, $t \mapsto \int_0^t [(T-\tau)u(\tau) + (t-\tau)u(\tau)^2]\,\mathrm{d}\tau$ is Lipschitz continuous and thus a.e. differentiable. If possibly (5.1) does not hold just for $t := T$, we can pass to the limit with $t \to T$ in (5.1), which gives $T = \int_0^T u(\tau)^2\,\mathrm{d}\tau$. This would be possible only if $|u| = 1$ a.e. on $[0,T]$. Coming back to (5.1), we would get $t = (T-t)u(t) + t$, which would give $u = 0$, a contradiction showing that a solution (u,y) to (P_1) cannot exist.

Of course, (P_1) must somehow violate the assumptions of the above presented theory. We consider $n = m = 1$, p, q arbitrary, $\varphi(t,r,s) := (2r - t^2)^2$ (we can add the term like $\varepsilon \max(|s|^p, 1)$ to satisfy formally (1.3b)), $\vartheta := 0$, and $S(t) := [-1,1]$. Then, we can think of choosing $l = 1$ and thus have to put

$$f(t,\tau,r,s) := (T-\tau)s + (t-\tau)s^2, \;\; K(t,\tau,r) = \begin{cases} 1 & \text{if } t \geq \tau, \\ 0 & \text{if } t < \tau. \end{cases} \tag{5.2}$$

This makes the orientor field $f(t,\tau,r,S)$ always convex compact, but evidently such f does not have the form required in Theorems 1 and 2 because it depends also on t. Alternatively, we can choose $l = 2$ and then put

$$f(t,r,s) := (s, s^2), \;\; K(t,\tau,r) = \begin{cases} (T-t, t-\tau) & \text{if } t \geq \tau, \\ (0,0) & \text{if } t < \tau. \end{cases} \tag{5.3}$$

It has already the above considered form, but the orientor field

$$Q(t,r) = \{(q_0, s, s^2) \in \mathbb{R}^3;\; q_0 \geq 0,\; s \in [-1,1]\} \tag{5.4}$$

is evidently nonconvex and (1.6) is violated so that Theorem 1 cannot be used. Analyzing the maximum principle (4.2)–(4.5), we get simply $\lambda_0 = 1$, $\lambda_1 = 0$, and $\lambda = 4(2y - t^2)$. For the (unique) relaxed optimal control $\nu_t = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, one gets $y = \frac{1}{2}t^2$, and hence $\lambda = 0$ and the Hamiltonian $\mathcal{H}(t,\cdot)$ constant for a.a. $t \in [0,T]$, so that necessarily $M(t) = S(t)$ and (4.6) thus coincides with (1.6).

5.2. Mordukhovich's example with a nonconvex orientor field modified

To present the examples from Mordukhovich (1988), Sect. 20 or Mordukhovich (1999) in context of integral equations, we consider the following problem:

$$\left.\begin{array}{l}\text{Minimize} \displaystyle\int_\Omega y(x)^2\,\mathrm{d}x \\ \quad\text{subject to } y(x) = y_0(x) \\ \quad + \displaystyle\int_\Omega (K_1(x,\xi)u_1(\xi) + K_2(x,\xi)u_2(\xi))\,\mathrm{d}\xi, \\ \quad u(x) \equiv (u_1(x), u_2(x)) \in \{1,-1\}\times[-1,1] \text{ for a.a. } x\in\Omega, \\ \quad y\in L^2(\Omega),\ u\in L^\infty(\Omega;\mathbb{R}^2), \end{array}\right\} \qquad (\mathsf{P}_2)$$

This is a very special case of (P), with $m = l = 2$, $n = 1$, $k = 0$, $\varphi(x,r,s) = r^2$, $f(x,r,s) = s$, $K(x,\xi,r) = (K_1(x,\xi), K_2(x,\xi)) \in \mathbb{R}^{n\times l} \equiv \mathbb{R}^2$, $\vartheta = 0$, and $S(x) = \{1,-1\}\times[-1,1]$ having two disconnected convex components. Our assumptions are rather trivially fulfilled with $q = 2$, $p\in[1,+\infty)$ arbitrary and $\ell = \ell_1 = 0$, provided K satisfies (1.1f,g), which means here $K_{1,2}(x,\cdot)\in L^{\gamma/(\gamma-1)}(\Omega)$ for a.a. $x\in\Omega$ and $|K_{1,2}(x,\xi)| \le \sum_{j=1}^J a_{q,j}(x)a_{1,j}(\xi)$, the growth with respect to s as well as the coercivity (1.3b) being irrelevant since $S(x)$ is a priori bounded. As there are no state constraints, we have simply $\lambda_0 = 1$ and $\lambda_1 = 0$, and the adjoint equation (4.4) then gives $\lambda = -2y$. Then, the Hamiltonian (4.3) results in $\mathcal{H}_y(x,s) \equiv \mathcal{H}_y(x,s_1,s_2) = -2\sum_{i=1,2} s_i \int_\Omega y(\xi)K_i(\xi,x)\,\mathrm{d}\xi$. On certain conditions, it may happen that the set of maximizers of $\mathcal{H}_y(x,\cdot)$ is contained only in one component of $S(x)$. E.g., assuming

