

## Convergence of Toland's critical points for sequences of D.C. functions and application to the resolution of semilinear elliptic problems

by

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**Abstract:** We prove that if a sequence  $(f_n)_n$  of D.C. functions (Difference of two Convex functions) converges to a D.C. function  $f$  in some appropriate way and if  $u_n$  is a critical point of  $f_n$ , in the sense described by Toland, and is such that  $(u_n)_n$  converges to  $u$ , then  $u$  is a critical point of  $f$ , still in Toland's sense. We also build a new algorithm which searches for this critical point  $u$  and then apply it in order to compute the solution of a semilinear elliptic equation.

**Keywords:** non-convex optimization, D.C. functions, Toland's critical point, normalized D.C. decomposition, semilinear elliptic problem, proximal algorithm.

### 1. Introduction

The purpose of this work is to study the stability property of Toland's critical points with respect to the convergence of sequences of D.C. functions (Difference of two Convex functions) and its application to the resolution of semi-linear elliptic problems. In the sequel, critical point always means critical in the sense of Toland. Section 2 is dedicated to some preliminary results. In Section 3, we study the normalized D.C. decomposition in a space having an infinite dimension and prove that every D.C. function  $f : X \rightarrow \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$  has a normalized D.C. decomposition. In Section 4, we present the conditions under which the convergence of a sequence  $(f_n)_n$  of D.C. functions to  $f$  and the convergence of a sequence  $(u_n)_n$  where  $u_n$  is a critical point of  $f_n$  to  $u$ , imply that  $u$  is a critical

properties. In Section 6, the resolution of a semi-linear elliptic problem by this algorithm is presented.

## 2. Preliminary results

Let  $X$  be a Banach space,  $X^*$  its dual space and  $\Gamma_0(X)$  the cone of proper, lower semi-continuous (lsc), convex functions on  $X$ .

**DEFINITION 2.1** 1. A function  $f$  is called a D.C. function if there exist two convex functions  $g$  and  $h$  such that  $f = g - h$ . The couple  $(g, h)$  is called a D.C. decomposition of  $f$ . If  $g$  and  $h$  are lower semi-continuous, this decomposition is called a lsc D.C. decomposition of  $f$ .

2. A point  $x^*$  in  $X$  is a local minimizer of  $f = g - h$ , if  $f(x^*) = g(x^*) - h(x^*)$  is finite and if there exists a neighbourhood  $V$  of  $x^*$  such that:

$$f(x^*) \leq f(x), \forall x \in V \Leftrightarrow g(x^*) - h(x^*) \leq g(x) - h(x), \forall x \in V.$$

3.  $\bar{x}$  is a critical point of  $f = g - h$  if:

$$\partial g(\bar{x}) \cap \partial h(\bar{x}) \neq \emptyset \Leftrightarrow 0 \in \partial g(\bar{x}) - \partial h(\bar{x}).$$

**DEFINITION 2.2** Let  $X$  be a reflexive Banach space and  $(f_n)_n$  be any sequence of lsc functions defined on  $X$ . The sequence  $(f_n)_n$  Mosco-epi-converges to a lsc function  $f$  (we write:  $f = M - \text{elm} f_n$ ) if:

$$\forall x \in X : w - \text{el} f_n(x) \geq f(x) \geq s - \text{el} f_n(x),$$

where:

$$w - \text{el} f_n(x) = \inf_{x_n \rightarrow x} \liminf_n f_n(x_n),$$

$$s - \text{el} f_n(x) = \inf_{x_n \rightarrow x} \limsup_n f_n(x_n).$$

**DEFINITION 2.3** Let  $f : X \rightarrow \overline{\mathbf{R}}$  be a D.C. function. The decomposition  $(g, h)$  is a normalized D.C. decomposition of  $f$  if:  $\inf_{x \in X} h(x) = 0$ .

Notice that this condition implies:  $h(x) \geq 0, \forall x \in X$ .

