Hence

$$
y_{ni\alpha} - y_{ni\alpha'} = h_n \sum_{j=1}^{i} [f(t_{n,j-1}, y_{n,j-1,\alpha}, r_{nj\alpha}) - f(t_{n,j-1}, y_{n,j-1,\alpha'}, r_{nj\alpha'})]
$$

$$
+ \frac{h_n}{2} [f(t_{ni}, y_{ni\alpha}, r_{ni\alpha}) - f(t_{ni}, y_{ni\alpha'}, r_{ni\alpha'})]
$$

$$
- \frac{h_n}{2} [f(t_{n0}, y_{n0\alpha}, r_{n0\alpha}) - f(t_{n0}, y_{n0\alpha'}, r_{n0\alpha'})]
$$

$$
= h_n \sum_{j=1}^{i} [f(t_{n,j-1}, y_{n,j-1,\alpha}, r_{nj\alpha}) - f(t_{n,j-1}, y_{n,j-1,\alpha'}, r_{nj\alpha})]
$$

$$
+ h_n \sum_{j=1}^{i} [f(t_{n,j-1}, y_{n,j-1,\alpha'}, r_{nj\alpha}) - f(t_{n,j-1}, y_{n,j-1,\alpha'}, r_{nj\alpha'})]
$$

$$
+ \frac{h_n}{2} [f(t_{ni}, y_{ni\alpha}, r_{ni\alpha}) - f(t_{ni}, y_{ni\alpha'}, r_{ni\alpha})]
$$

$$
- \frac{h_n}{2} [f(t_{n0}, y_{n0\alpha}, r_{n0\alpha}) - f(t_{n0}, y_{n0\alpha'}, r_{n0\alpha})]
$$

$$
+ \frac{h_n}{2} [f(t_{ni}, y_{ni\alpha'}, r_{ni\alpha}) - f(t_{ni}, y_{ni\alpha'}, r_{ni\alpha'})]
$$

$$
- \frac{h_n}{2} [f(t_{n0}, y_{n0\alpha'}, r_{n0\alpha}) - f(t_{n0}, y_{n0\alpha'}, r_{n0\alpha'})].
$$

It follows that

$$
\left(1 - \frac{h_n L}{2}\right) \|y_{ni\alpha} - y_{ni\alpha'}\| \leq h_n L \sum_{j=1}^{i} \|y_{n,j-1,\alpha} - y_{n,j-1,\alpha'}\|
$$

$$
+ 2MT|\alpha - \alpha'| + 2h_n M|\alpha - \alpha'|.
$$

Since $h_n < 2/L$, it then follows from the discrete Bellman–Gronwall inequality that

$$
\|y_{ni\alpha} - y_{ni\alpha'}\| \leq c|\alpha - \alpha'|, \quad i = 0, \ldots, N.
$$

The Lipschitz equicontinuity of $G_n(r_{n\alpha})$ follows as in Theorem 3.3. ∎

The following theorem gives error estimates relative to a given discrete re-laxed control:

THEOREM 3.5 *For a given discrete relaxed control* $r_n \in R_n$, *let* y_n, $\widehat{y}_n$ *be the corresponding discrete states and* $\widetilde{y}_n$ *the corresponding solution of the continuous state equation. If the function* f *is Lipschitz continuous w.r.t.* (t, y) *on* D, *then* (c *denoting various constants*)

$$
\|\widehat{y}_n - \widetilde{y}_n\|_\infty < ch_n,
$$

and if f is C^2 w.r.t. (t, y) on D, with partial derivatives continuous on D, then

$$\|\widehat{y}_n - \widetilde{y}_n\|_\infty \le ch_n^2,$$
$$|G_n(r_n) - G(r_n)| \le ch_n^2.$$

Moreover, if $n' > n$ and $N_n \ne N_{n'}$

$$|G_n(r_n) - G_{n'}(r_n)| \le ch_n \ (or \ ch_n^2).$$

Proof. We write

$$\widehat{y}_n(t) = y^0 + \int_0^t f(s, \widehat{y}_n(s), r_n(s))\, ds + \int_0^{t'} \alpha_n(s)\, ds + \int_{t'}^t \alpha_n(s)\, ds,$$

where $t' := \max_{t_{ni} \le t} t_{ni}$ and $\alpha_n(t)$ was defined in the proof of Theorem 3.2. If f is Lipschitz continuous w.r.t. (t, y) with constant L'', we have

