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Upper bounds on the integrated tail of the reliability function, with applications

by

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Abstract: Simple upper bounds for the integrated tail of the reliability function for the following classes of life distributions: IFR, IFRA, DMRL, NBU, NBUE and HNBUE are presented. These bounds are very useful for calculation of the mean time to failure of an item with prescribed accuracy, and to obtain refined upper bounds on the mean residual life function.

Keywords: reliability, upper bounds, ageing classes, mean time to failure.

1. Introduction

Let τ be a lifetime, i.e., a non-negative random variable with right-continuous distribution function $F(t)$, and reliability (survival) function $\overline{F}(t) = 1 - F(t)$, with $F(0) = 0$. If the distribution F is absolutely continuous, then its failure density and failure rate (hazard rate) functions will be denoted by f and $h = f/\overline{F}$, respectively. We suppose that $\mu = E[\tau]$ is finite, but its value is unknown in advance. Let $V(t) = \int_t^\infty \overline{F}(x)\,\mathrm{d}x$ be the integrated tail of the reliability function $\overline{F}(x)$, also known as the integrated distribution function, Bernhard (2000). The function $V(t)$ is of importance in reliability applications because the following expressions hold true:

$$\mu = \int_0^\infty \overline{F}(x)\,\mathrm{d}x = \int_0^t \overline{F}(x)\,\mathrm{d}x + V(t) \tag{1}$$

$$m(t) = \begin{cases} \dfrac{V(t)}{\overline{F}(t)}, & \text{if } \overline{F}(t) > 0 \\ 0, & \text{if } \overline{F}(t) = 0 \end{cases} \tag{2}$$

where $m(t) = E[\tau - t \mid \tau > t]$ is the mean residual life (MRL) at time t, Hall and Wellner (1981).

Due to the unbounded interval of integration, it is not easy to calculate

Therefore, it is of interest to have some simple upper bounds, say $B(t)$, for $V(t)$. It is clear that in general case it is impossible to find an upper bound for $V(t)$ based on the knowledge of its values at finite number of points $t_1, t_2, \ldots, t_n$, $n > 0$, or on its behaviour on a bounded interval only. This is because the tail of a distribution (with finite mean) can be arbitrarily long. Therefore, we restrict our consideration to some classes of life distributions, based on the notion of ageing: IFR, IFRA, DMRL, NBU, NBUE and HNBUE, see Abouammoh and Qamber (1994), Barlow and Proschan (1975), Klefsjø (1982), Marshall and Proschan (1972).

In this paper, we derive several simple upper bounds on $V(t)$ for these classes of life distributions. Relations between the bounds are discussed, and some refinements and generalizations are proposed as well. In view of equations (1) and (2), the bounds obtained may be used:

- to calculate the mean time to failure, μ, of an item, with prescribed accuracy, Korczak (1999);
- to improve general bounds on the MRL, $m(t)$, reported in Hall and Wellner (1981).

As an application of the results, an example of numerical computation of the mean time to failure of a system is presented.

2. Basic definitions and assumptions

Throughout the paper we use the term "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing".

A life distribution F and its reliability function $\overline{F}$ with $S = \{t : \overline{F}(t) > 0\}$ are said to be (or to have)

1. Increasing Failure Rate (IFR) if $t \mapsto \overline{F}(t+x)/\overline{F}(t)$ is a decreasing function of the age t whenever $x > 0$ and $t \in S$. If F is absolutely continuous, then the IFR property is equivalent to the increasing failure rate $h(t)$.
2. Decreasing Mean Residual Life (DMRL) if $\mu < \infty$ and $t \mapsto m(t)$ is a decreasing function of the age $t \in S$.
3. Increasing Failure Rate Average (IFRA) if $t \mapsto \ln(\overline{F}(t))/t$ is increasing on $S - \{0\}$, i.e., if $t \mapsto (\overline{F}(t))^{1/t}$ is decreasing on $(0, \infty)$.
4. New Better than Used (NBU) if $\overline{F}(x)\overline{F}(y) \geq \overline{F}(x+y)$ for $x, y \geq 0$.
5. New Better than Used in Expectation (NBUE) if $\mu < \infty$ and $V(t) \leq \mu\overline{F}(t)$ for $t \geq 0$.
6. Harmonic New Better than Used in Expectation (HNBUE) if $\mu < \infty$ and $V(t) \leq \mu \exp(-t/\mu)$ for $t \geq 0$.

