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On uniformly approximate convex and strongly $\alpha(\cdot)$-paraconvex functions

by

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Abstract: In the paper the equivalence of the notions of uniformly approximate convex functions and strongly $\alpha(\cdot)$-paraconvex functions will be presented. As a consequence we obtain that uniformly approximate convex functions are Fréchet differentiable on residual sets.

Keywords: approximate convex and paraconvex functions, Fréchet differentiability.

Let $(X, \|\cdot\|)$ be a real Banach space. Let $f(\cdot)$ be a function defined on X with values in $\mathbb{R} \cup \{+\infty\}$. Let ε be a fixed positive number. We say that the function $f(\cdot)$ is *ε-convex* (see Jofré, Luc and Théra, 1998, and Luc, Ngai and Théra, 1999) if for every $x, y \in X$ and real t, $0 \le t \le 1$, the following inequality holds

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x-y\|. \tag{1}$$

Luc, Ngai and Théra (2000) introduced the following notions. We say that the function $f(\cdot)$ is *approximate convex at a point* $x_0 \in X$, if there is $\delta > 0$ such that (1) holds for every $x, y \in X$ such that $\|x - x_0\| \le \delta$, $\|y - x_0\| \le \delta$ and every t, $0 \le t \le 1$. We say that the function $f(\cdot)$ is *approximate convex on a set* $C \subset X$ if it is approximate convex at each point $x_0 \in C$. In particular, when $C = X$, we say that the function $f(\cdot)$ is *approximate convex*. In those definitions the choice of δ depends on ε and x_0 as well.

Now we shall give a uniform version of the notion of approximate convex functions.

We say that the function $f(\cdot)$ is *uniformly approximate convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that (1) holds for every $x, y \in X$ with $\|x - y\| \le \delta$ and every t, $0 \le t \le 1$. Of course each uniformly approximate convex function is approximate convex.

It is obvious that for every $\varepsilon > 0$ the domain of an ε-convex function $f(\cdot)$, $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ is convex. This is not valid for every approxi-

The domain of uniformly approximate convex function need not be convex. Indeed, let $x_0 \in X$ be a fixed point such that $\|x_0\| > 3$. Let

$$f(x) = \begin{cases} 0 & \text{if } \|x\| \le 1, \\ 1 & \text{if } \|x - x_0\| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that if $\|x - y\| \le 1$, then

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

This trivially implies that $f(\cdot)$ is uniformly approximate convex. Observe that in this case the domain of $f(\cdot)$ is not connected. One can construct a uniformly approximate convex function $f(\cdot)$ such that its domain is connected but it is not convex. Indeed, let $X = \mathbb{R}^2$ and let

$$f(x,y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y = 0,\ 2n \le x \le 2n+1,\ n = 1, 2, \ldots, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that in this case the domain of $f(\cdot)$ is connected and non-convex, however it is not closed.

We say that $A \subset X$ is *locally convex* if for each $x \in A$ there is a neighbourhood V of x such that the set $A \cap V$ is convex (see Tietze, 1928, Matsumura, 1928, Klee, 1951). We say that a set $A \subset X$ is *uniformly locally convex* if there is a neighbourhood V of 0 such that for each $x \in X$ the set $A \cap (x + V)$ is convex. Just from the definitions it follows that domains of uniformly approximate convex functions are uniformly locally convex. Of course, each uniformly locally convex set is also locally convex. The converse is not true. For example every open set is locally convex. For uniformly locally convex sets we have

PROPOSITION 1 *Let X be a locally convex topological space. Let $A \subset X$ be a uniformly locally convex set. Then, its closure $\bar{A}$ is uniformly locally convex.*

Proof. By the definition there is a convex closed neighborhood of zero W such that for all $x \in X$, $A \cap (x + W)$ is a convex set. Let $V = \frac{1}{2}W$. It is easy to see that every $\bar{x} \in \bar{A}$ $(V + x) \cap A$ is a convex set. Let $y, z \in \bar{A} \cap (x + V)$ and let t, $0 \le t \le 1$. Since $y, z \in \bar{A} \cap (x + V)$ for every convex neighbourhood of zero, U, there are $y_U, z_U \in A \cap (x + V)$ such that $y_U \in (y + U)$ and $z_U \in (z + U)$. By the convexity of U $ty_U + (1-t)z_U \in (ty + (1-t)z + U)$. The arbitrariness of U and the closedness of $\bar{A} \cap V$ imply that $tx + (1-t)y \in \bar{A} \cap (x + V)$. Thus, the set $\bar{A}$ is uniformly locally convex. ■

Of course, Proposition 1 is not valid for locally convex sets. Indeed, let C be an open non-convex set. The set C is locally convex, but its closure is not.