$$\left\| \int_0^{t'} \alpha_n(s)\, ds \right\| \le L''(M + 1)Th_n,$$

and if f is C^2 w.r.t. (t, y) on D, using the error estimate of the trapezoidal rule

$$\left\| \int_0^{t'} \alpha_n(s)\, ds \right\| \le cTh_n^2.$$

In both cases, we have

$$\left\| \int_{t'}^t \alpha_n(s)\, ds \right\| \le L''(M + 1)h_n^2.$$

Consequently

$$\|\widehat{y}_n(t) - \widetilde{y}_n(t)\| \le \int_0^t \|f(s, \widehat{y}_n, r_n) - f(s, \widetilde{y}_n, r_n)\|\, ds + ch_n \ (or \ ch_n^2)$$

$$\le L \int_0^t \|\widehat{y}_n(s) - \widetilde{y}_n(s)\|\, ds + ch_n \ (or \ ch_n^2).$$

By Gronwall's inequality

$$\|\widehat{y}_n(t) - \widetilde{y}_n(t)\| \le ch_n \ (or \ ch_n^2) \ in \ I.$$

It follows that

$$|G_n(r_n) - G(r_n)| = |g(y_{nN}) - g(\widetilde{y}_n(T))| \le ch_n \ (or \ ch_n^2).$$

Finally, let $n' > n$ with $N_n \ne N_{n'}$ and let $\widetilde{y}_{n'}$ denote the continuous state corresponding to the discrete control r_n considered as an element of $R_{n'}$. Then $\widetilde{y}_n = \widetilde{y}_{n'}$ and

$$|G_n(r_n) - G_{n'}(r_n)| = |g(y_{nN_n}) - g(y_{n'N_{n'}})|$$

$$\le |g(y_{nN_n}) - g(\widetilde{y}_n(T))| + |g(\widetilde{y}_{n'}(T)) - g(y_{n'N_{n'}})|$$

Next we define, for given $r_n \in R_n$, with corresponding discrete state y_n, the approximate discrete adjoint state z_n as the unique solution (which clearly exists for h_n sufficiently small) of the following linear implicit trapezoidal scheme

$$z_{n,i-1} = z_{ni} + \frac{h_n}{2}[z_{ni}f'_y(t_{ni}, y_{ni}, r_{ni}) + z_{n,i-1}f'_y(t_{n,i-1}, y_{n,i-1}, r_{ni})],$$
$$i = N, \ldots, 1,$$
$$z_{nN} = g'_y(y_{nN}),$$

and for given $r_n, \overline{r}_n \in R_n$, with y_n, z_n corresponding to r_n, the approximate discrete directional derivative of G by (with obvious notations)

$$D_n G(r_n, \overline{r}_n - r_n)$$
$$:= \frac{h_n}{2} \sum_{i=1}^{N} [z_{ni}f(t_{ni}, y_{ni}, \overline{r}_{ni} - r_{ni}) + z_{n,i-1}f(t_{n,i-1}, y_{n,i-1}, \overline{r}_{ni} - r_{ni})]$$
$$= \frac{1}{2} \int_0^T [z_n^+(t)f(t_n^+(t), y_n^+(t), \overline{r}_n(t) - r_n(t))$$
$$+ z_n^-(t)f(t_n^-(t), y_n^-(t), \overline{r}_n(t) - r_n(t))] \, dt.$$

We suppose that h_1 is chosen sufficiently small so that the approximate discrete adjoint scheme has a solution for every n and $r_n \in R_n$.

THEOREM 3.6 Let $(r_n \in R_n)$, $(\overline{r}_n \in R_n)$ be sequences of discrete relaxed controls such that $r_n \to r$ and $\overline{r}_n \to \overline{r}$ in R as $n \to \infty$. Then $\widehat{z}_n \to z$, $z_n^- \to z$, $z_n^+ \to z$, with $z = z_r$, uniformly, and $D_n G(r_n, \overline{r}_n - r_n) \to DG(r, \overline{r} - r)$, as $n \to \infty$. Moreover, we have (for the two smoothness cases of Theorem 3.5)

$$\|\widehat{z}_n - \widetilde{z}_n\|_\infty \le ch_n \ (or \ ch_n^2),$$

where $\widetilde{z}_n$ denotes the solution of the continuous adjoint equation corresponding to r_n and $\widetilde{y}_n$, and

$$|D_n G(r_n, \overline{r}_n - r_n) - DG(r_n, \overline{r}_n - r_n)| \le ch_n \ (or \ ch_n^2).$$