**PROPOSITION 2.4** Let  $X$  be a Banach space and  $f$  a lsc D.C. function, then  $f$  has a normalized D.C. decomposition.

**Proof.** Let  $f$  be a D.C. function. If  $f$  is lsc, there exist two lsc convex functions  $g$  and  $h$  such that  $f = g - h$ , Elhilali Alaoui (1996), with  $\text{dom}(\partial h) \neq \emptyset$ , Bronsted and Rockafellar (1965). Let  $x_0$  be any element of  $\text{dom}(\partial h)$  and  $x_0^*$  be any element of  $\partial h(x_0)$ . We define  $\widehat{g}$  and  $\widehat{h} : X \rightarrow \overline{\mathbf{R}}$  as:

$$\widehat{g}(x) = g(x) - h(x_0) - \langle x_0^*, x - x_0 \rangle,$$

$$\widehat{h}(x) = h(x) - h(x_0) - \langle x_0^*, x - x_0 \rangle.$$

One immediately proves that  $\widehat{g}$  and  $\widehat{h}$  are convex,  $\widehat{h}(x) \geq 0, \forall x \in X, \widehat{h}(x_0)$

**COROLLARY 2.5** *Every D.C. function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  has a normalized D.C. decomposition.*

*Proof.* Since  $\text{dom} f = \mathbf{R}^n$ ,  $f$  is continuous on  $\mathbf{R}^n$ , because every convex function is continuous on the interior of its domain. Then,  $f$  has a continuous decomposition, Elhilali Alaoui (1996). The above property is a direct consequence of the previous Proposition 2.4. ■

### 3. Coercivity

**DEFINITION 3.1** *Let  $X$  be a Banach space,  $f_n$  be any element of  $\Gamma_0(X)$ , for every  $n$  in  $\mathbf{N}$ .  $(f_n)_n$  is  $\|\cdot\|$ -equi-continuous at  $x_0$  in  $X$  if:*

$$\forall \epsilon > 0, \exists r_\epsilon > 0, \forall x \in B(x_0, r_\epsilon), \forall n \in \mathbf{N} : |\varphi_n(x) - \varphi_n(x_0)| < \epsilon.$$

**DEFINITION 3.2** *Let  $X$  be a Banach space and  $(f_n)_n$  be any sequence of functions in  $X$  with values in  $\overline{\mathbf{R}}$ .*

1.  $(f_n)_n$  is diagonal-coercive if:

$$\|x_n\| \rightarrow +\infty \Rightarrow f_n(x_n) \rightarrow +\infty.$$

2.  $(f_n)_n$  is strongly diagonal-coercive if:

$$\|x_n\| \rightarrow +\infty \Rightarrow \frac{f_n(x_n)}{\|x_n\|} \rightarrow +\infty.$$

3.  $(f_n)_n$  is equi-coercive if there exist two nonnegative constants  $\alpha$  and  $\beta$  such that:

$$\forall x \in X, \forall n \in \mathbf{N} : f_n(x) \geq \alpha\|x\| - \beta.$$

**REMARK 3.3** *Any equi-coercive sequence (or strongly diagonal-coercive) is diagonal-coercive.*

1. *An equi-coercive sequence is not necessarily strongly diagonal-coercive. For example, let  $f : X \rightarrow \overline{\mathbf{R}}$  defined by:  $f(x) = \|x\| + r$ , where  $r$  is a fixed real.  $f$  is equi-coercive but it is not strongly diagonal-coercive. Indeed:*

$$\frac{f(x)}{\|x\|} = 1 + \frac{r}{\|x\|} \xrightarrow{\|x\| \rightarrow +\infty} 1.$$

2. *If  $(f_n)_n$  is diagonal-coercive (resp. strongly diagonal-coercive) then, for any sequence  $(x_n)_n$  of  $X$ , if  $(f_n(x_n))_n$  (resp.  $(f_n(x_n)/\|x_n\|)_n$ ) is bounded from above, then  $(x_n)_n$  is bounded from above.*

Let  $X$  be a Banach space and  $f$  be any function in  $\Gamma_0(X)$ . The continuity of  $f$  on  $X$  implies the coercivity of  $f^*$ , Moreau (1965), where:

$$f^*(y) = \sup\{\langle x, y \rangle - f(x) : x \in X\},$$

is the conjugate function of  $f$ . This result has been generalized in Elghali (1988) for sequences of functions in  $\Gamma_0(X)$ .