It can be shown that a closed connected locally convex sets are convex (see Tietze, 1928, and Matsumura, 1928, for $\mathbb{R}^n$, and Klee, 1951, for topological linear spaces). In particular, if the domain of an approximate convex function $f(\cdot)$ is simultaneously connected and closed then it is convex. Using the results

PROPOSITION 2 *Let X be a locally convex space. Let $A \subset X$ be an open uniformly locally convex connected set. Then it is convex.*

Proof. By Proposition 1, $\bar{A}$, the closure of the set A, is convex. Thus, its interior $A = \operatorname{Int} \bar{A}$ is also convex. ∎

COROLLARY 3 *Let X be a locally convex space. Let $A \subset X$ be an open uniformly locally convex set. Then it is a union of disjoint convex open sets.*

As a consequence we obtain that if the domain of a uniformly approximate convex function $f(\cdot)$ is open, then it is a union of disjoint open convex sets.

In Rolewicz (2000) the notions of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions were introduced. We recall them below.

Let $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that $\alpha(0) = 0$ and

$$\limsup_{t \downarrow 0} \frac{\alpha(t)}{t} < +\infty. \tag{2}$$

Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a convex subset of X. Let $f(\cdot)$ be a real valued function defined on Ω. We say that the function $f(\cdot)$ is $\alpha(\cdot)$-*paraconvex* if there is a constant $C > 0$ such that for all $x, y \in \Omega$ and $0 \leq t \leq 1$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C\alpha(\|x - y\|). \tag{3}$$

For $\alpha(t) = t^2$ this definition was introduced in Rolewicz (1979a) and the t^2-paraconvex functions were called simply paraconvex functions. In Rolewicz (1979b) the notion was extended of the case $\alpha(t) = t^\gamma$, $1 \leq \gamma$, and t^γ-paraconvex functions were called γ-paraconvex functions. For $\alpha(t) = t^\gamma$, $2 < \gamma$ each t^γ-paraconvex function is convex. Moreover, if

$$\limsup_{t \downarrow 0} \frac{\alpha(t)}{t^2} = 0,$$

then each $\alpha(\cdot)$-paraconvex function is convex (Rolewicz, 2000).

We say that the function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex* if there is a constant $C > 0$ such that for all $x, y \in \Omega$ and $0 \leq t \leq 1$ we have

$$\begin{aligned} &f(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) + C\min[t, (1-t)]\alpha(\|x - y\|). \end{aligned} \tag{4}$$

The notions of $\alpha(\cdot)$-paraconvexity and strong $\alpha(\cdot)$-paraconvexity are not equivalent. A Reader can find some sufficient and necessary conditions for the equivalence of two notions in Rolewicz (2000). It can be shown that for $1 < \gamma$ the t^γ-paraconvex functions are strongly t^γ-paraconvex.

In this note we shall show that the notions of uniformly approximate convex

For this purpose we shall extend the notions of $\alpha(\cdot)$-paraconvex and strong $\alpha(\cdot)$-paraconvex functions to the case of functions, whose domains are uniformly locally convex sets. Let Ω be a uniformly locally convex set, and let $\delta > 0$ be such that for all $x \in X$, the sets $\Omega \cap B(x, \delta)$, where $B(x, \delta)$ denotes a closed ball with the center at x and the radius δ, are convex. We say that a function $f(\cdot)$ is *$\alpha(\cdot)$-paraconvex* if there is a constant $C > 0$ such that for all $x, y \in \Omega$ such that $\|x - y\| \leq \delta$ and $0 \leq t \leq 1$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C\alpha(\|x - y\|). \tag{3'}$$

We say that a function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex* if there is a constant $C > 0$ such that for all $x, y \in \Omega$ such that $\|x - y\| \leq \delta$ and $0 \leq t \leq 1$ we have

$$\begin{aligned} &f(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) + C\min[t, (1-t)]\alpha(\|x - y\|). \end{aligned} \tag{4'}$$

Basing on this extended notion of strongly $\alpha(\cdot)$-paraconvex functions we can formulate

THEOREM 4 *Let $(X, \|\cdot\|)$ be a real Banach space. Let $f(\cdot)$ be a function defined on X with values in $\mathbb{R} \cup \{+\infty\}$. Then, the function $f(\cdot)$ is uniformly approximate convex if and only if there is an $\alpha(\cdot)$ such that*

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0 \tag{5}$$

and such that the function is strongly $\alpha(\cdot)$-paraconvex on its domain $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$.