Proof (sketch). The convergences of $\widehat{z}_n$, z_n^-, z_n^+ are proved similarly to Theorem 3.2 and using Theorem 3.2. The last convergence is straightforward, considering the second expression of $D_n G$. The last two inequalities follow easily from Theorem 3.5 and the error estimate of the trapezoidal rule. ∎

Theorems 3.5 and 3.6 show that if f is C^2 w.r.t. (t, y) in D, then, for *given* discrete controls, the correspondent discrete states and approximate adjoints and cost derivatives are approximations of order h_n^2 to the correspondent exact ones, i.e. to those corresponding to control discretization only (control parametrization). This result justifies to some extent the discrete approximations used.

The following control approximation theorem is proved in Chryssoverghi et

THEOREM 3.7 *For every* $r \in R$, *there exists a sequence* $(w_n \in W_n \subset R_n)$ *of discrete classical controls such that* $w_n \to r$ *in* R.

4. Approximate relaxed descent method

Consider the following algorithm. The implementation of this algorithm will be described later.

ALGORITHM

Step 1. Set $n := 1$ and choose $r_1 \in R_1$.

Step 2. Find $\bar{r}_n \in R_n$ such that

$$d_n := D_n G(r_n, \bar{r}_n - r_n) = \min_{\bar{r}'_n \in R_n} D_n G(r_n, \bar{r}'_n - r_n).$$

Step 3. Find $\alpha_n \in [0,1]$ such that

$$G_n(r_n + \alpha_n(\bar{r}_n - r_n)) = \min_{\alpha \in [0,1]} G_n(r_n + \alpha(\bar{r}_n - r_n)).$$

Step 4. Choose any $r'_n \in R_n$ such that

$$G_n(r'_n) \leq G_n(r_n + \alpha_n(\bar{r}_n - r_n)).$$

Set $r_{n+1} := r'_n$, $n := n + 1$ and go to Step 2.

We suppose that the function f is at least Lipschitz continuous w.r.t. (t, y) on D.

THEOREM 4.1 *Every accumulation point* $r \in R$ *(and such points always exist) of the sequence* (r_n) *generated by the Algorithm satisfies the necessary conditions for optimality*

$$DG(r, \tilde{r} - r) \geq 0, \text{ for every } \tilde{r} \in R.$$

Moreover, $d_n \to 0$ *(in Step 2) for the whole sequence* (d_n).

Proof. Since $D_n G(r_n, \bar{r}'_n - r_n)$ is clearly (linear) continuous w.r.t. $\bar{r}'_n$ on the compact set $R_n \equiv [M_1(U)]^N$ and $G_n(r_n + \alpha(\bar{r}_n - r_n))$ is continuous w.r.t. α on $[0, 1]$ (Theorem 3.4), there exist $\bar{r}_n \in R_n$ and $\alpha \in [0, 1]$ satisfying Steps 2 and 3, respectively. Since R is compact, let $(r_n)_{n \in K}$, $(\bar{r}_n)_{n \in K}$, $K \subset \{1, 2, 3, \ldots\}$, be subsequences of the sequences generated by the Algorithm that converge to $r, \bar{r} \in R$, respectively. Clearly, by Step 2, $d_n \leq 0$ for every n. By Theorem 3.6

$$d_n := D_n G(r_n, \bar{r}_n - r_n) \to d := DG(r, \bar{r} - r) \leq 0, \text{ as } n \to \infty, n \in K.$$

We shall prove that $d = 0$. Suppose, by contradiction, that $d < 0$. One can easily see that the function

defined on $[0,1]$, has the derivative

$$\Phi'(\alpha) = DG(r_n + \alpha(\overline{r}_n - r_n), \overline{r}_n - r_n).$$