$$K_1(x,\xi)\begin{cases} < 0 & \text{if } y_0(x) > 0, \\ > 0 & \text{if } y_0(x) < 0 \end{cases} \tag{5.5}$$

for a.a. $\xi\in\Omega$, we have the following chain of implications

$$\begin{aligned} &|y_0(x)| > \int_\Omega [|K_1(x,\xi)| + |K_2(x,\xi)|]\,\mathrm{d}\xi \Rightarrow |y(x)| > 0 \\ &\Rightarrow \forall s_2 : s_1 \mapsto \mathcal{H}_y(x,s_1,s_2) \text{ increasing} \\ &\text{or } \forall s_2 : s_1 \mapsto \mathcal{H}_y(x,s_1,s_2) \text{ decreasing,} \end{aligned}$$

for x fixed (but arbitrary). This ensures that, for a.a. $x\in\Omega$, the set $M_y(x) := \{s\in S(x);\ \mathcal{H}_y(x,s) = \max\mathcal{H}_y(x,S(x))\}$ satisfies

$$M_y(x)\subset\{1\}\times[-1,1] \text{ or } M_y(x)\subset\{-1\}\times[-1,1].$$

Realizing that $\mathcal{H}_y(x,\cdot)$ is affine, we can see that $M_y(x)$ is convex for a.a. $x\in\Omega$, and thus (4.6) satisfied. Then, Theorem 2 with $M = M_y$ yields existence of the solution to (P_2) provided (5.5) holds and $|y_0| > \int_\Omega(|K_1(\cdot,\xi)| + |K_2(\cdot,\xi)|)\,\mathrm{d}\xi$ is

Let us emphasize that $Q(x,r) = (r^2 + \mathbb{R}^+) \times S(x)$ is nonconvex in our case so that the classical Filippov–Roxin-type condition (1.6) cannot be used. Also, the cost functional is strictly convex with respect to state and thus the relaxed problem to (P_2) has no concave structure, so that also the results from Roubíček and Schmidt (1997) relying on Bauer's principle cannot be used. Moreover, contrary to the original example Mordukhovich (1988, 1999), $M_y(x)$ need not be a singleton so that we cannot apriori say that the Young-measure representation of the optimal relaxed control is a.e. a Dirac measure, which would yield the existence of an optimal control of (P_2) straightforwardly. Besides, nontriviality of this example relies also on the fact that the solution to (P_2) indeed does not exist if, e.g., $y_0 = 0$ and $K_1 \geq 0$ does not vanish, which in fact covers also the well-known Bolza example.

To obtain the original Mordukhovich's (1988, 1999) example, one can put simply $\Omega := (0,T)$, $K_2(t,\tau) = 0$, $K_1(t,\tau) = 1$ for $t \geq \tau$, otherwise $K_1(t,\tau) = 0$, and y_0 constant.

5.3. Example with a boundary-value problem and an isoperimetric inequality

This example illustrates a simple, but rather nontrivial application of Theorems 1 and 2 to optimal control of elliptic equations (as announced in Remark 4). Let us consider the following problem:

$$\left.\begin{array}{l}
\text{Minimize } \displaystyle\int_0^1 u^2(x)\,\mathrm{d}x \\
\quad \text{subject to } y(x) = a(1-x) + bx \\
\quad + \displaystyle\int_0^x (1-x)\xi(\sin u(\xi) - cy(\xi))\,\mathrm{d}\xi \\
\quad + \displaystyle\int_x^1 x(1-\xi)(\sin u(\xi) - cy(\xi))\,\mathrm{d}\xi, \\
\quad \displaystyle\int_0^1 y(x)^3\,\mathrm{d}x \leq 0, \\
\quad y \in L^3(0,1),\ u \in L^2(0,1).
\end{array}\right\} \qquad (\mathsf{P}_3)$$

This falls into the previous considerations if $k = l = m = 1$, $p = 2$ (determined by the growth of $\varphi(x,r,\cdot)$), $q = 3$ (in agreement with the growth of $\vartheta(x,\cdot,s)$), $y_0(x) = a(1-x) + bx$, $f(x,r,s) = -cr + \sin s$, $\varphi(x,r,s) = s^2$, $\vartheta(x,r,s) = r^3$, $S = \mathbb{R}$, and the kernel K is symmetric:

$$K(x,\xi,r) = \begin{cases} (1-\xi)x & \text{for } x \leq \xi, \\ \xi(1-x) & \text{for } x > \xi. \end{cases} \qquad (5.6)$$

The constant c should satisfy

in order to fulfill (1.2). However, using the monotonicity argument instead of the Banach-fixed-point one, we can admit arbitrary c positive, too.

