Control and Cybernetics

vol. **30** (2001) No. 3

On nonzero-sum stopping game related to discrete risk processes

by

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Abstract: We consider two-person nonzero-sum stopping game. The players (insurers) observe discrete time risk processes until one of them decides to stop his process. Strategies of the players are stopping times. The aim of each player is to maximize his expected gain. We find Nash equilibrium point for this game under certain assumptions on reward sequences.

Keywords: Nash equilibrium, risk process, stopping game.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which all the considered random variables are determined. Let U_n^i, $n = 0, 1, 2, \ldots$, $i = 1, 2$, denote the discrete time risk process representing the surplus of the i-th insurer at time n, i.e.

$$U_n^i = u^i + nc^i - S_n^i, \; i = 1, 2,$$

where $u^i > 0$ is the initial capital of the i-th insurer, $c^i > 0$ is the amount of premium received at each period, $S_n^i = V_1^i + \ldots + V_n^i$ is the sum of all claims in the first n periods.

We also assume that the distribution of the claim sequences $V_1^1, V_2^1, \ldots,$ $V_1^2, V_2^2, \ldots$ depends on an unobserved random time θ, similarly as in the disorder problem considered by Shiryaev (1978) and generalized by Bojdecki (1979). θ represents the random moment at which the environment changes. Until that moment the subsequent claims of each insurer are iid random variables and at time θ the common distribution switches to another one. We assume that the premiums are established correctly till the moment θ, i.e. that average claims insurers and there is an urgent need to recalculate the premiums, i.e. to stop the process.

Let $\mathcal{F}_n = \sigma(V_1^1, \dots, V_n^1, V_1^2, \dots, V_n^2)$ denote the σ-field of the events observed at the n-th moment, $n = 1, 2, \dots$, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let Λ be the class of stopping times with respect to $(\mathcal{F}_n)_{n\in\mathcal{N}}$, i.e. $\tau \in \Lambda$ iff $\{\omega \in \Omega : \tau(\omega) = n\} \in \mathcal{F}_n$. For $n \in \mathcal{N}$ let us denote by Λ_n the class of all stopping times τ with respect to $(\mathcal{F}_n)_{n\in\mathcal{N}}$ such that $\tau \geq n$ a.s.

We shall consider the following stopping game. There are two players (insurers) observing their risk processes U_n^i until one of them decides to stop his process. Strategies of the players are the $\mathcal{F}_n$-stopping times. For the pair of strategies $(\tau^1, \tau^2) \in \Lambda \times \Lambda$, the ith player reward is

$$g^i(\tau^1, \tau^2) = X^i_{\tau^i} I(\tau^i < \tau^j) + Y^i_{\tau^j} I(\tau^j < \tau^i)$$
$$+ W^i_{\tau^i} I(\tau^i = \tau^j < \infty) + \limsup_n W_n^i I(\tau^i = \tau^j = \infty),\ i, j = 1, 2,\ i \neq j,$$

where $I(A)$ is the indicator function of the set $A \in \mathcal{F}$,

$$X_n^i = U_n^i - k_1^i P(\theta < n \mid \mathcal{F}_n),\ Y_n^i = U_n^i - k_2^i P(\theta < n \mid \mathcal{F}_n),$$
$$W_n^i = U_n^i - k_3^i P(\theta < n \mid \mathcal{F}_n),$$

$n = 1, 2, \dots$, and k_1^i, k_2^i, k_3^i are nonnegative constants representing the stopping costs, $i = 1, 2$.

The aim of the ith player, $i = 1, 2$, is to maximize the expected gain $E(g^i(\tau^1, \tau^2))$ with respect to τ^i in Λ. So, we face the problem of finding Nash equilibrium strategies for this game.

Let us recall that $(\widehat{\tau}^1, \widehat{\tau}^2) \in \Lambda \times \Lambda$ is a Nash equilibrium point if for any other strategy $(\tau^1, \tau^2) \in \Lambda \times \Lambda$ we have

$$E(g^1(\widehat{\tau}^1, \widehat{\tau}^2)) \geq E(g^1(\tau^1, \widehat{\tau}^2)),\ E(g^2(\widehat{\tau}^1, \widehat{\tau}^2)) \geq E(g^2(\widehat{\tau}^1, \tau^2)).$$

The pair $(E[g^1(\widehat{\tau}^1, \widehat{\tau}^2)], E[g^2(\widehat{\tau}^1, \widehat{\tau}^2)])$ of values is called the equilibrium value corresponding to $(\widehat{\tau}^1, \widehat{\tau}^2)$. We say that $(\widehat{\tau}^1, \widehat{\tau}^2) \in \Lambda_n \times \Lambda_n$ is a Nash equilibrium point at n if

$$E(g^1(\widehat{\tau}^1, \widehat{\tau}^2)) = \operatorname*{ess\,sup}_{\tau^1 \in \Lambda_n} E(g^1(\tau^1, \widehat{\tau}^2) \mid \mathcal{F}_n) \text{ a.s.},$$
$$E(g^2(\widehat{\tau}^1, \widehat{\tau}^2)) = \operatorname*{ess\,sup}_{\tau^2 \in \Lambda_n} E(g^2(\widehat{\tau}^1, \tau^2) \mid \mathcal{F}_n) \text{ a.s.}$$

Let us note that the described game may be applied by one insurer who manages two surplus processes (e.g. two different kinds of insurance policies). So, he/she should care for each risk process. The assumption that $\mathcal{F}_n$ is the observed σ-field at n is natural then.

In Section 2 we describe the stopping game investigated by Ohtsubo (1987) which is the generalized version of the Dynkin's stopping problem presented by sequences under which Ohtsubo's results on existence of Nash equilibrium points are still valid and which are essential for our risk model. Section 3 contains the detailed analysis of our game associated with risk processes.

Stopping games with various reward function structures and various modifications of stopping (selection) strategies were analyzed by many authors, e.g. Yasuda (1985), Enns and Ferenstein (1987), Sakaguchi (1991), Ferenstein (1993) and Szajowski (1993), to list a few. A broad survey of such games has been presented recently by Nowak and Szajowski in (1999).

A single player optimal stopping time problem for the risk process with change in claim distribution, similar to that of our paper, was investigated by Bobecka, Danielak and Ferenstein in (2002). Optimal stopping of continuous time risk processes was analyzed by Ferenstein and Sierociński (1997) and Jensen (1997). General optimal stopping time theory is presented in the excellent monographs by Chow, Robbins, Siegmund (1971) and Shiryaev (1978).

2. General Dynkin's game

Let $(X_n^i)_{n=0,1,\dots}$, $(Y_n^i)_{n=0,1,\dots}$, $(W_n^i)_{n=0,1,\dots}$, $i = 1,2$, be six sequences of real-valued random variables defined on $(\Omega, \mathcal{F}, P)$ and adapted to $(\mathcal{F}_n)_{n\in\mathcal{N}}$. We assume the following:

(a) $X_n^i \geq W_n^i \geq Y_n^i$ for every $n \in \mathcal{N}$ and each $i = 1,2$,
(b) $E|X_n^i| < \infty$ and $E|Y_n^i| < \infty$ for each $i = 1,2$,
(c) $E[\sup_n (X_n^i)^+] < \infty$ for each $i = 1,2$.