The purpose of this section is to study the conditions to be imposed on the sequence  $(f_n)_n$  in  $\Gamma_0(X)$  which imply the strong equi-coercivity of the sequence

**PROPOSITION 3.4** *Let  $X$  be a Banach space and  $(f_n)_n$  be any sequence of functions in  $\Gamma_0(X)$ . If  $(f_n)_n$  is  $\|\cdot\|$ -equi-continuous at 0 and  $\sup_{n \in \mathbb{N}} f_n(0) < +\infty$ , then  $(f_n^*)_n$  is equi-coercive.*

*Proof.* Because  $(f_n)_n$  is strongly equi-continuous at 0, there exist  $\alpha > 0$  and  $r > 0$  such that:

$$f_n(x) \leq \alpha, \quad \forall x \in rB, \quad \forall n \in \mathbb{N},$$

(see Elghali, 1988), where  $B = B(0, 1)$  is the unit ball centered at the origin. Then:

$$\begin{aligned} f_n^*(x^*) &= \sup_{x \in X} \{ \langle x, x^* \rangle - f_n(x) \} \geq \sup_{x \in rB} \{ \langle x, x^* \rangle - \alpha \} = r \|x^*\|_* - \alpha \\ &\quad \forall x^* \in X^*, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where  $\|\cdot\|_*$  denotes the norm of the dual space  $X^*$ . Hence,  $(f_n^*)_n$  is equi-coercive. ■

We deduce from the above result that when  $X$  has a finite dimension, if  $(f_n)_n$  is a sequence of convex finite functions, that is if for every  $n$  in  $\mathbb{N}$  :  $\text{dom}(f_n) = X$ , which converges to some convex finite function  $f$ , then  $(f_n^*)_n$  is strongly diagonal-coercive.

**PROPOSITION 3.5** *Let  $(f_n)_n$  ( $f_n : \mathbf{R}^p \rightarrow \mathbf{R}$ ) be any sequence of convex and finite functions, which converges to  $f$ . Then  $(f_n^*)_n$  is strongly diagonal-coercive.*

*Proof.* Let  $(x_n^*)_n$  be any sequence of vectors in  $\mathbf{R}^p$  such that:  $\|x_n^*\| \rightarrow +\infty$ . Using the definition of the conjugate function, we obtain:

$$\frac{f_n^*(x_n^*)}{\|x_n^*\|} = \sup_{x \in X} \left\{ \left\langle x, \frac{x_n^*}{\|x_n^*\|} \right\rangle - \frac{f_n(x)}{\|x_n^*\|} \right\},$$

which implies:

$$\begin{aligned} \liminf_n \frac{f_n^*(x_n^*)}{\|x_n^*\|} &= \liminf_n \sup_{x \in X} \left\{ \left\langle x, \frac{x_n^*}{\|x_n^*\|} \right\rangle - \frac{f_n(x)}{\|x_n^*\|} \right\} \\ &\geq \sup_{x \in X} \liminf_n \left\{ \left\langle x, \frac{x_n^*}{\|x_n^*\|} \right\rangle - \frac{f_n(x)}{\|x_n^*\|} \right\}. \end{aligned}$$

We choose a subsequence such that:

$$\liminf_n \left\{ \left\langle x, \frac{x_n^*}{\|x_n^*\|} \right\rangle - \frac{f_n(x)}{\|x_n^*\|} \right\} = \lim_{n'} \left\{ \left\langle x, \frac{x_{n'}^*}{\|x_{n'}^*\|} \right\rangle - \frac{f_{n'}(x)}{\|x_{n'}^*\|} \right\}. \quad (1)$$