The relations between the classes are as follows:

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE} \Rightarrow \text{HNBUE}$$

If $\mu = E[\tau] = \infty$, then $V(t) = \infty$ for all t. On the other hand, if $\mu < \infty$ and its value is known, $V(t)$ may be easily calculated by integrating $\overline{F}$ numerically over $[0, t]$ (see (1)), and in this case the upper bounds for $V(t)$ seem to be rather not needed.

Let $b_F = \sup\{t \geq 0 : \overline{F}(t) > 0\}$. If $b_F < \infty$ (i.e., $Pr\{\tau \leq b_F\} = 1$), then $\mu = V(b_F)$. Furthermore, in this case we may apply (if it is at all necessary) the following trivial upper bound:

$$V(t) \leq \overline{F}(t)(b_F - t), \quad t < b_F. \tag{3}$$

In order to exclude the cases mentioned above, we assume throughout that:

ASSUMPTION 2.1 *$\mu = E[\tau] < \infty$ and the exact value of μ is not known in advance.*

ASSUMPTION 2.2 *$\overline{F}(t) > 0$ for $t \geq 0$, i.e., $b_F = \infty$.*

Note that these assumptions also exclude distributions degenerated at 0, and at ∞.

3. Main results

We start with two fairly general bounds.

PROPOSITION 3.1 (General bound 1) *Let the failure rate h exist on $[x, \infty)$, $x \geq 0$. If $h(t) \geq \alpha$ for $t \geq x$, where $\alpha > 0$, then for any $t \geq x$:*

$$V(t) \leq \overline{F}(t)/\alpha \overset{\text{symb.}}{\equiv} B_{\text{GEN1}}(t). \tag{4}$$

Proof. The inequality $\alpha \leq h(u)$ is equivalent to $\overline{F}(u) \leq f(u)/\alpha$, from which (4) follows by integration over $[t, \infty)$. ■

PROPOSITION 3.2 (General bound 2) *Let $\overline{F}$ have the following property: $\alpha t \leq -\ln(\overline{F}(t))$ for $t \geq x$, where $\alpha > 0$ and $x \geq 0$. Then, for any $t \geq x$:*

$$V(t) \leq \exp(-\alpha t)/\alpha \overset{\text{symb.}}{\equiv} B_{\text{GEN2}}(t). \tag{5}$$

Proof. The inequality $\alpha u \leq -\ln(\overline{F}(u))$ is equivalent to $\overline{F}(u) \leq \exp(-\alpha u)$, from which (5) follows. ■

These bounds may be applied to lifetimes with decreasing failure rate or decreasing failure rate average. However, they require some knowledge of the behaviour of the failure rate function $h(t)$ or cumulative failure rate function $H(t) = -\ln(\overline{F}(t))$. Furthermore, their convergence to 0 may be very slow. More useful and better bounds may be obtained within some classes of life distributions based on notions of ageing: IFR, IFRA, DMRL, NBU, NBUE and HNBUE. First, we consider the basic (and the simplest) bounds. Some

Proposition 3.3 (IFR bound) *If F is IFR with the failure rate h, then for $h(t) > 0$:*

$$V(t) \leq \overline{F}(t)/h(t) \overset{\text{symb.}}{\equiv} B_{\text{IFR}}(t). \tag{6}$$

The bound is sharp and decreases to 0 as $t \to \infty$.