Assumption (a) corresponds to the "Case I" studied by Ohtsubo (1987), but instead of (b) and (c) he assumed that

(b)* $E[\sup_n |X_n^i|] < \infty$ and $E[\sup_n |Y_n^i|] < \infty$ for each $i = 1,2$.

Let us consider the following noncooperative stopping game. There are two players and the first player and the second one choose (as their strategies) the stopping times $\tau^1 \in \Lambda$ and $\tau^2 \in \Lambda$, respectively, without mutual cooperation. Then the ith player, $i = 1,2$, gets the reward

$$\begin{aligned} g^i(\tau^1,\tau^2) &= X_{\tau^i}^i I(\tau^i < \tau^j) + Y_{\tau^j}^i I(\tau^j < \tau^i) \\ &+ W_{\tau^i}^i I(\tau^i = \tau^j < \infty) + \limsup_n W_n^i I(\tau^i = \tau^j = \infty),\ j = 1,2,\ i \neq j. \end{aligned}$$

The aim of the ith player, $i = 1,2$, is to maximize the expected gain $E(g^i(\tau^1,\tau^2))$ with respect to τ^i in Λ.

We will now present a constructive way of finding an equilibrium value and the Nash equilibrium points given in Ohtsubo in (1987). For each $m \in \mathcal{N}$ let us define a pair $(\beta_n^m, \gamma_n^m)_{n=0,1,2,\dots,m}$ of sequences of random variables by backward induction

$$(\beta_m^m, \gamma_m^m) = (W_m^1, W_m^2),$$

$$(\beta_n^m, \gamma_n^m) = \ldots\Big((W_n^1, W_n^2) \quad (X_n^1, Y_n^2) \quad \Big)$$

$n = m-1, m-2, \ldots, 0$, where $val(A)$ denotes the set of all Nash equilibrium values in the bimatrix game A with two pure strategies. In other words, for a 2×2 bimatrix $A = [(a_{ij}, b_{ij})]$, $i, j = 1, 2$, $(a_{i_0 j_0}, b_{i_0 j_0}) \in val(A)$ iff $a_{i_0 j_0} \geq a_{i j_0}$ for all i and $b_{i_0 j_0} \geq b_{i_0 j}$ for all j.

Under the assumption (a) the above value relation is equivalent to:

$$(\beta_m^m, \gamma_m^m) = (W_m^1, W_m^2),$$
$$(\beta_n^m, \gamma_n^m) = \begin{cases} (E(\beta_{n+1}^m \mid \mathcal{F}_n), E(\gamma_{n+1}^m \mid \mathcal{F}_n)) & \text{on } A_n^m \\ (W_n^1, W_n^2) & \text{off } A_n^m, \end{cases}$$

$n = m-1, m-2, \ldots, 0$, where

$$A_n^m = \{(X_n^1, X_n^2) < (E(\beta_{n+1}^m \mid \mathcal{F}_n), E(\gamma_{n+1}^m \mid \mathcal{F}_n))\}.$$

Under the assumptions (a), (b) and (c) the following Lemmas 1–4 (Lemmas 3.1, 3.2, 3.4 and 3.5 in Ohtsubo, 1987) are satisfied:

LEMMA 1 *For each $n, m \in \mathcal{N}$ with $n \leq m$,*

$$(W_n^1, W_n^2) = (\beta_n^n, \gamma_n^n) \leq (\beta_n^m, \gamma_n^m) \leq (\beta_n^{m+1}, \gamma_n^{m+1}).$$

Thus, we can define $\beta_n = \lim_{m\to\infty} \beta_n^m$ and $\gamma_n = \lim_{m\to\infty} \gamma_n^m$. Then β_n and γ_n are $\mathcal{F}_n$-measurable.

LEMMA 2 *The bisequence $(\beta_n, \gamma_n)_{n=0,1,\ldots}$ satisfies the following relation: for each $n \in \mathcal{N}$,*

$$(\beta_n, \gamma_n) = \begin{cases} (E(\beta_{n+1} \mid \mathcal{F}_n), E(\gamma_{n+1} \mid \mathcal{F}_n)) & \text{on } A_n \\ (W_n^1, W_n^2) & \text{off } A_n, \end{cases}$$

where $A_n = \{(X_n^1, X_n^2) < (E(\beta_{n+1} \mid \mathcal{F}_n), E(\gamma_{n+1} \mid \mathcal{F}_n))\}$.

LEMMA 3 $\limsup_n \beta_n = \limsup_n W_n^1$ *and* $\limsup_n \gamma_n = \limsup_n W_n^2$.

For each $n \in \mathcal{N}$, define two random variables, $\overline{\tau}_n$ and $\overline{\sigma}_n$, by

$$\overline{\tau}_n = \inf\{k \geq n : \beta_k = W_k^1\}, \ \overline{\sigma}_n = \inf\{k \geq n : \gamma_k = W_k^2\},$$

where we suppose that the infimum over empty set is equal to $+\infty$.

LEMMA 4 (i) *For each $n \in \mathcal{N}$, $\beta_n = W_n^1$ iff $\gamma_n = W_n^2$.*
(ii) *For each $n \in \mathcal{N}$, $\overline{\tau}_n = \overline{\sigma}_n$.*

The above results have been proved in Ohtsubo (1987) under the stronger assumption (b)* instead of (b) and (c). The proofs of Lemmas 1, 2 and 4 are exactly the same under the weaker assumptions. Below we give the modified

Proof of Lemma 3. Let n, m be arbitrary fixed integers in $\mathcal{N}$, $n \geq m$. From the assumptions (b) and (c) W_m^1 and $\sup_{k\in\mathcal{N}}(W_k^1)^+$ are integrable and

$$W_m^1 \leq \sup_{k\geq m} W_k^1 \leq \sup_{k\in\mathcal{N}}(W_k^1)^+.$$

Hence $\sup_{k\geq m} W_k^1$ is integrable. Moreover $E(\sup_{k\geq m} W_k^1 \mid \mathcal{F}_n)$ is an $\mathcal{F}_n$-martingale, so it is a regular martingale. Since $\sup_{k\geq m} W_k^1$ is $\mathcal{F}_\infty$-measurable, where $\mathcal{F}_\infty$ is the σ-field generated by $\cup_{n=0}^{\infty}\mathcal{F}_n$, by letting $n \to \infty$ (under fixed m) we get

$$\lim_{n\to\infty} E(\sup_{k\geq m} W_k^1 \mid \mathcal{F}_n) = E(\sup_{k\geq m} W_k^1 \mid \mathcal{F}_\infty) = \sup_{k\geq m} W_k^1 \text{ a.s.}$$

Similarly as Ohtsubo we can prove that

$$\beta_n \leq E(\sup_{k\geq m} W_k^1 \mid \mathcal{F}_n), \text{ a.s., } n \geq m. \tag{2.1}$$

So, we have $\limsup_n \beta_n \leq \sup_{k\geq m} W_k^1$, a.s. Letting $m \to \infty$ we obtain the inequality $\limsup_n \beta_n \leq \limsup_n W_n^1$. The reverse follows from the fact that $\beta_n \geq W_n^1$ for all $n \in \mathcal{N}$. Similarly, it is proved that $\limsup_n \gamma_n = \limsup_n W_n^2$. ■

The theorem below gives the Nash equilibrium strategies and inequalities for the corresponding game values.

