Control and Cybernetics

vol. **30** (2001) No. 3

Robust stability of a family of matrices

by

Long Wang[1], **Lin Zhang**[2] **and Zhizhen Wang**[1]

[1]Center for Systems and Control
State Key Laboratory of Turbulence and Complex Systems
Department of Mechanics and Engineering Science, Peking University
Beijing 100871, People's Republic of China
E-mail: longwang@mech.pku.edu.cn

[2]Department of Automation, Tsinghua University
Beijing 100084, People's Republic of China

Abstract: For a family of matrices with multilinear uncertainty structure, stability of the entire matrix family can be guaranteed by stability of its vertex matrices.

Keywords: robust stability, matrix families, multilinear uncertainty structure, algebra theory, vertex results.

Stability of matrix families is an important research subject in robust control, and has received much attention in the past few years, see Horn and Johnson (1985), Bhattacharyya et al. (1995), Barmish (1994), Ackermann (1993), Bartlett et al. (1988), Kaczorek (1993), Ackermann (1991, 1992), Anderson et al. (1995), Polyak and Kogan (1995), Wang (1997, 1998a, 1998b), Wang and Yu (2001a), Duan et al. (2001), Wang and Yu (2001b).

The robust stability problem with linear uncertainty structure has been completely resolved by the celebrated Edge Theorem of Bartlett, Hollot and Huang (1988). Multilinear uncertainty structure is a special class of *nonlinear* uncertainty structure, see Ackermann (1991, 1992), Anderson et al. (1995), Polyak and Kogan (1995), Wang (1997, 1998a, 1998b), Wang and Yu (2001a), Duan et al. (2001), Wang and Yu (2001b). It is more complicated than the linear uncertainty structure, but has some intrinsic properties in robust stability analysis, see Ackermann (1992), Anderson et al. (1995), Polyak and Kogan (1995), Wang (1998a, 1998b). This is due to the fact that a multilinear function is a *linear* function of some variable when all other variables are fixed. Ackermann (1992), Anderson et al. (1995), Wang (1998a) have investigated robust stability

Supported by the National Key Project of China and the National Natural Science Fo...

problem with multilinear uncertainty structure, revealed some interesting phenomena, and established some easily testable criteria. In this brief paper, we exploit the multilinear uncertainty structure and by using some matrix properties, we show that, for a family of matrices with multilinear uncertainty structure, D-stability of the entire matrix family can be guaranteed by D-stability of its vertex matrices.

Consider the matrix family

$$\mathcal{F} = \Big\{ \sum_{i=1}^{m} f_i(q_1, q_2, \ldots, q_l) A_i \mid q_i \in [\underline{q}_i, \overline{q}_i],\ i = 1, 2, \ldots, l \Big\} \tag{1}$$

where $A_i \in \mathcal{C}^{n\times n}$, $i = 1, 2, \ldots, m$ are fixed matrices, and $f_i(q_1, q_2, \ldots, q_l)$, $i = 1, 2, \cdots, m$ are multiaffine functions of the uncertain parameters $q_1, q_2, \ldots, q_l$.

Denote the vertex set of $\mathcal{F}$ as

$$\mathcal{F}_V = \Big\{ \sum_{i=1}^{m} f_i(q_1, q_2, \ldots, q_l) A_i \mid q_i \in \{\underline{q}_i, \overline{q}_i\},\ i = 1, 2, \ldots, l \Big\} \tag{2}$$

and suppose that all matrices in $\mathcal{F}_V$ are normal matrices.

The set of all eigenvalues of a square matrix A is denoted as $eigen[A]$. For a family $\mathcal{A}$ of square matrices, define

$$eigen[\mathcal{A}] = \{eigen[A] \mid A \in \mathcal{A}\}. \tag{3}$$

The convex hull of a set Λ is denoted as $conv[\Lambda]$.