Proof. Since h is increasing on $[0, \infty)$, $h(s) = f(s)/\overline{F}(s) \geq h(t)$, i.e., $\overline{F}(s) \leq f(s)h(t)$ for $s \geq t$, from which (6) follows by integrating over $[t, \infty)$. The bound is attained by an exponential distribution, hence "$\leq$" cannot be replaced by "$<$". Its monotone convergence to 0 is obvious. ∎

Proposition 3.4 (DMRL bounds) *Let F be DMRL.*

(a) *If $0 < x \leq t$ and $\overline{F}(t-x) - \overline{F}(t) > 0$, then*

$$V(t) \leq \frac{\overline{F}(t)}{\overline{F}(t-x) - \overline{F}(t)} \int_{t-x}^{t} \overline{F}(u)\mathrm{d}u \overset{\text{symb.}}{\equiv} B_{1,\text{DMRL}}(t, x). \tag{7}$$

The bound is sharp.

(b) *If $x > 0$ and $\overline{F}(t) - \overline{F}(t+x) > 0$, then*

$$V(t) \leq \frac{\overline{F}(t)}{\overline{F}(t) - \overline{F}(t+x)} \int_{t}^{t+x} \overline{F}(u)\mathrm{d}u \overset{\text{symb.}}{\equiv} B_{2,\text{DMRL}}(t, x) \tag{8}$$

The bound is sharp.

If $\limsup_{t\to\infty} \overline{F}(t+x)/\overline{F}(t) < 1$, then both bounds (7) and (8) tend to 0 as $t \to \infty$. In particular, if F is IFR, then the convergence to 0 of these bounds is monotone.

(c) *If F is absolutely continuous and $h(t) > 0$, then:*

$$V(t) \leq B_{\text{IFR}}(t). \tag{9}$$

The bound is sharp. It tends to 0 as $t \to \infty$, provided that $\liminf_{t\to\infty} h(t) > 0$.

Proof. From definition of DMRL it follows that

$$\int_{t_2}^{\infty} \overline{F}(u)\,\mathrm{d}u \leq \frac{\overline{F}(t_2)}{\overline{F}(t_1) - \overline{F}(t_2)} \int_{t_1}^{t_2} \overline{F}(u)\,\mathrm{d}u, \; 0 \leq t_1 < t_2.$$

Substituting $t_1 = t - x$ and $t_2 = t$, $0 < x \leq t$, gives (7). The proof of (8) is similar. If $\limsup_{t\to\infty} \overline{F}(t+x)/\overline{F}(t) < 1$, then the multipliers before the integral sign in (7) and (8) are both bounded for t large enough, and hence the bounds tend to 0. If F is IFR, then both $\overline{F}(t)/(\overline{F}(t-x)-\overline{F}(t))$ and $\overline{F}(t)/(\overline{F}(t)-\overline{F}(t+x))$ are decreasing in t for any $x > 0$, see Barlow and Proschan (1965). Hence the bounds also decrease to 0 as $t \to \infty$ in this case. Now observe that (7) and the absolute continuity of F imply that

$$V(t) \leq \lim_{x\to 0} \frac{\overline{F}(t)x}{\overline{F}(t) - \overline{F}(t+x)} \frac{1}{x} \int_{t}^{t+x} \overline{F}(u)\,\mathrm{d}u = \frac{\overline{F}(t)}{h(t)},$$

Since $\int_a^b \overline{F}(u)\,\mathrm{d}u \le \overline{F}(a)(b-a)$, $0 \le a < b$, we have:

COROLLARY 3.1 (Simplified DMRL bounds) *Let F be DMRL.*

(a) *If $0 < x \le t$ and $\overline{F}(t-x) - \overline{F}(t) > 0$, then*

$$V(t) \le \frac{\overline{F}(t-x)\overline{F}(t)x}{\overline{F}(t-x) - \overline{F}(t)} \stackrel{\text{symb.}}{\equiv} B_{1\mathrm{S,DMRL}}(t,x). \tag{10}$$

(b) *If $x > 0$ and $\overline{F}(t) - \overline{F}(t+x) > 0$, then*

$$V(t) \le \frac{[\overline{F}(t)]^2 x}{\overline{F}(t) - \overline{F}(t+x)} \stackrel{\text{symb.}}{\equiv} B_{2\mathrm{S,DMRL}}(t,x) \tag{11}$$