THEOREM 1 *Under assumptions (a), (b) and (c) the following statements are true:*

(i) *For every* $n \in \mathcal{N}$,
$$\overline{\beta}_n := E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) = E(g^1(\overline{\tau}_n, \overline{\sigma}_n) \mid \mathcal{F}_n) \geq \beta_n,$$
and
$$\overline{\beta}_n \geq E(\beta_{\tau\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) \geq E(g^1(\tau, \overline{\sigma}_n) \mid \mathcal{F}_n) \text{ for all } \tau \in \Lambda_n.$$
(ii) *For every* $n \in \mathcal{N}$,
$$\overline{\gamma}_n := E(\gamma_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) = E(g^2(\overline{\tau}_n, \overline{\sigma}_n) \mid \mathcal{F}_n) \geq \gamma_n,$$
and
$$\overline{\gamma}_n \geq E(\gamma_{\overline{\tau}_n\wedge\sigma} \mid \mathcal{F}_n) \geq E(g^2(\overline{\tau}_n, \sigma) \mid \mathcal{F}_n) \text{ for all } \sigma \in \Lambda_n.$$
(iii) *For each* $n \in \mathcal{N}$, *a pair* $(\overline{\tau}_n, \overline{\sigma}_n)$ *is an equilibrium point at* n, *and a Nash equilibrium value corresponding to* $(\overline{\tau}_0, \overline{\sigma}_0)$ *is equal to* $(E(\beta_{\overline{\tau}_0\wedge\overline{\sigma}_0}), E(\gamma_{\overline{\tau}_0\wedge\overline{\sigma}_0}))$.
(iv) *If* $\limsup_n W_n^1 = -\infty$ *a.s. then, for each* $n \in \mathcal{N}$, $\overline{\tau}_n = \overline{\sigma}_n < \infty$ *a.s.*

Proof of Theorem 1. (i) Let $n \in \mathcal{N}$ be fixed arbitrarily. We will show first that $\{\beta_{k\wedge\overline{\sigma}_n}, k \geq n\}$ is a regular submartingale. From Lemma 2 it follows that

$$\beta_k = E(\beta_{k+1} \mid \mathcal{F}_k) \text{ a.s. if } n \leq k < \overline{\sigma}_n. \tag{2.2}$$

Moreover

$$\beta_{k\wedge\overline{\sigma}_n} = \beta_k I(k < \overline{\sigma}_n) + \beta_{\overline{\sigma}_n} I(k \geq \overline{\sigma}_n)$$

and

$$\beta_{(k+1)\wedge\overline{\sigma}_n} = \beta_{k+1} I(k \le \overline{\sigma}_n - 1) + \beta_{k\wedge\overline{\sigma}_n} I(k \ge \overline{\sigma}_n). \tag{2.4}$$

Now, from (2.2), (2.3), (2.4), we have that $\{\beta_{k\wedge\overline{\sigma}_n}, k \ge n\}$ is an $(\mathcal{F}_k)$-martingale.

To show that it is a regular submartingale it suffices to prove that (Proposition IV-5-24 in Neveu, 1975)

$$E[\sup_{k\ge n}(\beta_{k\wedge\overline{\sigma}_n})^+] < \infty. \tag{2.5}$$

From (2.1) it follows that

$$\beta_k^+ \le E[\sup_l (W_l^1)^+ \mid \mathcal{F}_k], \text{ a.s.} \tag{2.6}$$

and hence we have

$$E[\sup_k(\beta_k)^+] \le E[\sup_l (W_l^1)^+] < \infty, \text{ a.s.}$$

and thus (2.5). Thus, the sequence $\{\beta_{k\wedge\overline{\sigma}_n}, k \ge n\}$ is a regular submartingale. By the optional sampling theorem for regular submartingales (Corollary IV-2-25 in Neveu, 1975) we have $\beta_n = \beta_{n\wedge\overline{\sigma}_n} \le E(\beta_{\tau\wedge\overline{\sigma}_n} \mid \mathcal{F}_n)$ for any $\tau \in \Lambda_n$. In particular for $\tau = \overline{\tau}_n$ we obtain $\beta_n \le E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) = \overline{\beta}_n$. Moreover, since $\overline{\tau}_n = \overline{\sigma}_n$, from Lemma 4: $\beta_{\tau\wedge\overline{\sigma}_n} \le E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_{\tau\wedge\overline{\sigma}_n})$, and hence we have $E(\beta_{\tau\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) \le E[E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_{\tau\wedge\overline{\sigma}_n}) \mid \mathcal{F}_n] = E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_n)$. Now, let us note that $\beta_{\overline{\sigma}_n} = W^1_{\overline{\sigma}_n} \ge Y^1_{\overline{\sigma}_n}$ if $\overline{\sigma}_n < \infty$ and, from Lemma 2, $\beta_k \ge X_k^1$ if $\overline{\sigma}_n > k$. Hence from Lemma 3 we get for any $\tau \in \Lambda_n$,

$$\begin{aligned}
&\overline{\beta}_n \ge E(\beta_{\tau\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) \\
&= E[\beta_\tau I(\tau < \overline{\sigma}_n) + \beta_{\overline{\sigma}_n} I(\overline{\sigma}_n < \tau) \\
&+ \beta_{\overline{\sigma}_n} I(\tau = \overline{\sigma}_n < \infty) + \limsup_n \beta_n I(\tau = \overline{\sigma}_n = \infty) \mid \mathcal{F}_n] \\
&\ge E[X_\tau^1 I(\tau < \overline{\sigma}_n) + Y^1_{\overline{\sigma}_n} I(\overline{\sigma}_n < \tau) \\
&+ W^1_{\overline{\sigma}_n} I(\tau = \overline{\sigma}_n < \infty) + \limsup_n W_n^1 I(\tau = \overline{\sigma}_n = \infty) \mid \mathcal{F}_n] \\
&= E(g^1(\tau, \overline{\sigma}_n) \mid \mathcal{F}_n).
\end{aligned}$$

In particular, for $\tau = \overline{\tau}_n$ we have, from Lemma 4,

$$\begin{aligned}
&\overline{\beta}_n = E(\beta_{\overline{\tau}_n\wedge\overline{\sigma}_n} \mid \mathcal{F}_n) \\
&= E(W^1_{\overline{\tau}_n} I(\overline{\tau}_n = \overline{\sigma}_n < \infty) + \limsup_n W_n^1 I(\overline{\tau}_n = \overline{\sigma}_n = \infty) \mid \mathcal{F}_n) \\
&= E(g^1(\overline{\tau}_n, \overline{\sigma}_n) \mid \mathcal{F}_n).
\end{aligned}$$

Therefore the proof of (i) is complete.

(iii) Immediately follows from (i) and (ii).