THEOREM 1

$$eigen[\mathcal{F}] \subset conv[eigen[\mathcal{F}_V]]. \tag{4}$$

Proof. For any $\lambda \in eigen[\mathcal{F}]$, we must show that $\lambda \in conv[eigen[\mathcal{F}_V]]$. Since $\lambda \in eigen[\mathcal{F}]$, there exist $q_i^0 \in [\underline{q}_i, \overline{q}_i]$, $i = 1, 2, \ldots, l$ such that λ is an eigenvalue of $\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0) A_i \doteq A_0$. Namely, there exists $x \in \mathcal{C}^n$, $x \neq 0$ such that

$$A_0 x = \lambda x. \tag{5}$$

Without loss of generality, suppose the eigenvector x is normalized, i.e., $x^H x = 1$ (the superscript H denotes conjugate transpose). Then

$$\lambda = \lambda x^H x = x^H A_0 x = x^H \Big[\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0) A_i \Big] x. \tag{6}$$

We now prove that

$$\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0) A_i \in conv[\mathcal{F}_V] \tag{7}$$

First, note that $f_i(q_1, q_2^0, \ldots, q_l^0)$, $i = 1, 2, \ldots, m$ are affine functions of q_1. So, $\sum_{i=1}^{m} f_i(q_1, q_2^0, \ldots, q_l^0)A_i$ is also an affine function of q_1. Thus, there exists $\eta \in [0, 1]$ such that

$$\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0)A_i$$
$$= \eta\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, q_2^0, \ldots, q_l^0)A_i\Big) + (1-\eta)\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, q_2^0, \ldots, q_l^0)A_i\Big). \tag{8}$$

Second, when $q_1^* = \underline{q}_1$ or $\overline{q}_1$, $f_i(q_1^*, q_2, q_3^0, \ldots, q_l^0)$, $i = 1, 2, \cdots, m$ are affine functions of q_2. So, $\sum_{i=1}^{m} f_i(q_1^*, q_2, q_3^0, \ldots, q_l^0)A_i$ is also an affine function of q_2. Thus, there exists $\mu_1 \in [0, 1]$ (for $\underline{q}_1$) and $\mu_2 \in [0, 1]$ (for $\overline{q}_1$) such that

$$\sum_{i=1}^{m} f_i(\underline{q}_1, q_2^0, q_3^0, \ldots, q_l^0)A_i = \mu_1\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, \underline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)$$
$$+ (1-\mu_1)\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, \overline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big) \tag{9}$$
$$\sum_{i=1}^{m} f_i(\overline{q}_1, q_2^0, q_3^0, \ldots, q_l^0)A_i = \mu_2\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, \underline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)$$
$$+ (1-\mu_2)\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, \overline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big). \tag{10}$$

Henceforth, we have

$$\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0)A_i$$
$$= \eta\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, q_2^0, \ldots, q_l^0)A_i\Big) + (1-\eta)\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, q_2^0, \ldots, q_l^0)A_i\Big)$$
$$= \eta\Big[\mu_1\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, \underline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)$$
$$+ (1-\mu_1)\Big(\sum_{i=1}^{m} f_i(\underline{q}_1, \overline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)\Big]$$
$$+ (1-\eta)\Big[\mu_2\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, \underline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)$$
$$+ (1-\mu_2)\Big(\sum_{i=1}^{m} f_i(\overline{q}_1, \overline{q}_2, q_3^0, \ldots, q_l^0)A_i\Big)\Big] \tag{11}$$

Continuing this process, we can prove that

$$\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0) A_i \in conv[\mathcal{F}_V]. \tag{12}$$

For notational simplicity, let

$$\mathcal{F}_V = \{F_i \mid i = 1, 2, \ldots, p\},\ p = 2^l. \tag{13}$$

Then, there exist $\eta_i \geq 0$, $i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p} \eta_i = 1$ such that

$$\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \ldots, q_l^0) A_i = \sum_{i=1}^{p} \eta_i F_i. \tag{14}$$