If $\limsup_{t\to\infty} \overline{F}(t+x)/\overline{F}(t) < 1$, then both bounds (10) and (11) tend to 0 as $t \to \infty$. In particular, if F is IFR, then the convergence to 0 of these bounds is monotone. ■

PROPOSITION 3.5 (IFRA bound) *If F is IFRA and $\overline{F}(t) < 1$, then:*

$$V(t) \le \frac{-t\overline{F}(t)}{\ln(\overline{F}(t))} \stackrel{\text{symb.}}{\equiv} B_{\mathrm{IFRA}}(t). \tag{12}$$

The bound is sharp and decreases to 0 as $t \to \infty$.

Proof. As F is IFRA, $x \mapsto (\overline{F}(x))^{1/x}$ is decreasing on $(0,\infty)$. Since $t > 0$ (because $\overline{F}(t) < 1$), $(\overline{F}(x))^{1/x} \le (\overline{F}(t))^{1/t}$, i.e., $\overline{F}(x) \le [(\overline{F}(t))^{1/t}]^x$ for $x \ge t$, from which (12) follows. Its monotone convergence to 0 follows from the IFRA defining property. ■

PROPOSITION 3.6 (NBU bound) *If F is NBU and $\overline{F}(t) < 1$, then:*

$$V(t) \le \frac{1}{1-\overline{F}(t)} \int_t^{2t} \overline{F}(u)\,\mathrm{d}u \stackrel{\text{symb.}}{\equiv} B_{1,\mathrm{NBU}}(t). \tag{13}$$

The bound is sharp and tends to 0 as $t \to \infty$. The convergence is monotone for $t \ge F^{-1}(0.5) = \inf\{x : F(x) \ge 0.5\}$.

Proof. Integration of the NBU defining inequality, $\overline{F}(t+u) \le \overline{F}(t)\overline{F}(u)$, with respect to u in the interval (t,∞), yields:

$$\int_{2t}^{\infty} \overline{F}(u)\,\mathrm{d}u \le \overline{F}(t) \int_t^{\infty} \overline{F}(u)\,\mathrm{d}u. \tag{14}$$

This inequality can be written as:

$$\int_t^{\infty} \overline{F}(u)\,\mathrm{d}u \int_t^{2t} \overline{F}(u)\,\mathrm{d}u \le \overline{F}(t) \int_t^{\infty} \overline{F}(u)\,\mathrm{d}u, \tag{15}$$

from which (13) follows. Its convergence to 0 is evident. Now observe that $t \mapsto \int_t^{2t} \overline{F}(u)\,\mathrm{d}u$ is decreasing on $[F^{-1}(0.5),\infty)$ iff $2\overline{F}(2t) \le \overline{F}(t)$ for $t \ge F^{-1}(0.5)$.

PROPOSITION 3.7 (NBUE bound) *If F is NBUE and $\overline{F}(t) < 1$, then:*

$$V(t) \le \frac{\overline{F}(t)}{1-\overline{F}(t)} \int_0^t \overline{F}(u)\,\mathrm{d}u \overset{\text{symb.}}{\equiv} B_{\mathrm{NBUE}}(t). \tag{16}$$

The bound is sharp and tends to 0 as $t \to \infty$.

Proof. Inequality (16) is a direct consequence of the NBUE defining property. Since $\mu < \infty$ and $\overline{F}(t) \to 0$ as $t \to \infty$, the bound tends to 0. ■

COROLLARY 3.2 (Simplified NBUE bound) *If F is NBUE and $\overline{F}(t) < 1$, then:*

$$V(t) \le \frac{t\overline{F}(t)}{1-\overline{F}(t)} \overset{\text{symb.}}{\equiv} B_{\mathrm{S,NBUE}}(t). \tag{17}$$

The bound tends to 0 as $t \to \infty$. ■

Observe that (17) can also be obtained from (13) by replacing the integral by its upper bound, $t\overline{F}(t)$. The next proposition gives another NBU bound, which is a result of both NBU and NBUE properties.