Because  $\|x_{n'}^*\| \rightarrow +\infty$  and  $(f_{n'}(x))_{n'}$  converges to  $f(x) \in \mathbf{R}$ , the equality (1) may be written down as:

$$\liminf_n \left\langle x, \frac{x_n^*}{\|x_n^*\|} \right\rangle = \lim_{n'} \left\langle x, \frac{x_{n'}^*}{\|x_{n'}^*\|} \right\rangle$$



The sequence  $(x_{n'}^*/\|x_{n'}^*\|)_{n'}$  being bounded, there exists a subsequence such that:

$$\lim_{n''} \frac{x_{n''}^*}{\|x_{n''}^*\|} = u \in \mathbf{R}, \text{ with } \|u\| = 1.$$

We then deduce

$$\liminf_n \frac{f_n^*(x_n^*)}{\|x_n^*\|} \geq \sup_{x \in X} \langle x, u \rangle = +\infty,$$

which ends up the proof. ■

#### 4. Critical points of sequences of D.C. functions

Since its introduction by Toland in 1978, the notion of critical point of a D.C. function has attracted attention of several researchers such as, for example, Correa and Lemaréchal (1993), Pham Dinh Tao and El Bernoussi (1986), Lemaire (1988), Yassine (1988, 1999). Our purpose is to prove that if  $(f_n)_n$  is a sequence of D.C. functions which converges in an appropriate way to a D.C. function  $f$  and if  $u_n$  is a critical point of  $f_n$  such that  $(u_n)_n$  converges to  $u$ , then  $u$  is a critical point of  $f$ .

Let us first recall some results we will use in the remaining parts of the paper.

**DEFINITION 4.1** *Let  $X$  be a Banach space,  $f: X \rightarrow \overline{\mathbf{R}}$  be a D.C. function and  $(g, h)$  be a D.C. decomposition of  $f$ . We define the function  $f_*: X^* \rightarrow \overline{\mathbf{R}}$  as:*

$$\forall x^* \in X^*: f_*(x^*) = h^*(x^*) - g^*(x^*).$$

Notice that  $f_*$  is not the conjugate function of  $f$ . The link between  $f_*$  and the conjugate function  $f^*$  of  $f$  can be found in Ellaia and Hiriart-Urruty (1986) and Ellaia (1984).

**DEFINITION 4.2** *Let  $X$  be a reflexive Banach space,  $(f_n)_n$  be a sequence of D.C. functions of  $X$  in  $\overline{\mathbf{R}}$  and  $(g_n, h_n)$  be a D.C. decomposition of  $f_n$ . We say that  $(f_n)_n$  D.C. Mosco-converges to  $f = g - h$  and we note  $f_n \xrightarrow{D.C.M} f$ , if  $(g_n)_n$  Mosco-converges to  $g$  and  $(h_n)_n$  Mosco-converges to  $h$  (see the above Definition 2.2).*

Before announcing the main result of this section, let us recall the following theorem.

**THEOREM 4.3** (Attouch, 1989) *Let  $X$  be a reflexive Banach space. For every sequence  $(\varphi_n)_n$  of proper, lsc and convex functions  $X \rightarrow \overline{\mathbf{R}}$ , the following properties are equivalent:*

2.  $\partial\varphi_n \xrightarrow{G} \partial\varphi$ , which means that:

$$\forall x, x^* \in \partial\varphi, \exists (x_n, x_n^*) \in \partial\varphi_n : x_n \rightarrow x, x_n^* \rightarrow x^*, \varphi_n(x_n) \rightarrow \varphi(x).$$

REMARK 4.4 This above defined graph-convergence is also called the Kuratowski–Painlevé convergence.