By the Mean Value Theorem and the continuity of DG (Theorem 2.2), we then have, for $\alpha \in [0,1]$

$$G(r_n + \alpha(\overline{r}_n - r_n)) - G(r_n) = \alpha DG(r_n + \alpha'(\overline{r}_n - r_n), \overline{r}_n - r_n)$$
$$= \alpha(d + \varepsilon_{\alpha n}),$$

for some $\alpha' \in [0, \alpha]$, where $\varepsilon_{\alpha n} \to 0$ as $n \to \infty$, $n \in K$, and $\alpha \to 0^+$. Hence

$$G(r_n + \alpha(\overline{r}_n - r_n)) - G(r_n) \le \alpha \frac{d}{2},$$

for $\alpha \in [0, \delta]$ (with $\delta > 0$), $n \ge J$, $n \in K$. Since

$$(\alpha \mapsto G(r_n + \alpha(\overline{r}_n - r_n)) - G(r_n))$$

is a bounded sequence of equicontinuous functions (Theorem 3.3) that converge pointwise to the function $\alpha \mapsto G(r + \alpha(\overline{r} - r)) - G(r)$ (Theorem 2.1, continuity of G), the convergence is uniform on $[0,1]$. Hence

$$G(r + \alpha(\overline{r} - r)) - G(r) \le \alpha \frac{d}{2} + \zeta_n,$$

for $\alpha \in [0, \delta]$, $n \ge J$, $n \in K$, where $\zeta_n \to 0$. Similarly, by Theorems 3.4 and 3.2, the sequence

$$(\alpha \mapsto G_n(r_n + \alpha(\overline{r}_n - r_n)) - G_n(r_n))$$

also converges uniformly on $[0,1]$ to $\alpha \mapsto G(r + \alpha(\overline{r} - r)) - G(r)$. Hence

$$G_n(r_n + \alpha(\overline{r}_n - r_n)) - G_n(r_n) \le \alpha \frac{d}{2} + \theta_n,$$

for $\alpha \in [0, \delta]$, $n \ge J$, $n \in K$, where $\theta_n \to 0$. By Steps 4 and 3

$$G_n(r'_n) - G_n(r_n) \le G_n(r_n + \alpha_n(\overline{r}_n - r_n)) - G_n(r_n) \le \delta \frac{d}{2} + \theta_n \le \delta \frac{d}{3},$$

for $n \ge J'$, $n \in K$. Now, the control r_{n+1} is equal to the control $r'_n \in R_n \subset R_{n+1}$, but considered as an element of R_{n+1}. Therefore, if $h_{n+1} \le h_n/2$ (resp. $h_{n+1} = h_n$), we have (Theorem 3.5, in both smoothness cases)

$$G_{n+1}(r_{n+1}) - G_n(r_n) \le \delta \frac{d}{3} + ch_n \quad \left(\text{resp.} \le \delta \frac{d}{3}\right),$$

for $n \ge J'$, $n \in K$. Since, for *every* $n \in \mathbf{N}$, we have by Steps 4 and 3

$$G_n(r'_n) - G_n(r_n) \le G_n(r_n + \alpha_n(\overline{r}_n - r_n)) - G_n(r_n)$$
$$= \min\nolimits C \, (r_n + \alpha(\overline{r}_n - r_n)) \, - \, C \, (r_n) < 0$$

it follows similarly that if $h_{n+1} \leq h_n/2$ (resp. $h_{n+1} = h_n$), then

$$G_{n+1}(r_{n+1}) - G_n(r_n) \leq ch_n \text{ (resp. } \leq 0).$$

Hence

$$G_{n+1}(r_{n+1}) - G_1(r_1) \leq$$

$$\leq c \sum_{1 \leq k \leq n} h_k + \sum_{\substack{1 \leq k \leq n \\ k \in K}} \frac{\delta d}{3} \leq 2ch_1 + \sum_{\substack{1 \leq k \leq n \\ k \in K}} \frac{\delta d}{3} \rightarrow -\infty,$$

which contradicts the fact that the sequence $(G_n(r_n) := g(y_{nN}))$ is bounded (Theorem 3.1). Therefore $d = 0$. Since this limit is unique, we also conclude that $d_n \rightarrow 0$ for the whole sequence. Now, we have by Step 2

$$D_n G(r_n, \overline{r}'_n - r_n) \geq D_n G(r_n, \overline{r}_n - r_n) = d_n, \text{ for every } \overline{r}'_n \in R_n.$$

Let $\tilde{r} \in R$ be any control and $(\tilde{r}_n \in R_n)$ a sequence converging to $\tilde{r}$ (Theorem 3.7). Then

$$D_n G(r_n, \tilde{r}_n - r_n) \geq D_n G(r_n, \overline{r}_n - r_n) = d_n,$$

and passing to the limit, $n \in K$, we find

$$DG(r, \tilde{r} - r) \geq DG(r, \overline{r} - r) = d = 0,$$

and since $\tilde{r}$ is arbitrary, r satisfies therefore the necessary conditions for optimality. ∎