PROPOSITION 3.8 (NBU bound) *If F is NBU and $\overline{F}(2t) < 1$, then:*

$$V(t) \le \frac{\overline{F}(t)+\overline{F}(2t)}{1-\overline{F}(2t)} \int_0^t \overline{F}(u)\,\mathrm{d}u \overset{\text{symb.}}{\equiv} B_{2,\mathrm{NBU}}(t). \tag{18}$$

The bound is sharp and converges to 0 as $t \to \infty$.

Proof. We have:

$$\int_t^\infty \overline{F}(u)\,\mathrm{d}u = \int_t^{2t} \overline{F}(u)\,\mathrm{d}u + \int_{2t}^\infty \overline{F}(u)\,\mathrm{d}u. \tag{19}$$

By integrating the NBU defining inequality, $\overline{F}(u+t) \le \overline{F}(u)\overline{F}(t)$, with respect to u in the interval $(0,t)$, we obtain:

$$\int_t^{2t} \overline{F}(u)\,\mathrm{d}u \le \overline{F}(t) \int_0^t \overline{F}(u)\,\mathrm{d}u. \tag{20}$$

Application of (20) and NBUE bound, $B_{\mathrm{NBUE}}(2t)$, to (19) yields:

$$\int_t^\infty \overline{F}(u)\,\mathrm{d}u \le \overline{F}(t) \int_0^t \overline{F}(u)\,\mathrm{d}u + \frac{\overline{F}(2t)}{1-\overline{F}(2t)} \int_0^{2t} \overline{F}(u)\,\mathrm{d}u. \tag{21}$$

Using (20) once again, we obtain:

$$\int_0^{2t} \overline{F}(u)\,\mathrm{d}u = \int_0^t \overline{F}(u)\,\mathrm{d}u + \int_t^{2t} \overline{F}(u)\,\mathrm{d}u$$

Hence:

$$\int_t^\infty \overline{F}(u)\,\mathrm{d}u \le \overline{F}(t)\int_0^t \overline{F}(u)\,\mathrm{d}u + \frac{\overline{F}(2t)(1+\overline{F}(t))}{1-\overline{F}(2t)}\int_0^t \overline{F}(u)\,\mathrm{d}u. \qquad (23)$$

Then, a simple algebra gives (18). Since $\mu < \infty$ and $\overline{F}(t) \to 0$ as $t \to \infty$, the bound tends to 0. ■

COROLLARY 3.3 (Simplified NBU bound) *If F is NBU and $\overline{F}(2t) < 1$, then:*

$$V(t) \le \frac{t(\overline{F}(t)+\overline{F}(2t))}{1-\overline{F}(2t)} \overset{\text{symb.}}{\equiv} B_{2\mathrm{S,NBU}}(t). \qquad (24)$$

The bound tends to 0 as $t \to \infty$. ■

For HNBUE class of lifetime distributions one can obtain the following bound, which is rather poor, and is given for completeness only:

PROPOSITION 3.9 (HNBUE bound) *Let F be HNBUE with $\mu > 0$. If $t > 0$ and $\overline{F}(t-) < 1$, then:*

$$V(t) < \frac{(\int_0^t \overline{F}(u)\,\mathrm{d}u)^2}{t-\int_0^t \overline{F}(u)\,\mathrm{d}u} \overset{\text{symb.}}{\equiv} B_{\mathrm{HNBUE}}(t). \qquad (25)$$

The bound converges to 0 as $t \to \infty$.

Proof. From the HNBUE defining property and the inequality, $\exp(-x) < 1/(1+x)$, $x > 0$, it follows that $V(t) \le \mu\exp(-t/\mu) < \mu^2/(t+\mu)$ for $t > 0$. Then, after some algebra, we obtain (25). ■

It is easy to see that the following relations hold between the bounds introduced above:

1. Suppose that F is absolutely continuous. Then $B_{\mathrm{IFR}}(t) \le B_{\mathrm{IFRA}}(t) \Leftrightarrow F \in \{\mathrm{IFRA}\}$. However, if $F \in \{\mathrm{IFRA}\} - \{\mathrm{IFR}\}$, then $B_{\mathrm{IFR}}(t)$ may not be an upper bound on $V(t)$ for some values of t.