THEOREM 4.5 Let  $X$  be a reflexive Banach space,  $(f_n)_n$  be a sequence of D.C. functions and  $(g_n, h_n)$  be a normalized D.C. decomposition of  $f_n$ . Let us assume that the following conditions are verified:

1. The sequence  $(f_n)_n$  D.C-Mosco-converges to  $f = g - h$ .
2. The sequence  $(h_n^*)_n$  is strongly diagonal-coercive.
3. For every  $n$  in  $\mathbb{N}$ ,  $u_n$  is a critical point of  $f_n$  and  $(u_n)_n$  converges to  $u$  in the strong topology of  $X$ .

Then,  $u$  is a critical point of  $f$ .

Proof. According to assumption 3., for every  $n$  in  $\mathbb{N}$ , we have:  $\partial g_n(u_n) \cap \partial h_n(u_n) \neq \emptyset$ . Hence one can choose  $u_n^*$  in  $\partial g_n(u_n) \cap \partial h_n(u_n)$ . According to Fenchel's equality, we have:

$$h_n(u_n) + h_n^*(u_n^*) = \langle u_n, u_n^* \rangle.$$

Because  $(g_n, h_n)$  is a normalized decomposition of  $f_n$ , one has:  $h_n(u_n) \geq 0$ , which implies:

$$h_n^*(u_n^*) \leq \langle u_n, u_n^* \rangle \leq \|u_n\| \|u_n^*\| \Rightarrow \frac{h_n^*(u_n^*)}{\|u_n^*\|} \leq \|u_n\|, \quad (2)$$

for every  $\|u_n^*\| \neq 0$ . Because the sequence  $(u_n)_n$  is supposed to be convergent, the inequality (2) and the hypothesis 2. imply that  $(u_n^*)_n$  is bounded. Let  $(u_{s(n)}^*)_n$  be any subsequence  $^*$ -weakly convergent to  $u^* \in X^*$ . We have:

$$u_{s(n)} \xrightarrow{\|\cdot\|} u, u_{s(n)}^* \rightharpoonup^* u^*, g_n \xrightarrow{M} g, h_n \xrightarrow{M} h,$$

which prove, according to Theorem 4.3 and the definition of the graph-convergence, that  $u^*$  belongs to  $\partial g(u)$  and  $u^*$  belongs to  $\partial h(u)$ . This completes the proof.  $\blacksquare$

It is sometimes better to impose some conditions on the D.C. function  $f$  itself instead of imposing some other ones on a decomposition  $(g, h)$  of  $f$ . Accordingly, we prove the following theorem, which looks like the above Theorem 4.5, changing the hypothesis 2. of this theorem as follows.

THEOREM 4.6 Let  $X$  be a reflexive Banach space,  $(f_n)_n$  be any sequence of D.C. functions and  $(g_n, h_n)$  be a normalized D.C. decomposition of  $f_n$ . Let us suppose that the following conditions are verified:

2. The sequence  $((f_n)_*)_n$  is diagonal-coercive.
3. For every  $n$  in  $\mathbf{N}$ ,  $u_n$  is a critical point of  $f_n$ ,  $(u_n)_n$  converges to  $u$  in the strong topology of  $X$  and  $(g_n(u_n))_n$  converges to  $g(u)$ .

Then,  $u$  is a critical point of  $f$ .

Proof. According to the assumption 3., one has, for every  $n$  in  $\mathbf{N}$ :  $h_n(u_n) \cap \partial g_n(u_n) \neq \emptyset$ . Upon choosing  $u_n^*$  in  $\partial g_n(u_n) \cap \partial h_n(u_n)$ , we have, according to Fenchel's equality:

$$\begin{aligned} h_n(u_n) + h_n^*(u_n^*) &= \langle u_n, u_n^* \rangle; \quad g_n(u_n) + g_n^*(u_n^*) = \langle u_n, u_n^* \rangle, \\ \Rightarrow (f_n)_*(u_n^*) &= g_n(u_n) - h_n(u_n) \leq g_n(u_n), \end{aligned} \quad (3)$$

because  $h_n(u_n) \geq 0$ , for every  $n$  in  $\mathbf{N}$ . Thanks to the above assumption 3., the sequence  $(g_n(u_n))_n$  converges, from which we deduce thanks to the inequality (3) that:

$$\liminf_n (f_n)_*(u_n^*) < +\infty.$$

This ensures, thanks to this hypothesis 2., that  $(u_n^*)_n$  is bounded. Let  $(u_{s(n)}^*)_n$  be a subsequence  $*$ -weakly convergent to some  $u^*$  in  $X^*$ . We have

$$u_{s(n)} \xrightarrow{\|\cdot\|} u, \quad u_{s(n)}^* \rightharpoonup u^*, \quad g_n \xrightarrow{G} g, \quad h_n \xrightarrow{M} h,$$

which imply, using Theorem 4.3 and the definition of the graph-convergence, that  $u^*$  belongs to  $\partial g(u)$  and to  $\partial h(u)$ . ■

## 5. Algorithm

In the following,  $H$  denotes a real Hilbert space,  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  are respectively the scalar product and the norm on  $H$ . In this paragraph, our purpose is to study the algorithm:

$$\begin{cases} \text{Given } u_0 \in H, \\ \text{find } p_n \in \partial h_n(u_n); \quad u_{n+1} = \text{prox}_{\lambda g_n}(u_n + \lambda p_n), \end{cases} \quad (\text{ALG})$$

where  $w = \text{prox}_{\lambda \varphi}(u)$  is the unique point where the function  $\frac{1}{2\lambda} \|\cdot - u\|^2 + \varphi(\cdot)$  reaches its minimum. We will prove that if  $(g_n)_n$  and  $(h_n)_n$  are two sequences in  $\Gamma_0(H)$ , Mosco-converging, respectively, to  $g$  and  $h$  and if the sequence  $(u_n)_n$  converges to  $u$  in  $H$ , then the sequence  $(u_n)_n$  built in the algorithm (ALG) converges to some critical point  $u$  of  $f = g - h$ . Notice that Lemaire (1988) proved a similar result, assuming that the sequence  $(g_n)_n$  decreases to  $g$  and that the sequence  $(h_n)_n$  increases to  $h$ . We recall that a sequence  $(\varphi_n)_n$  increases (resp. decreases) to a function  $\varphi$  if:

$$\forall n \in \mathbf{N}, \quad \forall x \in H : \varphi_n(x) \leq \varphi_{n+1}(x) \quad (\text{resp. } \varphi_{n+1}(x) \leq \varphi_n(x)).$$

**THEOREM 5.1** *Let  $(u_n)_n$  be the sequence defined by the algorithm (ALG). Assume that the following conditions are verified:*

1. *For every  $n \in \mathbb{N}$ ,  $h_n$  is nonnegative.*
2. *The sequence  $(h_n^*)_n$  is strongly diagonal-coercive in  $H$ .*
3. *The sequences  $(g_n)_n$  and  $(h_n)_n$  are Mosco-converging respectively to  $g$  and  $h$ .*
4. *The sequence  $(u_n)_n$  converges to  $u$  in  $H$ .*

*Then  $u$  is a critical point of  $f$ .*

**Proof.** Because  $p_n$  belongs to  $\partial h_n(u_n)$ , Fenchel's equality implies that:

$$h_n(u_n) + h_n^*(p_n) = \langle u_n, p_n \rangle.$$

Because  $h_n(u_n) \geq 0$ , we have:

$$h_n^*(p_n) \leq \langle u_n, p_n \rangle \leq \|u_n\| \|p_n\| \Rightarrow \frac{h_n^*(p_n)}{\|p_n\|} \leq \|u_n\|, \quad (4)$$

for every  $p_n \neq 0$ . Because the sequence  $(u_n)_n$  is supposed to be convergent, the inequality (4) and the hypothesis 2. imply that  $(p_n)_n$  is bounded in  $H$ . Let  $(p_{s(n)})_n$  be a subsequence  $*$ -weakly convergent to some  $p$  in  $H$ . We have:

$$u_{s(n)} \xrightarrow{\|\cdot\|} u, p_{s(n)} \rightharpoonup p, h_n \xrightarrow{M} h.$$