(iv) From Lemma 4 it is sufficient to prove that $\overline{\sigma}_n < \infty$, a.s. From (2.6) it follows that $\sup_{k \geq n} E[(\beta_{k \wedge \overline{\sigma}_n})^+] < \infty$. Hence a martingale $\{\beta_{k\wedge\overline{\sigma}_n},\ k \geq n\}$ converges a.s. to a limit which is an integrable r.v. (Theorem IV-1-2 in Neveu, 1975). Thus, the sequence $\{\beta_n,\ n \geq 0\}$ converges to a finite limit a.s. on the set $\{\overline{\sigma}_n = \infty\}$. Therefore, since $\limsup_n \beta_n = \limsup_n W_n^1$ a.s., we have $\limsup_n W_n^1 > -\infty$ a.s. on $\{\overline{\sigma}_n = \infty\}$. But we have assumed that $\limsup_n W_n^1 = -\infty$ a.s. Hence we get $P(\overline{\sigma}_n = \infty) = 0$. ■

REMARK 1 *Theorem 1 corresponds to Theorem 3.1 in Ohtsubo (1987). But in that paper Ohtsubo proved that* $\overline{\beta}_n = \beta_n$ *and* $\overline{\gamma}_n = \gamma_n$. *Under our weaker assumptions we can prove only that* $\overline{\beta}_n \geq \beta_n$ *and* $\overline{\gamma}_n \geq \gamma_n$. *This results from the fact that in Ohtsubo's case the sequence* $\{\beta_{k\wedge\overline{\sigma}_n},\ k \geq n\}$ *is a regular martingale while in ours it is only a regular submartingale.*

3. The Markov model

We will present the stopping game described in Section 2 in the case when the reward sequences are functions of states of a homogeneous Markov process, similarly as in Ohtsubo. Ohtsubo in (1987) studied the noncooperative stopping game in the following Markov model.

Let $(Z_n, \mathcal{F}_n, \mathcal{P}_x)$ be a homogeneous Markov process on the phase space $(E, \mathcal{B})$. Let $B(E)$ be the set of bounded $\mathcal{B}$-measurable functions on $(E, \mathcal{B})$. Let $\phi^i, \psi^i, h^i \in B(E)$, $i = 1, 2$. For stopping times τ^1 and τ^2 in Λ, the mean reward of the ith player is

$$E_x[g^i(\tau^1,\tau^2)] = E_x[\phi^i(Z_{\tau^i})I(\tau^i < \tau^j) + \psi^i(Z_{\tau^j})I(\tau^j < \tau^i) + h^i(Z_{\tau^i})I(\tau^i = \tau^j < \infty) + \limsup_n h^i(Z_n)I(\tau^i = \tau^j = \infty)],$$

where $i, j = 1, 2$, $i \neq j$, $x \in E$, and E_x denotes the expectation operator with respect to $\mathcal{P}_x$.

Under the assumption that $\psi^i \leq h^i \leq \phi^i$ on E for each $i = 1, 2$, let us define the bisequence $\{(\alpha_n^1, \alpha_n^2),\ n \in \mathcal{N}\}$ of functions by

$$(\alpha_0^1(x), \alpha_0^2(x)) = (h^1(x), h^2(x)),\ x \in E,$$
$$(\alpha_{n+1}^1(x), \alpha_{n+1}^2(x)) = \begin{cases} (T\alpha_n^1(x), T\alpha_n^2(x)) & \text{on } A \\ (h^1(x), h^2(x)) & \text{off } A, \end{cases}$$

$x \in E$, $n \in \mathcal{N}$, where the operator T is such that $Tf(x) = E_x[f(Z_1)]$, $x \in E$, for $f \in B(E)$, and

$$A = \{x \in E : (\phi^1(x), \phi^2(x)) < (T\alpha_n^1(x), T\alpha_n^2(x))\}.$$

Then, $\{\alpha_n^i\}_{n=0}^{\infty}$, $i = 1, 2$, are increasing in n and we denote the limits of α_n^i

Define the stopping times $\overline{\tau}^i$, $i = 1, 2$, by

$$\overline{\tau}^i = \inf\{n \geq 0 : Z_n \in \Gamma^i\}$$

($= +\infty$ if no such n exists), where $\Gamma^i = \{x \in E : \alpha^i(x) = h^i(x)\}$. The theorem below is the immediate consequence of Theorem 1:

THEOREM 2 *A pair $(\overline{\tau}^1, \overline{\tau}^2)$ of stopping times is a Nash equilibrium point and the Nash value function $(\overline{\beta}, \overline{\gamma})$ corresponding to $(\overline{\tau}^1, \overline{\tau}^2)$ is such that $\overline{\beta} \geq \alpha^1$ and $\overline{\gamma} \geq \alpha^2$.*

Let us note that for Theorem 2 to be satisfied it is sufficient to assume that the functions ϕ^i, ψ^i, h^i, $i = 1, 2$, are such that $E|\phi^i(Z_n)| < \infty$, $E|\psi^i(Z_n)| < \infty$ and $E[\sup_n(\phi^i(Z_n))^+] < \infty$, which corresponds to our assumptions (b) and (c).

4. Equilibrium strategies for the risk process model

Now, we apply the results of Section 2 to find Nash equilibrium strategies for our game associated with the risk processes.

First, let us make precise assumptions on the distributions of random variables defining the processes $U_n^i = u^i + nc^i - (V_1^i + \ldots + V_n^i)$, $n = 0, 1, 2, \ldots$, $i = 1, 2$. Let $\theta, V_1^1, V_2^1, \ldots, V_1^2, V_2^2, \ldots$ have the distribution as follows

$$P(\theta = 0) = \pi,\ P(\theta = n) = (1 - \pi)(1 - p)^{n-1}p,\ n = 1, 2, \ldots,$$

where $\pi \in [0, 1]$, $p \in (0, 1]$ are fixed and known and

$$(V_n^1, V_n^2) = \begin{cases} (V_n^{1,I}, V_n^{2,I}) & \text{if } n < \theta \\ (V_n^{1,II}, V_n^{2,II}) & \text{if } n \geq \theta, \end{cases} \quad n = 1, 2, \ldots,$$

where $V_n^{1,I}, V_n^{2,I}$ are iid with the density f_0, and $V_n^{1,II}, V_n^{2,II}$ are iid with the density f_1, $f_0 \neq f_1$.

We also assume that $\mu_1 > c^i \geq \mu_0$, $i = 1, 2$, where μ_j is the expectation of the random variables with the density f_j, $j = 0, 1$. Let us note that the inequalities for $\mu' s$ mean that the premiums are established correctly for the distribution with the density f_0 and then after the change into f_1 the situation is unfavorable for the insurers.

Let, for any n, Π_n denote the conditional probability that the change in distribution of claims has occurred not later than at n, given the observations till that moment, i.e.

$$\Pi_n = P(\theta \leq n \mid \mathcal{F}_n),\ n = 1, 2, \ldots.$$

Using the Bayes formula (Shiryaev, 1978) we obtain:

$$\Pi_n =$$

$$\Pi_{n-1} f_1(V_n^1, V_n^2) + (1 - \Pi_{n-1}) p f_1(V_n^1, V_n^2)$$

which may be rewritten in a more convenient form

$$\Pi_n = \frac{\lambda(V_n^1, V_n^2)}{\lambda(V_n^1, V_n^2) + w(\Pi_{n-1})},$$

where the functions λ and w are defined as follows

$$\lambda(V_n^1, V_n^2) = \frac{f_1(V_n^1, V_n^2)}{f_0(V_n^1, V_n^2)}, \ w(\Pi_{n-1}) = \frac{(1-\Pi_{n-1})(1-p)}{\Pi_{n-1} + (1-\Pi_{n-1})p}.$$

From the Bayes formula it also follows that (Bojdecki, 1979)

$$P(\theta = n \mid \mathcal{F}_n) = \frac{\frac{p}{1-p}\lambda(V_n^1, V_n^2)}{p\xi_n + 1},$$

where

$$\xi_n = \sum_{i=1}^{n} (1-p)^{i-n-1} \lambda(V_n^1, V_n^2) \ldots \lambda(V_i^1, V_i^2), \ \xi_0 = 0.$$

Let us define $Z_n = (\Pi_n, U_n^1, U_n^2, \xi_n)$ for $n = 1, 2, \ldots$, $Z_0 = (\Pi_0, U_0^1, U_0^2, \xi_0) = (\pi, u^1, u^2, 0)$.