2. Let F be IFR. Then $B_{2,\mathrm{DMRL}}(t,x) \le B_{\mathrm{IFR}}(t)$ for any $t \ge 0$ and $x > 0$, and $B_{\mathrm{IFR}}(t) \le B_{1,\mathrm{DMRL}}(t,x)$ for any $0 < x \le t$.

3. If F is IFRA, then $B_{\mathrm{IFRA}}(t) \le B_{\mathrm{NBUE}}(t)$ for any t. However, if $F \in \{\mathrm{NBUE}\} - \{\mathrm{IFRA}\}$, then $B_{\mathrm{IFRA}}(t)$ may not be an upper bound on $V(t)$ for some values of t.

4. If F is IFRA, then $B_{1,\mathrm{NBU}}(t) \le B_{\mathrm{IFRA}}(t)$ for any t.

5. If F is NBU, then $B_{1,\mathrm{NBU}}(t) \le B_{\mathrm{NBUE}}(t)$ and $B_{2,\mathrm{NBU}}(t) \le B_{\mathrm{NBUE}}(t)$ for any t.

6. For any F, $B_{\mathrm{IFRA}}(t) \le B_{\mathrm{S,NBUE}}(t)$.

The bounds introduced so far may be improved and generalized in many ways. We can use the "brute force" improvement consisting of replacing a bound, say $B_{\mathrm{X}}(t)$, by:

$$B_{\mathrm{bf,X}}(t;s) = \int_t^s \overline{F}(u)\,\mathrm{d}u + B_{\mathrm{X}}(s),\ t < s.$$

Furthermore, the simplified bounds may be improved by applying higher order upper Darboux sum for corresponding integral, e.g., through replacing the integral $\int_a^b \overline{F}(u)\,\mathrm{d}u$ by the sum $(\sum_{k=1}^{n-1} \overline{F}(a+kh))/h$, where $n > 1$ and $h = (b-a)/n$. At last, some bounds may be generalized by considering multiple time-points $t = t_0 < t_1 < t_2 < \ldots < t_n < \infty$, $n \geq 0$. Some such generalizations are presented below for information only. The proofs are omitted, as they are similar to those of corresponding basic bounds.

PROPOSITION 3.10 (Generalized IFR bound) *Let F be IFR with the failure rate h. Let $t = t_0 < t_1 < \ldots < t_n < \infty$, $n \geq 0$. If $h(t) > 0$, then:*

$$V(t) \leq \frac{\overline{F}(t_n)}{h(t_n)} + \sum_{k=1}^{n} \frac{\overline{F}(t_{k-1}) - \overline{F}(t_k)}{h(t_{k-1})} \overset{\text{symb.}}{\equiv} B_{\mathrm{IFR}}^{(n)}(t, t_1, \ldots, t_n).$$

The bound is sharp. Furthermore,

$$\begin{aligned}
&B_{\mathrm{IFR}}^{(0)}(t) = B_{\mathrm{IFR}}(t),\\
&B_{\mathrm{IFR}}^{(n+1)}(t, t_1, \ldots, t_n, t_{n+1}) \leq B_{\mathrm{IFR}}^{(n)}(t, t_1, \ldots, t_n),\ t_{n+1} > t_n,\ n \geq 0,\\
&\lim_{t\to\infty} B_{\mathrm{IFR}}^{(n)}(t, t_1, \ldots, t_n) = \lim_{x\to\infty} B_{\mathrm{IFR}}^{(n)}(t+x, t_1+x, \ldots, t_n+x) = 0. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 3.11 (Generalized IFRA bound) *Let F be IFRA and $t = t_0 < t_1 < \ldots < t_n < \infty$, $n \geq 0$. If $\overline{F}(t) < 1$, then:*

$$V(t) \leq \sum_{k=1}^{n-1} \frac{t_k \overline{F}(t_k)\left(1 - (\overline{F}(t_k))^{\frac{t_{k+1}-t_k}{t_k}}\right)}{-\ln(\overline{F}(t_k))} + \frac{t_n \overline{F}(t_n)}{-\ln(\overline{F}(t_n))}$$
$$\overset{\text{symb.}}{\equiv} B_{\mathrm{IFRA}}^{(n)}(t, t_1, \ldots, t_n).$$