Therefore

$$\begin{aligned} \lambda &= x^H \Big[\sum_{i=1}^{m} f_i(q_1^0, q_2^0, \cdots, q_l^0) A_i \Big] x \\ &= x^H \Big[\sum_{i=1}^{p} \eta_i F_i \Big] x = \sum_{i=1}^{p} \eta_i x^H F_i x. \end{aligned} \tag{15}$$

Since F_i, $i = 1, 2, \ldots, p$ are normal matrices, there exist unitary matrices U_i, $i = 1, 2, \ldots, p$ such that

$$U_i^H F_i U_i = diag\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}\}. \tag{16}$$

Hence

$$x^H F_i x = x^H U_i diag\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}\} U_i^H x. \tag{17}$$

Let

$$y_i = U_i^H x = [y_{i1}, y_{i2}, \ldots, y_{in}]^T \in \mathcal{C}^n. \tag{18}$$

Then

$$x^H F_i x = y_i^H diag\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}\} y_i = \sum_{j=1}^{n} \overline{y_{ij}} y_{ij} \lambda_{ij}. \tag{19}$$

Since $\overline{y_{ij}} y_{ij} = |y_{ij}|^2 \geq 0$ and

$$\sum_{j=1}^{n} \overline{y_{ij}} y_{ij} = y_i^H y_i = x^H U_i U_i^H x = x^H x = 1 \tag{20}$$

we have

Furthermore, since $\eta_i \geq 0$, $i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p} \eta_i = 1$, we have

$$\lambda = \sum_{i=1}^{p} \eta_i x^H F_i x \in conv[eigen[\mathcal{F}_V]]. \tag{22}$$

This completes the proof. ■

Given an open convex region $\mathcal{D}$ in the complex plane, a matrix is termed $\mathcal{D}$-stable, if all of its eigenvalues lie in $\mathcal{D}$. A matrix family is termed $\mathcal{D}$-stable, if all of its members are $\mathcal{D}$-stable. Hurwitz stability and Schur stability are two special cases of $\mathcal{D}$-stability when the stability regions are the open left half plane and the open unit disk, respectively. By Theorem 1, we have

COROLLARY 1 *$\mathcal{F}$ is $\mathcal{D}$-stable if and only if $\mathcal{F}_V$ is $\mathcal{D}$-stable.*

EXAMPLE 1 *Consider the matrix family*

$$\mathcal{F} = \Big\{ \sum_{i=1}^{2} f_i(q_1, q_2) A_i \,\Big|\, q_1 \in [1, 2], q_2 \in [2, 4] \Big\} \tag{23}$$

where $f_1(q_1, q_2) = q_1 q_2 - 4$, $f_2(q_1, q_2) = 2q_1 - q_2$; *and*

$$A_1 = \begin{pmatrix} 0.2 & 0 & 0.1 \\ 0 & 0.1 & 0 \\ -0.1 & 0 & 0.2 \end{pmatrix} \tag{24}$$

$$A_2 = \begin{pmatrix} 0.179167 & 0.004167 & -0.083333 \\ 0.004167 & -0.029167 & 0.045833 \\ -0.083333 & 0.045833 & 0.029167 \end{pmatrix}. \tag{25}$$

It is easy to see that the four vertex (normal) matrices in $\mathcal{F}_V$ are

$$V_1 = \sum_{i=1}^{2} f_i(q_1, q_2) A_i|_{q_1=1, q_2=2} = \begin{pmatrix} -0.4 & 0 & -0.2 \\ 0 & -0.2 & 0 \\ 0.2 & 0 & -0.4 \end{pmatrix} \tag{26}$$

$$V_2 = \sum_{i=1}^{2} f_i(q_1, q_2) A_i|_{q_1=1, q_2=4}$$

$$= \begin{pmatrix} -0.358333 & -0.008333 & 0.166667 \\ -0.008333 & 0.058333 & -0.091667 \\ 0.166667 & -0.091667 & -0.058333 \end{pmatrix} \tag{27}$$