LEMMA 5 (Z_n) $_{n=0,1,\ldots}$ *forms a homogeneous Markov chain with respect to* $(\mathcal{F}_n)_{n=0,1,\ldots}$.

Proof. (1) Z_n is $\mathcal{F}_n$-measurable for $n = 1, 2, \ldots$

(2) For $n = 1, 2, \ldots$ there exists a measurable function χ such that

$$Z_{n+1} = \chi(Z_n, (V_{n+1}^1, V_{n+1}^2)).$$

Indeed

$$\begin{aligned}
Z_{n+1} &= (\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) \\
&= \left(\frac{\lambda(V_{n+1}^1, V_{n+1}^2)}{\lambda(V_{n+1}^1, V_{n+1}^2) + w(\Pi_n)}, U_n^1 + c^1 - V_{n+1}^1, \right. \\
& \left. U_n^2 + c^2 - V_{n+1}^2, \frac{(\xi_n + 1)\lambda(V_{n+1}^1, V_{n+1}^2)}{1-p} \right) \\
&= \chi(\Pi_n, U_n^1, U_n^2, \xi_n, (V_{n+1}^1, V_{n+1}^2)) = \chi(Z_n, (V_{n+1}^1, V_{n+1}^2)), \ n = 1, 2, \ldots.
\end{aligned}$$

(3) The conditional distribution of V_{n+1}^1, V_{n+1}^2 given $\mathcal{F}_n$ has the distribution function

$$\begin{aligned}
F_{\Pi_n}(x, y) &= P(V_{n+1}^1 \le x, V_{n+1}^2 \le y \mid \mathcal{F}_n) \\
&= P(V_{n+1}^1 \le x, V_{n+1}^2 \le y, \theta \le n \mid \mathcal{F}_n) \\
&+ P(V_{n+1}^1 \le x, V_{n+1}^2 \le y, \theta \ge n+1 \mid \mathcal{F}_n) \\
&= \Pi_n F_1(x) F_1(y) + P(V_{n+1}^1 \le x, V_{n+1}^2 \le y, \theta = n+1, \theta > n \mid \mathcal{F}_n) \\
&+ P(V_{n+1}^1 \le x, V_{n+1}^2 \le y, \theta > n+1, \theta > n \mid \mathcal{F}_n) \\
&= \Pi_n F_1(x) F_1(y) + (1-\Pi_n) p F_1(x) F_1(y) + (1-\Pi_n)(1-p) F_0(x) F_0(y),
\end{aligned}$$

Now we may apply Lemma II.17 of Shiryaev (1978) to obtain that $(Z_n)_{n=0,1,\dots}$ forms a homogeneous Markov chain with respect to $(\mathcal{F}_n)_{n=0,1,\dots}$. ■

Let us denote

$$\Pi_{n_-} = P(\theta < n \mid \mathcal{F}_n) = \Pi_n - P(\theta = n \mid \mathcal{F}_n) = \Pi_n - \frac{\frac{p}{1-p}\lambda(V_n^1, V_n^2)}{p\xi_n + 1}.$$

Now our reward sequences may be rewritten as follows:

$$X_n^i = U_n^i - k_1^i \Pi_{n_-} = \phi^i(Z_n),\ Y_n^i = U_n^i - k_2^i \Pi_{n_-} = \psi^i(Z_n),$$
$$W_n^i = U_n^i - k_3^i \Pi_{n_-} = h^i(Z_n),$$

$n = 1, 2, \dots,$ where k_1^i, k_2^i, k_3^i are nonnegative constants representing the stopping costs and ϕ^i, ψ^i, h^i are functions defined on a state space of the homogeneous Markov chain $(Z_n)_{n=0,1,\dots}$.

We assume that $k_2^i \geq k_3^i \geq k_1^i$, $i = 1, 2$. This is equivalent to $X_n^i \geq W_n^i \geq Y_n^i$ for every $n \in \mathcal{N}$ and each $i = 1, 2$, (or: $\psi^i \leq h^i \leq \phi^i$ on E for each $i = 1, 2$) and means that, for each of the players, it is more profitable to stop at the same time than to stop as the second one and the most profitable is to stop as the first one. Hence the assumption (a) is fulfilled. In what follows we assume that

$$\int_{-\infty}^{+\infty} x^2 f_j(x)\, dx < \infty,\ j = 0, 1, \tag{4.1}$$

which ensures that assumptions (b) and (c) are satisfied as the consequence of the following lemma in Bobecka, Danielak and Ferenstein (2002):

LEMMA 6 *If* $\mu_1 > c^i > \mu_0$, $i = 1, 2$, *and (4.1) then*

(1) $\lim_{n\to\infty} E(U_n^i) = -\infty$ *and*
(2) $E[\sup_n (U_n^i)^+] < \infty$.

Sequences $(\beta_n^m, \gamma_n^m)_{n=0,1,2,\dots,m}$ of the conditional game values in the finite horizon case, defined in Section 2, become for our game:

$$(\beta_m^m, \gamma_m^m) = (W_m^1, W_m^2) = (U_m^1 - k_3^1 \Pi_{m_-}, U_m^2 - k_3^2 \Pi_{m_-}),$$
$$(\beta_n^m, \gamma_n^m) = \begin{cases} (E(\beta_{n+1}^m \mid \mathcal{F}_n), E(\gamma_{n+1}^m \mid \mathcal{F}_n)) & \text{on } A_n^m \\ (W_n^1, W_n^2) & \text{off } A_n^m \end{cases}$$
$$= \begin{cases} (E(\beta_{n+1}^m \mid \mathcal{F}_n), E(\gamma_{n+1}^m \mid \mathcal{F}_n)) & \text{on } A_n^m \\ (U_n^1 - k_3^1 \Pi_{n_-}, U_n^2 - k_3^2 \Pi_{n_-}) & \text{off } A_n^m, \end{cases}$$

$n = m-1, m-2, \dots, 0$, where

$$A_n^m = \{(X_n^1, X_n^2) < (E(\beta_{n+1}^m \mid \mathcal{F}_n), E(\gamma_{n+1}^m \mid \mathcal{F}_n))\}$$