The bound is sharp. Furthermore,

$$\begin{aligned}
&B_{\mathrm{IFRA}}^{(0)}(t) = B_{\mathrm{IFRA}}(t),\\
&B_{\mathrm{IFRA}}^{(n+1)}(t, t_1, \ldots, t_n, t_{n+1}) \leq B_{\mathrm{IFRA}}^{(n)}(t, t_1, \ldots, t_n),\ t_{n+1} > t_n,\ n \geq 0,\\
&\lim_{t\to\infty} B_{\mathrm{IFRA}}^{(n)}(t, t_1, \ldots, t_n) = \lim_{x\to\infty} B_{\mathrm{IFRA}}^{(n)}(t+x, t_1+x, \ldots, t_n+x) = 0. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 3.12 (Generalized NBU bound) *If F is NBU and $\overline{F}(nt) < 1$, then for any integer $n \geq 1$:*

$$V(t) \leq \frac{(\int_0^t \overline{F}(u)\,\mathrm{d}u) \sum_{k=1}^{n} \overline{F}(kt)}{\ldots} \overset{\text{symb.}}{\equiv} B^{(n)} (t)$$

The bound is sharp. Increase of n above 2 does not necessarily improve the bound. However, $B_{\mathrm{NBU}}^{(n)}(t) \leq B_{\mathrm{NBUE}}(t)$ *for any* $n \geq 1$*, and hence the bound tends to* 0 *as* $t \to \infty$*. Of course,* $B_{\mathrm{NBU}}^{(1)}(t) \equiv B_{\mathrm{NBUE}}(t)$*.* ■

4. Example of application

The mean time to failure, μ, is an important reliability index for non-repaired items, and is given by the well known expression (1). The explicit and easily computed expressions for μ are available for few probability distribution laws only, Birolini (1994), and for coherent non-repaired systems composed of independent exponential components, Gaede (1977), Karpiński and Korczak (1990). For the latter case, the expressions are based on Poincaré's or a polynomial form of the system reliability structure, and may also be generalized for the case when the component lifetimes are Weilbull distributed with the same shape parameter. However, this approach cannot be applied for the general case, e.g. when the shape parameters of the components are different. Numerical computation of such kind of integral using numerical integration may be difficult. Integration by change of variable, e.g. $x = 1/t - 1$, requires individual consideration for each particular case, and in some cases may be inefficient, Lastman and Sinha (1989). Therefore, in practice, the mean time to failure is approximated by a finite limit integral:

$$\mu(0,t) = \int_0^t \overline{F}(x)\,\mathrm{d}x,$$

where t should be chosen so that the error caused by neglecting the "tail", $V(t)$, is small, for example, less than $\varepsilon = 0.01$.

Here we show how to use the upper bounds on $V(t)$ in practice to compute the mean time to failure with prescribed accuracy. Let us consider an item (e.g. a coherent system of independent components). To compute the mean time to failure, μ, of the item, with a given accuracy $\varepsilon > 0$ we proceed as follows:

1. Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\varepsilon_1 + \varepsilon_1 = \varepsilon$. Usually, we take $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$.
2. Select appropriate upper bound, say B, on V.
3. Choose a value t such that $B(t) < \varepsilon_2$.
4. Compute the proper integral $\mu(0,t)$ from one of the quadrature formulae to an accuracy of ε_1. For example, Romberg's quadrature may be used, see Lastman and Sinha (1989), Marciniak et al. (1992).