The proof will be based on the following lemma:

LEMMA 5 *A function $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex if and only if there is $C_1 > 0$ such that*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C_1 t(1-t)\alpha(\|x - y\|). \tag{4''}$$

Proof. This is a trivial consequence of the following inequality $t(1-t) \leq \min[t, (1-t)] \leq 2t(1-t)$ for every $t \in [0, 1]$. ∎

Proof of Theorem 4. Suppose that a function $f(\cdot)$ is uniformly approximate convex and $\delta > 0$ be such that (1) holds with $\varepsilon = 1$ for all x, y such that $\|x - y\| \leq \delta$. Let Ω denote the domain of the function $f(\cdot)$, $\operatorname{dom} f$. This domain is the uniformly locally convex set and for all $x \in X$, the sets $\Omega \cap B(x, \delta)$ are convex.

Let $\hat{\delta}(\varepsilon) > 0$ denote the supremum of the numbers δ such that (1) holds to see that the function $\hat{\delta}(\varepsilon)$ is non-decreasing. Now, let $\delta(\varepsilon)$ be an arbitrary continuous increasing function such that $\delta(\varepsilon) < \hat{\delta}(\varepsilon)$ for $\varepsilon > 0$ and such that

$$\lim_{\varepsilon \downarrow 0} \delta(\varepsilon) = 0. \tag{6}$$

Let $\varepsilon(\delta) = \delta^{-1}(\varepsilon)$. Observe that $\varepsilon(\delta)$ is a continuous increasing function such that

$$\lim_{\delta \downarrow 0} \varepsilon(\delta) = 0. \tag{7}$$

Let $\alpha(s) = s\varepsilon(s)$. By (7) we have (5). Let $x, y \in X$ be such that $\|x - y\| = \delta$. By the definition of $\delta(\varepsilon)$ and $\varepsilon(\delta)$ we have

$$\begin{aligned}
&f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon(\delta)t(1-t)\delta \\
&= tf(x) + (1-t)f(y) + t(1-t)\alpha(\delta) \\
&= tf(x) + (1-t)f(y) + t(1-t)\alpha(\|x-y\|),
\end{aligned}$$

i.e. by $(4'')$ the function $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex.

Suppose now that the function is strongly $\alpha(\cdot)$-paraconvex, i.e. there is $C > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + Ct(1-t)\alpha(\|x-y\|). \tag{4'}$$

Since (5) for every $\varepsilon > 0$ there is $\delta > 0$ such that for $0 < s \leq \delta$,

$$\alpha(s) \leq \frac{\varepsilon}{C} s. \tag{8}$$

Thus, by $(4'')$ and (8) we have

$$\begin{aligned}
&f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + Ct(1-t)\alpha(\|x-y\|) \\
&\leq tf(x) + (1-t)f(y) + \frac{\varepsilon}{C} Ct(1-t)\|x-y\| \\
&= tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x-y\|,
\end{aligned}$$

i.e. the function $f(\cdot)$ is uniformly approximate convex. ■

It can be shown that a strongly $\alpha(\cdot)$-paraconvex function (i.e. uniformly approximate convex function) $f(\cdot)$ defined on a convex open set Ω is locally Lipschitz (see Luc, Ngai and Théra, 2000, Rolewicz, 2001).

Let $f(\cdot)$ be a real-valued function defined on a convex set $\Omega \subset X$. We say that a linear functional $x^* \in X^*$ is a $\alpha(\cdot)$-*subgradient* of the function $f(\cdot)$ at a point x_0 if there is $C > 0$ such that

$$\langle x^*, h \rangle \leq f(x_0 + h) - f(x_0) + C\alpha(\|h\|) \tag{9}$$

for all h such that $x + h \in \Omega$.

The set of all $\alpha(\cdot)$-subgradients of the function $f(\cdot)$ at the point x_0 is called the $\alpha(\cdot)$-*subdifferential* of the function $f(\cdot)$ at the point x_0 and we shall denote

The notions of $\alpha(\cdot)$-subgradient and $\alpha(\cdot)$-subdifferential can be considered as a uniformization of the notion of subdifferential in nonconvex analysis, considered by several authors (see for example Fabian, 1989, Ioffe, 1983, 1984, 1989, 1990, Mordukchovich, 1980, 1988, and many more).