Then for $n = m-1, m-2, \ldots, 0$, we have

$$
\begin{aligned}
\beta_n^m &= E(\beta_{n+1}^m \mid \mathcal{F}_n) I(A_n^m) + (W_n^1) I(\overline{A}_n^m) \\
&= E(\beta_{n+1}^m \mid \mathcal{F}_n) I(X_n^1 < E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 < E(\gamma_{n+1}^m \mid \mathcal{F}_n)) \\
&+ (W_n^1) I(X_n^1 \geq E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 \geq E(\gamma_{n+1}^m \mid \mathcal{F}_n)) \\
&= [E(\beta_{n+1}^m \mid \mathcal{F}_n) - W_n^1] I(X_n^1 < E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 < E(\gamma_{n+1}^m \mid \mathcal{F}_n)) + W_n^1, \\
\gamma_n^m &= E(\gamma_{n+1}^m \mid \mathcal{F}_n) I(A_n^m) + (W_n^2) I(\overline{A}_n^m) \\
&= E(\gamma_{n+1}^m \mid \mathcal{F}_n) I(X_n^1 < E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 < E(\gamma_{n+1}^m \mid \mathcal{F}_n)) \\
&+ (W_n^2) I(X_n^1 \geq E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 \geq E(\gamma_{n+1}^m \mid \mathcal{F}_n)) \\
&= [E(\gamma_{n+1}^m \mid \mathcal{F}_n) - W_n^2] I(X_n^1 < E(\beta_{n+1}^m \mid \mathcal{F}_n)) I(X_n^2 < E(\gamma_{n+1}^m \mid \mathcal{F}_n)) + W_n^2,
\end{aligned}
$$

which may be rewritten in a more convenient form:

$$
\begin{aligned}
\beta_n^m &= \varphi_n^{1,m} I(\varphi_n^{1,m} > X_n^1 - W_n^1) I(\varphi_n^{2,m} > X_n^2 - W_n^2) + W_n^1 \\
&= \varphi_n^{1,m} I(\varphi_n^{1,m} > (k_3^1 - k_1^1)\Pi_{n-}) I(\varphi_n^{2,m} > (k_3^2 - k_1^2)\Pi_{n-}) + W_n^1, \\
\gamma_n^m &= \varphi_n^{2,m} I(\varphi_n^{1,m} > X_n^1 - W_n^1) I(\varphi_n^{2,m} > X_n^2 - W_n^2) + W_n^2 \\
&= \varphi_n^{2,m} I(\varphi_n^{1,m} > (k_3^1 - k_1^1)\Pi_{n-}) I(\varphi_n^{2,m} > (k_3^2 - k_1^2)\Pi_{n-}) + W_n^2,
\end{aligned}
$$

where for convenience we introduce the random variables $\varphi_n^{i,m}$, $i = 1, 2$, defined as follows

$$
\begin{aligned}
\varphi_n^{1,m} &= E(\beta_{n+1}^m \mid \mathcal{F}_n) - W_n^1 = E(\beta_{n+1}^m \mid \mathcal{F}_n) - (U_n^1 - k_3^1 \Pi_{n-}), \\
\varphi_n^{2,m} &= E(\gamma_{n+1}^m \mid \mathcal{F}_n) - W_n^2 = E(\gamma_{n+1}^m \mid \mathcal{F}_n) - (U_n^2 - k_3^2 \Pi_{n-}),
\end{aligned}
$$

$n = m-1, m-2, \ldots, 0$.

The lemma below gives the recurrence formula for the sequence $(\varphi_n^{1,m}, \varphi_n^{2,m})$:

Lemma 7

$$
\begin{aligned}
\varphi_{m-1}^{i,m} &= \varphi_{m-1}^{i,m}(z) = \varphi_{m-1}^{i,m}(\widetilde{\pi}, u^1, u^2, \xi) \\
&= c^i - \widetilde{\pi}(\mu_1 - \widetilde{\mu}_p) - \widetilde{\mu}_p - k_3^i \frac{\frac{p}{1-p}\lambda(v^1, v^2)}{p\xi + 1}, \quad i = 1, 2,
\end{aligned}
$$

and, for $n = m-2, m-3, \ldots, 0$,

$$
\varphi_n^{i,m} = \varphi_n^{i,m}(z) = L\varphi_{n+1}^{i,m}(z),
$$

where

$$
L\varphi_{n+1}^{i,m}(z) = \iint\limits_{A_{n+1}^m} \varphi_{n+1}^{i,m}\left[\frac{\lambda(v^1, v^2)}{\lambda(v^1, v^2) + w(\widetilde{\pi})}, u^1 + c^1 - v^1,\right.
$$

$$
\left. u^2 + c^2 - v^2, \frac{(\xi + 1)\lambda(v^1, v^2)}{1-p}\right] f_{\widetilde{\pi}}(v^1, v^2)\, dv^1\, dv^2
$$

$$
\ldots \frac{p}{1-p}\lambda(v^1, v^2) \ldots
$$

$$\widetilde{\mu}_p = p\mu_1 + (1-p)\mu_0,$$
$$A^m_{n+1} = \left\{(v^1,v^2) : \varphi^{i,m}_{n+1}\left[\frac{\lambda(v^1,v^2)}{\lambda(v^1,v^2)+w(\widetilde{\pi})}, u^1+c^1-v^1,\right.\right.$$
$$\left.\left. u^2+c^2-v^2, \frac{(\xi+1)\lambda(v^1,v^2)}{1-p}\right] > (k^i_3-k^i_1)\widetilde{\pi},\ i=1,2\right\}.$$

Proof.

$$\begin{aligned}
&\varphi^{1,m}_{m-1} = \varphi^{1,m}_{m-1}(Z_{m-1}) = \varphi^{1,m}_{m-1}(\Pi_{m-1}, U^1_{m-1}, U^2_{m-1}, \xi_{m-1})\\
&= E(\beta^m_m(\Pi_m, U^1_m, U^2_m, \xi_m) \mid \mathcal{F}_{m-1}) - (U^1_{m-1} - k^1_3\Pi_{(m-1)_-})\\
&= E(U^1_m - k^1_3\Pi_{m_-} \mid \mathcal{F}_{m-1}) - U^1_{m-1}\\
&+ k^1_3(\Pi_{m-1} - P(\theta = m-1 \mid \mathcal{F}_{m-1}))\\
&= E(U^1_{m-1} + c^1 - V^1_m - k^1_3P(\theta < m \mid \mathcal{F}_m) \mid \mathcal{F}_{m-1}) - U^1_{m-1}\\
&+ k^1_3(\Pi_{m-1} - P(\theta = m-1 \mid \mathcal{F}_{m-1}))\\
&= U^1_{m-1} + c^1 - E(V^1_m \mid \mathcal{F}_{m-1}) - k^1_3E(P(\theta < m \mid \mathcal{F}_m) \mid \mathcal{F}_{m-1}) - U^1_{m-1}\\
&+ k^1_3\Pi_{m-1} - k^1_3P(\theta = m-1 \mid \mathcal{F}_{m-1})\\
&= c^1 - [\Pi_{m-1}\mu_1 + (1-\Pi_{m-1})p\mu_1 + (1-\Pi_{m-1})(1-p)\mu_0]\\
&- k^1_3E(E(I(\theta < m) \mid \mathcal{F}_m) \mid \mathcal{F}_{m-1})\\
&+ k^1_3\Pi_{m-1} - k^1_3P(\theta = m-1 \mid \mathcal{F}_{m-1})\\
&= c^1 - [\Pi_{m-1}(\mu_1 - \widetilde{\mu}_p) + \widetilde{\mu}_p] - k^1_3E(I(\theta \le m-1) \mid \mathcal{F}_{m-1})\\
&+ k^1_3\Pi_{m-1} - k^1_3P(\theta = m-1 \mid \mathcal{F}_{m-1})\\
&= c^1 - \Pi_{m-1}(\mu_1 - \widetilde{\mu}_p) - \widetilde{\mu}_p - k^1_3P(\theta \le m-1 \mid \mathcal{F}_{m-1})\\
&+ k^1_3\Pi_{m-1} - k^1_3P(\theta = m-1 \mid \mathcal{F}_{m-1})\\
&= c^1 - \Pi_{m-1}(\mu_1 - \widetilde{\mu}_p) - \widetilde{\mu}_p - k^1_3\Pi_{m-1}\\
&+ k^1_3\Pi_{m-1} - k^1_3P(\theta = m-1 \mid \mathcal{F}_{m-1})\\
&= c^1 - \Pi_{m-1}(\mu_1 - \widetilde{\mu}_p) - \widetilde{\mu}_p - k^1_3\frac{\frac{p}{1-p}\lambda(V^1_{m-1}, V^2_{m-1})}{p\xi_{m-1}+1}.
\end{aligned}$$