$$V_3 = \sum_{i=1}^{2} f_i(q_1, q_2) A_i|_{q_1=2, q_2=2}$$

$$= \begin{pmatrix} 0.358333 & 0.008333 & -0.166667 \\ 0.008333 & -0.058333 & 0.091667 \\ \end{pmatrix} \tag{28}$$

$$V_4 = \sum_{i=1}^{2} f_i(q_1, q_2) A_i|_{q_1=2, q_2=4} = \begin{pmatrix} 0.8 & 0 & 0.4 \\ 0 & 0.4 & 0 \\ -0.4 & 0 & 0.8 \end{pmatrix}. \tag{29}$$

Moreover, it is easy to see that

$$eigen[V_1] = \{-0.4 + 0.2i, -0.4 - 0.2i, -0.2\} \tag{30}$$

$$eigen[V_2] = \{-0.434412, 0.127763, -0.051685\} \tag{31}$$

$$eigen[V_3] = \{0.434412, -0.127763, 0.051685\} \tag{32}$$

$$eigen[V_4] = \{0.8 + 0.4i, 0.8 - 0.4i, 0.4\} \tag{33}$$

Hence, we see that the vertex set $\mathcal{F}_V = \{V_1, V_2, V_3, V_4\}$ *is Schur stable. By Corollary 1, we conclude that the entire family* $\mathcal{F}$ *is Schur stable.*

References

HORN, R.A. and JOHNSON, C.R. (1985) *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

BHATTACHARYYA, S.P., CHAPELLAT, H. and KEEL, L.H. (1995) *Robust Control: The Parametric Approach.* Prentice Hall , New York.

BARMISH, B.R. (1994) *New Tools for Robustness of Linear Systems.* MacMillan Publishing Company, New York.

ACKERMANN, J. (1993) *Robust Control: Systems with Uncertain Physical Parameters.* Springer-Verlag, London.

BARTLETT, A.C., HOLLOT, C.V. and HUANG, L. (1988) Root locations for an entire polytope of polynomial: it suffices to check the edges. *Mathematics of Control, Signals, and Systems*, **1**, 61–71.

KACZOREK, T. (1993) *Linear Control Systems.* John Wiley & Sons, London.

ACKERMANN, J. (1991) Uncertainty structures and robust stability analysis. *Proc. of 1st European Control Conference*, 2318–2327.

ACKERMANN, J. (1992) Does it suffice to check a subset of multilinear parameters in robustness analysis? *IEEE Trans. on Automatic Control*, **37**, 487–488.

ANDERSON, B.D.O., KRAUS, F., MANSOUR, M. and DASGUPTA, S. (1995) Easily testable sufficient conditions for the robust stability of systems with multilinear parameter dependence. *Automatica*, **31**: 25–40.

POLYAK, B.T. and KOGAN, J. (1995) Necessary and sufficient conditions for robust stability of linear systems with multilinear uncertainty structure. *IEEE Trans. on Automatic Control*, **40**, 1255–1260.

WANG, L. (1997) Robust stability of a class of polynomial families under nonlinearly correlated perturbations. *Systems and Control Letters*, **30**, 25–30.

WANG, L. (1998a) A unified approach to robust performance of a class of transfer functions with multilinearly correlated perturbations. *Journal of Op-*

WANG, L. (1998b) On strict positive realness of multilinearly parametrized interval systems. *Scientia Sinica*, **41E**, 552–560.

WANG, L. and YU, W. (2001) On Hurwitz stable polynomials and strictly positive real rational functions. *IEEE Trans. on Circuits and Systems-I*, **48**, 127–128.

DUAN, Z., HUANG, L. and WANG, L. (2001) Robustness analysis and synthesis of SISO systems under both plant and controller perturbations. *Systems and Control Letters*, **42**, 201–216.

WANG, L. and YU, W. (2001) Anderson's claim on fourth-order SPR synthesis is true. *IEEE Trans. on Circuits and Systems-I*, **48**, 506–509.