It can be shown that for strongly $\alpha(\cdot)$-paraconvex functions the $\alpha(\cdot)$-subdifferentials and Clarke subdifferentials coincide (Rolewicz, 2001). Thus, using Theorem 9 of Rolewicz (1999) we obtain

THEOREM 6 *Let $(X, \|.\|)$ be a real Banach space, which has the separable dual X^*. Let $f(\cdot)$ be a strongly $\alpha(\cdot)$-paraconvex (i.e. uniformly approximate convex) function. Suppose that its domain is an open set Ω. Then there is a subset A_f of the first category such that on the set $\Omega \setminus A_f$ the function f is Fréchet differentiable†.*

Proof. Put $\Phi = X^*$. Since the domain Ω of the function $f(\cdot)$ is a uniformly locally convex set, which is simultaneously open, it is a union of disjoint open convex sets. Thus, without loss of generality we may assume that Ω is an open convex set. Thus, the function $f(\cdot)$ is locally Lipschitz (see Luc, Ngai and Théra, 2000, Rolewicz, 2001), i.e. for each $x_0 \in X$ there are a convex neighbourhood V_{x_0} and a constant L_{x_0} such that f satisfies on V_{x_0} the Lipschitz condition with a constant L_{x_0}. This implies that for every $y \in V_{x_0}$ the Clarke subdifferential $\partial f|_y$ at y is not empty. This subdifferential is equal to the $\alpha(\cdot)$-subdifferential. Hence $f(\cdot)$ is a continuous $\alpha(\cdot)$-X^*-subdifferentiable function on V_{x_0}. Since we can cover the whole set Ω by neighbourhoods V_{x_0}, $f(\cdot)$ is a continuous $\alpha(\cdot)$-X^*-subdifferentiable function on the whole Ω. Then by Theorem 9 of Rolewicz (1999) there is a subset $A_f \subset \Omega$ of the first category such that on the set $\Omega \setminus A_f$ the function f is Fréchet differentiable. ■

COROLLARY 7 *Let $(X, \|.\|)$ be a real Banach space, which has the separable dual X^*. Let $f(\cdot)$ be a uniformly approximate convex function. Suppose that $\Omega = \mathrm{Int}(\mathrm{dom}\, f) \neq \emptyset$. Then, there is a subset A_f of the first category such that on the set $\Omega \setminus A_f$ the function f is Fréchet differentiable.*

Proof. Let

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that $\hat{f}$ is a uniformly approximate convex function such that $\Omega = \mathrm{dom}\, \hat{f}$ and $\hat{f}(x) = f(x)$ for $x \in \Omega$. Thus, there is a subset $A_f \subset \Omega$ of the first category such that on the set $\Omega \setminus A_f$ the function $f = \hat{f}$ is Fréchet differentiable. ■

QUESTION 8 *Is Corollary 7 valid for the approximate convex functions?*

† Actually, in the present paper, in Corollary 7 and Proposition 9 the set A_f can be taken from a narrower class than the first category sets, namely the so called weakly angle-small

Till now we have considered only scalar valued functions. In a similar way we can define strongly $\alpha(\cdot)$-paraconvex functions for functions having values in a normed space Y ordered by a convex pointed cone Z with non-empty interior.

Let as before $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that $\alpha(0) = 0$ and

$$\limsup_{t \downarrow 0} \frac{\alpha(t)}{t} < +\infty. \tag{2}$$

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces. Let $(Y, \|\cdot\|_Y)$ be ordered by a convex pointed cone Z with non-empty interior. Let Ω be a convex subset of X. Let $f(\cdot)$ be a function mapping Ω into Y, $f : \Omega \to Y$. Let e be an arbitrary fixed element of the interior of Z, $e \in \operatorname{Int} Z$. We say that the function $f(\cdot)$ is $\alpha(\cdot)$-*paraconvex* if there is a constant $C > 0$ such that for all $x, y \in \Omega$ and $0 \leq t \leq 1$ we have

$$f(tx + (1-t)y) \leq_Z tf(x) + (1-t)f(y) + C\alpha(\|x - y\|)e, \tag{3_v}$$

where the inequality in (3_v) means the inequality in the sense of order $\leq_Z$ induced by the cone Z. This definition does not depend on the choice of $e \in \operatorname{Int} Z$. It follows trivially from the fact that if we have two arbitrary $e, g \in \operatorname{Int} Z$, then there are $c_1, c_2 > 0$ such that $c_1 g \leq e \leq c_2 g$.

A natural question arises: is Theorem 6 valid for the vector valued functions?