For $n = m-2, m-3, \ldots, 0$

$$\begin{aligned}
&\varphi^{1,m}_n = \varphi^{1,m}_n(Z_n) = \varphi^{1,m}_n(\Pi_n, U^1_n, U^2_n, \xi_n)\\
&= E[\beta^m_{n+1}(\Pi_{n+1}, U^1_{n+1}, U^2_{n+1}, \xi_{n+1}) \mid \mathcal{F}_n] - (U^1_n - k^1_3\Pi_{n_-})\\
&= E[\beta^m_{n+1}(\Pi_{n+1}, U^1_{n+1}, U^2_{n+1}, \xi_{n+1})I(\beta^m_{n+1}(\Pi_{n+1}, U^1_{n+1}, U^2_{n+1}, \xi_{n+1}) > 0)\\
&I(\gamma^m_{n+1}(\Pi_{n+1}, U^1_{n+1}, U^2_{n+1}, \xi_{n+1}) > 0) \mid \mathcal{F}_n]\\
&+ E(U^1_{n+1} - k^1_3\Pi_{(n+1)_-} \mid \mathcal{F}_n) - (U^1_n - k^1_3\Pi_{n_-}).
\end{aligned}$$

Moreover, we have

$$E(U^1_{n+1} - k^1_3\Pi_{(n+1)_-} \mid \mathcal{F}_n) - U^1_n + k^1_3\Pi_{n_-}$$

$$\begin{aligned}
&= U_n^1 + c^1 - E(V_{n+1}^1 \mid \mathcal{F}_n) \\
&- k_3^1 E(P(\theta < n+1 \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) - U_n^1 + k_3^1 \Pi_{n_-} \\
&= c^1 - [\Pi_n(\mu_1 - \widetilde{\mu}_p) + \widetilde{\mu}_p] - k_3^1 \Pi_n + k_3^1 \Pi_n - k_3^1 P(\theta = n \mid \mathcal{F}_n) \\
&= c^1 - \Pi_n(\mu_1 - \widetilde{\mu}_p) - \widetilde{\mu}_p - k_3^1 \frac{\frac{p}{1-p}\lambda(V_n^1, V_n^2)}{p\xi_n + 1},
\end{aligned}$$

and

$$\begin{aligned}
&E[\varphi_{n+1}^{1,m}(\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) I(\varphi_{n+1}^{1,m}(\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) > 0) \\
&I(\varphi_{n+1}^{2,m}(\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) > 0) \mid \mathcal{F}_n] \\
&= \iint\limits_{A_{n+1}^m} \varphi_{n+1}^{i,m} \left[\frac{\lambda(v^1, v^2)}{\lambda(v^1, v^2) + w(\Pi_n)}, U_n^1 + c^1 - v^1, \right. \\
&\left. U_n^2 + c^2 - v^2, \frac{(\xi_n + 1)\lambda(v^1, v^2)}{1-p} \right] f_{\Pi_n}(v^1, v^2)\, dv^1\, dv^2,
\end{aligned}$$

where

$$\begin{aligned}
&A_{n+1}^m = \{(v^1, v^2) : (U_{n+1}^1 - k_1^1 \Pi_{(n+1)_-}, U_{n+1}^2 - k_1^2 \Pi_{(n+1)_-}) \\
&< (E(\beta_{n+2}^m \mid \mathcal{F}_{n+1}), E(\gamma_{n+2}^m \mid \mathcal{F}_{n+1}))\} \\
&= \{(v^1, v^2) : \varphi_{n+1}^{1,m}(\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) > (k_3^1 - k_1^1)\Pi_{n-}, \\
&\varphi_{n+1}^{2,m}(\Pi_{n+1}, U_{n+1}^1, U_{n+1}^2, \xi_{n+1}) > (k_3^2 - k_1^2)\Pi_{n-}\}.
\end{aligned}$$

The equalities for $\varphi_{m-1}^{2,m}$ and $\varphi_n^{2,m}$ are given in the same way. ■

For the infinite horizon, from Theorem 1, we get that a Nash equilibrium value corresponding to $(\overline{\tau}_0, \overline{\sigma}_0)$, where

$$\begin{aligned}
\overline{\tau}_0 &= \inf\{n \geq 0 : \beta_n = U_n^1 - k_3^1 \Pi_{n_-}\}, \\
\overline{\sigma}_0 &= \inf\{n \geq 0 : \gamma_n = U_n^2 - k_3^2 \Pi_{n_-}\},
\end{aligned}$$

is equal to $(\overline{\beta}_0, \overline{\gamma}_0) = (E(\beta_{\overline{\tau}_0 \wedge \overline{\sigma}_0}), E(\gamma_{\overline{\tau}_0 \wedge \overline{\sigma}_0}))$, where

$$\overline{\beta}_0 \geq \beta_0 = \lim_{m \to \infty} \beta_0^m \text{ and } \overline{\gamma}_0 \geq \gamma_0 = \lim_{m \to \infty} \gamma_0^m.$$

Now we apply the results of Section 3 for the Markov model.

Let us denote by α^i the limit of α_n^i, as $n \to \infty$, $i = 1, 2$, where the bisequence $\{(\alpha_n^1, \alpha_n^2),\, n \in \mathcal{N}\}$ is defined as follows

$$\begin{aligned}
&(\alpha_0^1(z), \alpha_0^2(z)) = (h^1(z), h^2(z)), \\
&(\alpha_{n+1}^1(z), \alpha_{n+1}^2(z)) = \begin{cases} (T\alpha_n^1(z), T\alpha_n^2(z)) & \text{on } A \\ (h^1(z), h^2(z)) & \text{off } A, \end{cases}
\end{aligned}$$

$n = 0, 1, \ldots, z \in E$, $(E, \mathcal{B})$ is a phase space of the Markov chain (Z_n), $n =$

and

$$A = \{(\phi^1(z), \phi^2(z)) < (T\alpha_n^1(z), T\alpha_n^2(z))\}.$$

Moreover, let

$$\varphi^i(z) = T\alpha^i(z) - h^i(z),\ i = 1, 2.$$

Define the stopping times $\overline{\tau}^i$, $i = 1, 2$, by

$$\overline{\tau}^i = \inf\{n \geq 0 : \alpha^i(Z_n) = h^i(Z_n)\}$$

$(= +\infty$ if no such n exists).

The theorem below, which is the consequence of Theorem 2, gives Nash equilibrium point and inequalities for the corresponding game value.