The integral equation is then equivalent to the two-point boundary-value problem for the 2nd-order linear elliptic ordinary equation:

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + cy = \sin u \text{ on } (0,1),\ y(0) = a,\ y(1) = b, \tag{5.8}$$

which can be checked just by direct calculation, if the short-hand notation $w = \sin u - cy$ is used:

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a(1-x) + bx + \int_0^x (1-x)\xi w(\xi)\,\mathrm{d}\xi + \int_x^1 x(1-\xi)w(\xi)\,\mathrm{d}\xi\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(b - a - \int_0^x \xi w(\xi)\,\mathrm{d}\xi + \int_x^1 (1-\xi)w(\xi)\,\mathrm{d}\xi\right)\\ &= -xw(x) - (1-x)w(x) = -w(x).\end{aligned}$$

Let us mention an illustrative interpretation of (5.8) as a deflection of stretched, homogeneous, elastically supported (c determines the linear response of the support) string with fixed end points, loaded in a perpendicular direction by the force $\sin u$. This also shows that the solution $y \in L^2(0,1)$ is, in fact, smooth, namely $y \in W^{2,\infty}(0,1)$ if one uses the standard notation for Sobolev spaces. It is an interesting observation that, although $f(x,r,\cdot)$ is nonlinear, the condition (1.6) is valid, i.e. $Q(x,r)$, here independent of (x,r), is always convex. Thus Theorem 1 yields existence on an (unspecified) optimal control.

Let us now modify (P_3) by restricting the admissible control values to the set

$$S(x) = [-3\pi, -2\pi] \cup [-\pi, 0] \cup [2\pi, 3\pi]. \tag{5.9}$$

Then $Q(x,r)$, again independent of (x,r), is no longer convex so that Theorem 1 does not apply. Let us analyze the optimality conditions. The adjoint equation (4.4) now looks as

$$\lambda(x) = -c\int_0^1 K(x,\xi)\lambda(\xi)\,\mathrm{d}\xi - 3\lambda_1 y(\xi)^2 \tag{5.10}$$

for $K(x,\xi) \equiv K(x,\xi,r)$ defined by (5.6) and for some λ_0, λ_1 non-negative, $\lambda_0+\lambda_1 > 0$. We claim that

$$\Lambda(x) := \int_0^1 K(\xi,x)\lambda(\xi)\,\mathrm{d}\xi \le 0 \text{ for a.a. } x \in [0,1]. \tag{5.11}$$

For $c \le 0$, we can see directly from (5.10) that even $\lambda \le 0$ as a consequence of non-negativity of K and λ_1. Using again $K \ge 0$, we get (5.11). For $c > 0$, we must use finer argument: Λ defined by (5.10)–(5.11) solves, in fact, the adjoint two-point boundary-value problem

$\mathrm{d}^2\Lambda$

and then (5.11) follows from the maximum principle (in the sense used in theory of 2nd-order differential equation) by use of a contradiction argument, saying that, if $\max_{x\in[0,1]} \Lambda(x) > 0$, then at some $x \in [0,1]$ there must be $\Lambda(x) > 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Lambda(x) \leq 0$ simultaneously, which, however, contradicts (5.12).

Now, the Hamiltonian (4.3) takes the form $\mathcal{H}_{\lambda_0,\lambda_1,\lambda,y}(x,s) = \Lambda(x)\sin s - \lambda_0 s^2$ (up to a function constant in s-variable) and, having (5.11) at our disposal, we can see that its maximum cannot be attained for s positive. Thus, from the maximum principle (4.2), we can see that *any* solution ν to (RP_3) satisfies

$$\operatorname{supp}(\nu_x) \subset [-\pi, 0] \cup [-3\pi, -2\pi] =: M(x) \text{ for a.a. } x \in [0,1]. \tag{5.13}$$

However, taking this M, the condition (4.6) is satisfied! Therefore, Theorem 2 can be applied even without any specific knowledge about a solution to the relaxed problem (RP_3). Also, let us emphasize that we do not have any specific knowledge about the adjoint state λ (except for (5.11)) nor about the Lagrange multipliers λ_0 and λ_1.

REMARK 7 (Existence in ordinary-differential-equation problems) *One can find in the literature further examples for situations covered by Theorem 2 in context of systems governed by ordinary differential equations, which can be viewed as a very special Volterra integral equation. We refer to Gabasov and Mordukhovich (1974), Ioffe and Tikhomirov (1974), Sect. 9.2.2, Proposition 1 with Theorem 3, Mordukhovich (1988, 1999) or Muñoz and Pedregal (2001). A non-academic example with $M(x)$ a singleton for a.a. $x \in \Omega$ and thus (4.6) satisfied trivially was presented by Bittner (1998) for a flight optimal control problem.*

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