Let us now show how the Algorithm can be implemented. In order to satisfy Step 2, it is easily seen that we can choose a classical control $\overline{r}_n := (\overline{r}_{n1}, \ldots, \overline{r}_{nN})$, with $\overline{r}_{ni} := \delta_{\overline{w}_{ni}}$, such that

$$z_{ni} f(t_{ni}, y_{ni}, \overline{w}_{ni}) + z_{n,i-1} f(t_{n,i-1}, y_{n,i-1}, \overline{w}_{ni})$$
$$= \min_{u \in U} [z_{ni} f(t_{ni}, y_{ni}, u) + z_{n,i-1} f(t_{n,i-1}, y_{n,i-1}, u)], \quad i = 1, \ldots, N,$$

which is a nonlinear programming problem for each i. Suppose, by induction, that the discrete control r_n is of Gamkrelidze type (by choosing also the initial control r_1 of this type), i.e. of the form

$$r_{ni} = \sum_{j=0}^{p} \beta_{nij} \delta_{u_{nij}}, \text{ with } \sum_{j=0}^{p} \beta_{nij} = 1, \ i = 1, \ldots, N, \ \beta_{nij} \text{ nonnegative},$$

where p is the dimension of the system. Then, by Step 3 (which is a one dimensional minimization problem)

$$\tilde{r}_{ni} := r_{ni} + \alpha_n(\overline{r}_{ni} - r_{ni}) = \alpha_n \delta_{\overline{w}_{ni}} + \sum_{j=0}^{p} (1 - \alpha_n) \beta_{nij} \delta_{u_{nij}}.$$

Applying the control $\tilde{r}_n$ to the discrete state equation, we obtain

$$y_{ni} = y_{n,i-1} + \alpha_n \frac{h_n}{2}[f(t_{ni}, y_{ni}, \overline{w}_{ni}) + f(t_{n,i-1}, y_{n,i-1}, \overline{w}_{ni})]$$
$$+ \sum_{j=0}^{p}(1-\alpha_n)\beta_{nij}\frac{h_n}{2}[f(t_{ni}, y_{ni}, u_{nij}) + f(t_{n,i-1}, y_{n,i-1}, u_{nij})], i = 1, \ldots, N.$$

It then follows from properties of convex hulls of finite vector sets (convex polyhedra) that the above convex combination of $p+2$ vectors in $\mathbf{R}^p$ containing $\overline{w}_{ni}, u_{ni0}, \ldots, u_{nip}$ is also a convex combination of the vector containing $\overline{w}_{ni}$ and p among the $p+1$ vectors containing $u_{ni0}, \ldots, u_{nip}$. The coefficients of the later combination can be computed by checking for each i the feasibility of at most $p+1$ linear programming problems w.r.t. barycentric coordinates. Therefore, the control $\tilde{r}_n$ can be replaced by a control r'_n concentrated, for each i, at $\overline{w}_{ni}$ and p among the $p+1$ points $u_{ni0}, \ldots, u_{nip}$ and yielding the same state y_n, hence the same cost, as $\tilde{r}_n$. The equivalent control r'_n is thus of Gamkrelidze type

$$r'_{ni} = \sum_{j=0}^{p}\beta'_{nij}\delta_{u'_{nij}}, \text{ with } \sum_{j=0}^{p}\beta'_{nij} = 1, \ i = 1, \ldots, N, \ \beta'_{nij} \text{ nonnegative},$$

and we can choose the control $r_{n+1} := r'_n$ in Step 4.

Finally, the discrete Gamkrelidze controls r_n, thus computed by the Algorithm can be approximated, or simulated, by piecewise constant classical controls as follows. For each $i = 1, \ldots, N$, subdivide the interval I_{ni} into $p+1$ intervals I_{nij} of lengths $\beta_{nij}h_n$, $j = 0, \ldots, p$. Then define the associated piecewise constant classical control w_n by

$$w_n(t) := u_{nij}, \text{ for } t \in I_{nij}, \ j = 0, \ldots, p, \ i = 1, \ldots, N.$$

The following theorem is proved similarly to Theorem 6.1 in Chryssoverghi et al. (1999).