We know only the answer in the case of finite dimensional spaces $\mathbb{R}^n$ with the standard order. In this case an n-dimensional $\alpha(\cdot)$-paraconvex function $f(\cdot) = (f_1(\cdot), \ldots, f_n(\cdot))$ is $\alpha(\cdot)$-paraconvex if and only if all functions $f_i(\cdot)$, $i = 1, \ldots, n$, are $\alpha(\cdot)$-paraconvex in the real-valued case.

Thus, by Theorem 6, we trivially obtain

PROPOSITION 9 *Let $(X, \|.\|)$ be a real Banach space, which has a separable dual X^*. Let $f(\cdot)$ be an n-dimensional $\alpha(\cdot)$-paraconvex function. Suppose that its domain is an open set Ω. Then there is a subset A_f of the first category such that on the set $\Omega \setminus A_f$ the function f is Fréchet differentiable.*

References

FABIAN, M. (1989) Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss. *Acta Univ. Carolinae*, **30**, 51–56.

IOFFE, A.D. (1983) On subdifferentiability spaces. *Ann. N.Y. Sci.*, **410**, 107–119.

IOFFE, A.D. (1984) Approximate subdifferentials and applications I: The finite dimensional theory. *Trans. Amer. Math. Soc.*, **281**, 389–416.

IOFFE, A.D. (1989) Approximate subdifferentials and applications III: The met-

IOFFE, A.D. (1990) Proximal analysis and approximate subdifferentials. *J. London Math. Soc.*, **41**, 175–192.

JOFRÉ, A., LUC, D.T. and THÉRA, M. (1998) ε-subdifferential and ε-monotonicity. *Nonlinear Analysis, Theory, Methods and Appl.*, **33**, 71–90.

KLEE, V. (1951) Convex sets in linear spaces. *Duke Math. Jour.*, **18**, 443–466.

LUC, D.T., NGAI, H.V. and THÉRA, M. (1999) On ε-convexity and ε-monotonicity. In: *Calculus of Variations and Differential Equations*, eds.: Ioffe, A., Reich, S. and Shapiro, I. *Research Notes in Mathematics Series*, Chapman and Hall, 82–100.

LUC, D.T., NGAI, H.V. and THÉRA, M. (2000) Approximate convex functions. *Jour. Nonlinear and Convex Anal.*, **1**, 155–176.

MATSUMURA, S. (NAKAJIMA) (1928) Über convexe Kurven und Flächen. *Tohôku Math. Jour.*, **29**, 227–230.

MORDUKHOVICH, B.S. (1980) Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems. *Soviet Math. Doklady*, **22**, 526–530.

MORDUKHOVICH, B.S. (1988) *Approximation methods in Problems of Optimization and Control* (in Russian). Nauka, Moscow.

PALLASCHKE, D. and ROLEWICZ, S. (1997) *Foundation of mathematical optimization.* Mathematics and its Applications 388, Kluwer Academic Publishers, Dordrecht/Boston/London.

ROLEWICZ, S. (1979a) On paraconvex multifunctions. *Oper. Res. Verf. (Methods of Oper. Res.)*, **31**, 540–546.

ROLEWICZ, S. (1979b) On γ-paraconvex multifunctions. *Math. Japonica*, **24**, 293-300.

ROLEWICZ, S. (1997) Locally monotone operators in spaces without linearity. In: Lecture Notes in Economics and Mathematical Systems 452, eds.: Gritzmann, P., Horst, R., Sachs, E. and Tichatschke, R., Springer, 292–297.

ROLEWICZ, S. (1998) On uniformly Φ-convex functions and strongly monotone multifunctions. *Functiones et Approximatio*, **26**, 231–238.

ROLEWICZ, S. (1999) On $\alpha(\cdot)$-monotone multifunction and differentiability of γ-paraconvex functions. *Stud. Math.*, **133**, 29–37.

ROLEWICZ, S. (2000a) On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions. *Control and Cybernetics*, **29**, 367–377.

ROLEWICZ, S. (2000b) On cyclic $\alpha(\cdot)$-monotone multifunctions. *Stud. Math.*, **141**, 263–272.

ROLEWICZ, S. (2001) On equivalence of Clarke, Dini, $\alpha(\cdot)$-subgradients and local $\alpha(\cdot)$-subgradients for strongly $\alpha(\cdot)$-paraconvex functions. *Optimization*, **50**, 353–360.

TIETZE, H. (1928) Über Konvexheit im kleinem und im grossen und über gewisse der Punkten einer Menge zugeordnete Dimensionszahlen. *Math. Zeit.*, **28**, 697–707.