THEOREM 3 (i) *The pair* $(\overline{\tau}^1, \overline{\tau}^2)$ *is a Nash equilibrium point.*
(ii) *The corresponding game value is equal to* $(\overline{\beta}, \overline{\gamma})$*:*

$$\begin{aligned}
&\overline{\beta} \geq \beta_0 = \beta_0(z_0) = \alpha^1(z_0)\\
&= \varphi^1(z_0)I(\varphi^1(z_0) > 0)I(\varphi^2(z_0) > 0) - h^1(z_0),\\
&\overline{\gamma} \geq \gamma_0 = \gamma_0(z_0) = \alpha^2(z_0)\\
&= \varphi^2(z_0)I(\varphi^1(z_0) > 0)I(\varphi^2(z_0) > 0) - h^2(z_0),
\end{aligned}$$

where

$$\begin{aligned}
&\varphi^i(z_0) = T\alpha^i(z_0) - h^i(z_0)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha^i\left[\frac{\lambda(v^1, v^2)}{\lambda(v^1, v^2) + w(\pi)}, u^1 + c^1 - v^1,\right.\\
&\left. u^2 + c^2 - v^2, \frac{\lambda(v^1, v^2)}{1 - p}\right] f_\pi(v^1, v^2)dv^1dv^2 - u^i,\ i = 1, 2.
\end{aligned}$$

(iii) $\overline{\tau}^i < \infty$, *a.s.*, $i = 1, 2$.

Proof.

$$\begin{aligned}
&(\beta_0, \gamma_0) = (\beta_0(z_0), \gamma_0(z_0)) = (\alpha^1(z_0), \alpha^2(z_0))\\
&= \begin{cases} (T\alpha^1(z_0), T\alpha^2(z_0)) & \text{on } A\\ (h^1(z_0), h^2(z_0)) & \text{off } A, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
&A = \{z_0 \in E : (\phi^1(z_0), \phi^2(z_0)) < (T\alpha^1(z_0), T\alpha^2(z_0))\},\\
&\phi^i(z_0) = \phi^i(\Pi_0, U_0^1, U_0^2, \xi_0) = \phi^i((\pi, u^1, u^2, 0)) = u^i,\ i = 1, 2,\\
&T\alpha^i(z_0) = T\alpha^i(\Pi_0, U_0^1, U_0^2, \xi_0)\\
&= E[\alpha^i(\Pi_1, U_1^1, U_1^2, \xi_1) \mid (\Pi_0, U_0^1, U_0^2, \xi_0) = (\pi, u^1, u^2, 0)]\\
&= E\left[\alpha^i\left(\frac{\lambda(V_1^1, V_1^2)}{\lambda(V_1^1, V_1^2) + w(\Pi_0)}, U_0^1 + c^1 - V_1^1,\right.\right.
\end{aligned}$$

$$U_0^2 + c^2 - V_1^2, \frac{(\xi_0 + 1)\lambda(V_1^1, V_1^2)}{\ldots}\Big) \mid (\Pi_0, U_0^1, U_0^2, \xi_0) = (\pi, u^1, u^2, 0)\Big]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha^i\left[\frac{\lambda(v^1,v^2)}{\lambda(v^1,v^2)+w(\pi)}, u^1+c^1-v^1,\right.$$
$$\left. u^2+c^2-v^2, \frac{\lambda(v^1,v^2)}{1-p}\right] f_\pi(v^1,v^2)\,dv^1\,dv^2,\ i=1,2.$$

and

$$\begin{aligned} A &= \{z_0 \in E : (\phi^1(z_0),\phi^2(z_0)) < (T\alpha^1(z_0), T\alpha^2(z_0))\} \\ &= \{z_0 \in E : T\alpha^i(z_0) - h^i(z_0) > \phi^i(z_0) - h^i(z_0) = u^i - u^i = 0,\ i=1,2\} \\ &= \{z_0 \in E : \varphi^i(z_0) > 0,\ i = 1,2\}. \end{aligned}$$

Moreover, from Lemma 6 we have

$$\limsup_n h^i(Z_n) = \limsup_n W_n^1 = \limsup_n (U_n^1 - k_3^1\Pi_{n_-}) = -\infty \text{ a.s.}$$

So (iii) immediately follows from (iv) of Theorem 1. ■

REMARK 2 *We do not need to assume that claim distributions of each player are the same until the moment θ and at θ they switch to another common distribution. We may assume, instead, that the claim distributions of each player are different, i.e.*

$$V_n^i = \begin{cases} V_n^{i,I} & \text{if } n < \theta \\ V_n^{i,II} & \text{if } n \geq \theta, \end{cases} \quad n = 1,2,\ldots,$$

where $V_n^{i,I}$ are iid with the density f_0^i, and $V_n^{i,II}$ are iid with the density f_1^i, $i=1,2$. Then, under the assumption that $\mu_1^i > c^i \geq \mu_0^i$, where μ_j^i is the expectation of the random variable with the density f_j^i, $j=0,1$, $i=1,2$, all formulas and results are still true if we only replace the function λ with

$$\lambda(V_n^1, V_n^2) = \frac{f_1^1(V_n^1) f_1^2(V_n^2)}{f_0^1(V_n^1) f_0^2(V_n^2)}.$$

References

BOBECKA, K., DANIELAK, K. and FERENSTEIN, E.Z. (2002) On Optimal Stopping of a Discrete Risk Process. *Demonstratio Mathematica* (to appear).

BOJDECKI, T. (1979) Probability Maximizing Approach to Optimal Stopping and its Application to a Disorder Problem. *Stochastics*, **3**, 61–71.

CHOW, Y.S., ROBBINS, H. and SIEGMUND, D. (1971) *Great Expectations: The Theory of Optimal Stopping*. Houghton Mifflin, Boston.

DYNKIN, E.B. (1969) Game Variant of a Problem on Optimal Stopping. *Soviet Math. Dokl.*, **10**, 270–274.

ENNS, E.G. and FERENSTEIN, E.Z. (1987) On a Multi-Person Time-Sequential

FERENSTEIN, E.Z. (1993) A Variation of the Dynkin's Stopping Game. *Mathematica Japonica*, **38**, 2, 371–379.

FERENSTEIN, E.Z. and SIEROCIŃSKI, A. (1997) Optimal Stopping of a Risk Process. *Applicationes Mathematicae*, **24**, 3, 335–342.

JENSEN, U. (1997) An Optimal Stopping Problem in Risk Theory. *Scand. Actuarial J.*, **2**, 149–159.

NEVEU, J. (1975) *Discrete-Parameter Martingales.* North-Holland Publishing Company, Amsterdam, Oxford.

NOWAK, A.S. and SZAJOWSKI, K. (1999) Nonzero-Sum Stochastic Games. *Annals of the International Society of Dynamic Games*, **4**, 297–343.

OHTSUBO, Y. (1987) A Nonzero-Sum Extension of Dynkin's Stopping Problem. *Mathematics of Operations Research*, **12**, 277–296.

SAKAGUCHI, M. (1991) Sequential Games with Priority under Expected Value Maximization. *Mathematica Japonica*, **36**, 3, 545–562.

SHIRYAEV, A.N. (1978) *Optimal Stopping Rules.* Springer-Verlag, New York.

SZAJOWSKI, K. (1993) Markov Stopping Games with Random Priority. *Zeitschrift für Operations Research*, **3**, 69–84.

YASUDA, M. (1985) On Randomized Strategy in Neveu's Stopping Problem. *Stochastic Processes and their Applications*, **21**, 159–166.