THEOREM 4.2 *If the sequence of discrete relaxed controls $(r_n \in R_n)$ converges to r in R, then the associated sequence of piecewise constant classical controls (w_n) also converges to r.*

5. Numerical example

Set $I_- := [0, \frac{1}{2})$, $I_+ := [\frac{1}{2}, 1]$, $I := I_- \cup I_+$, and consider the following optimal control problem, with state equation

$$y' = f_1 := \begin{cases} y_2 + w_1^2 - \frac{1}{3}, & t \in I_-, \end{cases}$$

$$y_2' = f_2 := \begin{cases} y_1 + w_2^2 - \frac{1}{3}, & t \in I_-, \\ y_1 + w_2 - \frac{3}{10}, & t \in I_+, \end{cases}$$

$$y_3' = f_3 := y_1^2 + y_2^2, \ t \in I,$$

$$y_1(0) = y_2(0) = y_3(0) = 0,$$

nonconvex control constraint set

$$U := U' \cup U'',$$

where

$$U' := \{u := (u_1, u_2) | \ u_1 \geq 0, \ u_1 - u_2 \leq 0, \ 2u_1 + u_2 - 1 \leq 0\},$$
$$U'' := \{u| \ u_2 \geq 0, \ -u_1 + u_2 \leq 0, \ \tfrac{1}{2}u_1 + u_2 - \tfrac{1}{2} \leq 0\},$$

and cost to be minimized

$$G(w) := y_3(1).$$

The theory deployed in the paper with slight modifications, can clearly be applied to this problem since (f_1, f_2) is continuous, and Lipschitz continuous w.r.t. (t, y), on $\overline{I}_- \times \mathbf{R}^2 \times U$ and on $I_+ \times \mathbf{R}^2 \times U$, and f_3 is independent of y_3 and continuous and Lipschitz on $I \times B \times U$, for every closed ball $B \subset \mathbf{R}^2$. It is easily verified that a (non unique) optimal relaxed control is

$$r^*(t) := \begin{cases} \frac{1}{3}\delta_{(0,0)} + \frac{1}{3}\delta_{(1,0)} + \frac{1}{3}\delta_{(0,1)}, & t \in I_-, \\ \delta_{(\frac{1}{5}, \frac{3}{10})}, & t \in I_+, \end{cases}$$

with corresponding optimal state $y^* = 0$ and cost $G(r^*) = 0$. Note that this problem has no purely classical solutions since the zero cost can be approximated via a sequence of classical controls (W is dense in R and G is continuous), but the only ordinary control that yields this value does not satisfy the constraints for $t \in I_-$.

The discrete relaxed descent method was applied with successive step sizes $h = 2^{-k}0.1$, $k = 0, \ldots, 5$, and 15 iterations for each step size, i.e. 90 iterations in total, with the choice of the initial control $r_1(t) := \delta_{(1,0)}$, $t \in I$. Note that the computing time required with this progressive refinement is about 1/3 of that with constant step size $h = 2^{-5}0.1$. Step 2 reduces here to a minimization of a quadratic (for I_-) or a linear (for I_+) function on the union of two triangles $U' \cup U''$, for each $i = 1, \ldots, N$. The coefficient α_n in Step 3 was computed by the golden section method (note that here the cost is convex w.r.t. the relaxed control r), and the equivalent Gamkrelidze control r_n' in Step 4 was computed by checking the feasibility of at most 3 (instead of 4) linear programming problems, for each i, since f_3 is independent of u. For $n = 90$ ($N_n = 320$, $h_n = 1/320$), we obtained the relaxed control r_n' and the relaxed cost

Figs. 1 and 2 show the contribution of the computed relaxed control r'_n to the state equation , i.e. the two components of the vector function

$$(s_1(t), s_2(t)) := \begin{cases} \int_U (u_1^2, u_2^2) r'_n(t)(du) \approx (\frac{1}{3}, \frac{1}{3}), & t \in I_-, \\ \int_U (u_1, u_2) r'_n(t)(du) \approx (\frac{1}{5}, \frac{3}{10}), & t \in I_+. \end{cases}$$