Let $\mu^{\approx}(0,t)$ be an approximate value of this integral, i.e.,

$$|\mu(0,t) - \mu^{\approx}(0,t)| < \varepsilon_1.$$

Since $B(t) < \varepsilon_2$,

A more precise estimation is:

$$\mu^{\approx}(0,t) - \varepsilon_A < \mu < \mu^{\approx}(0,t) + \varepsilon_A + B(t),$$

where ε_A is actual error of numerical integration.

Let us now consider a coherent system of independent components with reliability block diagram of Fig. 1. We assume, for simplicity, that the components have the same constant failure rate $\lambda = 0.001$.

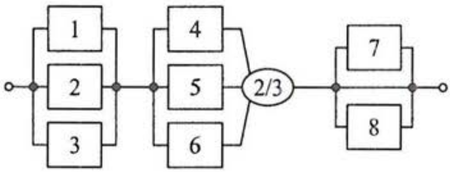

Figure 1. An example of a system

The system is a series composition of three independent subsystems (modules), each having a k-out-of-n structure. Hence, each subsystem has an IFR life, and in consequence, the system itself has an IFR life, see Barlow and Proschan (1965). Therefore IFR upper bound (6) applies. However, for comparison, we also compute some other upper bounds.

We have:

$$\overline{F}(t) = (3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda})(3e^{-2\lambda t} - 2e^{-3\lambda t})(2e^{-\lambda t} - e^{-2\lambda t})$$
$$= 18e^{-4\lambda t} - 39e^{-5\lambda t} + 33e^{-6\lambda t} - 13e^{-7\lambda t} + 2e^{-8\lambda t},$$
$$V(t) = \left(\frac{9}{2}e^{-4\lambda t} - \frac{39}{5}e^{-5\lambda t} + \frac{11}{2}e^{-6\lambda t} - \frac{13}{7}e^{-7\lambda t} + \frac{1}{4}e^{-8\lambda t}\right)/\lambda,$$
$$\mu = \frac{83}{140\lambda} \approx 592.857.$$

The results of computation of the upper bounds on $V(t)$ for some values of t are given in Table 1.

t	$V(t)$	$B_{\mathrm{IFR}}(t)$	$B_{\mathrm{IFRA}}(t)$	$B_{\mathrm{NBUE}}(t)$	$B_{\mathrm{S,NBUE}}(t)$
500	191.073	238.697	401.014	438.345	545.497
1000	41.888	45.241	69.313	87.848	159.443
1500	7.469	7.738	11.093	16.104	41.266
2000	1.188	1.210	1.648	2.651	8.960
2500	0.177	0.179	0.234	0.404	1.705
3000	0.025	0.025	0.032	0.059	0.298
3500	0.004	0.004	0.004	0.008	0.049

One can see that the bounds converge quickly to 0 and become close to true values of the "tail". Suppose that we want to compute the mean time to failure, μ, with an error of at most $\varepsilon = 0.1$. We choose $\varepsilon_1 = \varepsilon_2 = 0.05$. From Table 1 we see that $B_{\mathrm{IFR}}(3000) = 0.025 < 0.05$. Hence $t = 3000$ suffices for our purpose. Integrating $\overline{F}(u)$ over $[0, t]$ numerically gives:

$$\mu^{\approx}(0, t) = 592.83,$$

hence

$$592.78 < \mu < 592.91.$$

Actually,

$$|\mu - \mu^{\approx}(0, t)| \approx 0.03 < 0.1.$$

5. Conclusions

The paper gives some simple upper bounds on the integrated tail, V, of the reliability function, $\overline{F}$. These bounds allow the mean time to failure to be computed with prescribed accuracy, by replacing improper integral, $\int_0^\infty \overline{F}(x)\,\mathrm{d}x$ with the proper one, $\int_0^t \overline{F}(x)\,\mathrm{d}x$. A numerical example shows that the proposed upper bounds on V may be applied to fairly complex systems. In fact, these bounds have been used successfully in the Telecommunications Research Institute for numerical computation of the mean time to failure of complex electronic systems.

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