Finally, applying the approximation procedure by a piecewise constant classical control $\overline{w}_n$ described in Section 4, we obtained the approximate classical cost

$$\overline{G}_n(\overline{w}_n) = 0.00000169638,$$

which was computed using the 3×320 steps corresponding to the intervals $I_{nij}, i = 1, \ldots, N_n, \ j = 0, 1, 2$, defined in this procedure.
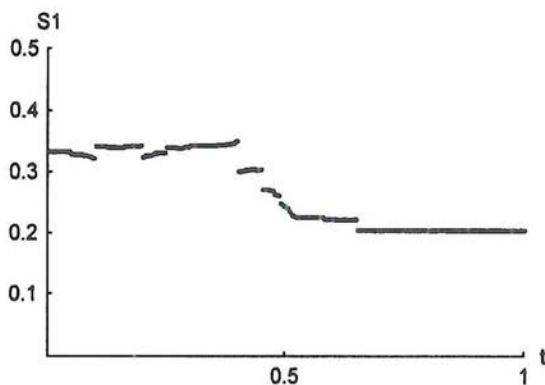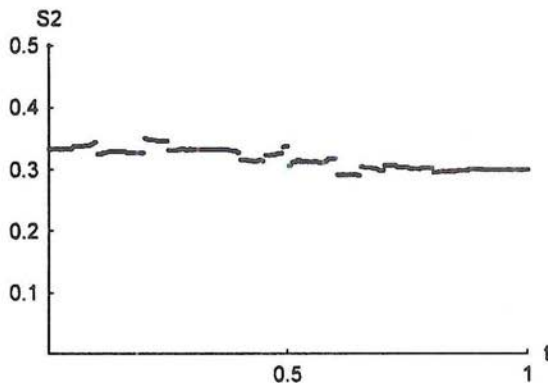

Figure 1.

6.   Final comments

A combined descent and discretization method using an approximate directional derivative has been applied to an optimal control problem involving nonlinear ordinary differential equations with control constraints. This method has the following general advantages:

- it is adapted to nonconvex constrained optimal control problems (whose solutions are often nonclassical) since it exploits their nonconvex structure at each iteration by using relaxed controls,
- it generates sequences that converge to the strong relaxed necessary conditions for optimality,
- the progressive refinement of the discretization reduces computing time and memory,
- it avoids the consideration of separate discrete problems thus generating a single, instead of a double or triple, sequence of controls,
- using various other efficient discrete schemes, the general procedure can be applied to problems whose direct discrete adjoint is either not defined, or involves heavy computations.

Finally, this approach can also be applied to some optimal control problems involving distributed systems, and to state constrained problems, using penalty functions (see Chryssoverghi et al., 1999).

References

CHRYSSOVERGHI, I. and BACOPOULOS, A. (1990) Discrete approximation of relaxed optimal control problems. *Journal of Optimization Theory and Applications*, **65**, 3, 395–407.

CHRYSSOVERGHI, I. and BACOPOULOS, A. (1993) Approximation of relaxed nonlinear parabolic optimal control problems. *Journal of Optimization Theory and Applications*, **77**, 1, 31–50.

CHRYSSOVERGHI, I., BACOPOULOS, A., KOKKINIS, B. and COLETSOS, J. (1997) Mixed Frank–Wolfe penalty method with applications to nonconvex optimal control problems. *Journal of Optimization Theory and Applications*, **94**, 2, 311–334.

CHRYSSOVERGHI, I., BACOPOULOS, A., COLETSOS, J. and KOKKINIS, B. (1998) Discrete approximation of nonconvex hyperbolic optimal control problems with state constraints. *Control and Cybernetics*, **27**, 1, 29–50.

CHRYSSOVERGHI, I., COLETSOS, J. and KOKKINIS, B. (1999) Discrete relaxed method for semilinear parabolic optimal control problems. *Control and Cybernetics*, **28**, 2, 157–176.

FATTORINI, H.O. (1999) *Infinite Dimensional Optimization and Control Theory*. Cambridge Univ. Press, Cambridge.

POLAK, E. (1997) *Optimization: Algorithms and Consistent Approximations*.

ROUBÍČEK, T. (1991) A convergent computational method for constrained optimal relaxed control problems. *Journal of Optimization Theory and Applications*, **69**, 589–603.

ROUBÍČEK, T. (1997) *Relaxation in Optimization Theory and Variational Calculus*. Walter de Gruyter, Berlin.

SCHWARTZ, A. and POLAK, E. (1996) Consistent approximations for optimal control problems based on Runge–Kutta integration. *SIAM Journal on Control and Optimization*, **34**, 4, 1235–1269.

THOMEE, V. (1997) *Galerkin Finite Element Methods for Parabolic Problems*. Springer, Berlin.

WARGA, J. (1972) *Optimal Control of Differential and Functional Equations*. Academic Press, New York.

WARGA, J. (1977) Steepest descent with relaxed controls. *SIAM Journal on Control and Optimization*, **15**, 674–682.