Using Theorem 4.3 and the definition of the graph-convergence, we thus prove that  $p$  belongs to  $\partial h(u)$ . Next we claim that  $p$  also belongs to  $\partial g(u)$ . According to the algorithm (ALG), we have:

$$\begin{aligned} u_{s(n)+1} &= J_{\partial g_{s(n)}}^\lambda(u_{s(n)} + \lambda p_{s(n)}) = (I + \lambda \partial g_{s(n)})^{-1}(u_{s(n)} + \lambda p_{s(n)}) \\ \Leftrightarrow v_{s(n)} &= \frac{u_{s(n)} - u_{s(n)+1}}{\lambda} + p_{s(n)} \in \partial g_{s(n)}(u_{s(n)+1}). \end{aligned} \quad (5)$$

We have

$$u_{s(n)} \xrightarrow{\|\cdot\|} u, u_{s(n)+1} \xrightarrow{\|\cdot\|} u,$$

and according to (5),  $(v_{s(n)})_n$  weakly converges to  $p$ . Because  $(g_n)_n$  Mosco-converges to  $g$ , we deduce, following Theorem 4.3 and the definition of the graph-convergence, that  $p$  belongs to  $\partial g(u)$ . This completes the proof.  $\blacksquare$

**REMARK 5.2** *The condition 1. is, in fact, not restrictive. Indeed, according to Proposition 2.4, every D.C. function admits a normal decomposition  $(g, h)$ , with*



## 6. Approximation of the solution of semilinear elliptic problems

In this paragraph, the algorithm (ALG) is used for the approximation of the critical points of the function  $f$ , defined on  $H = L^2(\Omega)$  by  $f = g - h$ , with:

$$\begin{aligned} g(u) &= \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} k u dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise} \end{cases} \\ h(u) &= \int_{\Omega} j(x, u(x)) dx, \end{aligned} \quad (6)$$

where  $\Omega$  is an open, smooth and bounded subset of  $\mathbf{R}^N$  ( $N \geq 1$ ),  $k$  belongs to  $L^2(\Omega)$  and  $j : \Omega \times \mathbf{R} \rightarrow \mathbf{R}^+$  is such that, for almost every  $x \in \Omega$ :

$$\begin{aligned} r &\mapsto j(x, r) \text{ is convex, lsc, proper,} \\ j(x, r) &\leq c(x)|r|^2, \text{ with } c \in L^\infty(\Omega), \\ \min_{r \in \mathbf{R}} j(x, r) &= j(x, 0) = 0. \end{aligned} \quad (7)$$

We observe that  $f$  is D.C. on  $L^2(\Omega)$ , because  $g$  and  $h$  belong to  $\Gamma_0(L^2(\Omega))$ , Brézis (1992). We then define the sequences  $(g_n)_n$  and  $(h_n)_n$  by:

$$\forall u \in L^2(\Omega) : \begin{cases} g_n(u) = g(u) \\ h_n(u) = \int_{\Omega} j_n(x, u(x)) dx, \end{cases} \quad (8)$$

where  $j_n(x, \cdot)$  is the Yoshida-approximation of  $j(x, \cdot)$ . Notice that  $(j_n(x, \cdot))_n$  increases to  $j(x, \cdot)$  and satisfies

$$\forall n : j_n(x, r) \leq j(x, r), \quad \left| \frac{\partial j_n}{\partial r}(x, r) \right| \leq n.$$

We know that:  $\partial h_n(u) = j'_n(\cdot, u)$ , Brézis (1992). Let us recall that:

$$w = \text{prox}_{\lambda g}(u) = (I + \lambda \partial g)^{-1}(u) \Leftrightarrow \begin{cases} w \in H^2(\Omega) \cap H_0^1(\Omega) \\ \frac{1}{\lambda} w - \Delta w - k = \frac{1}{\lambda} u, \end{cases}$$

in the distributional sense. The (ALG) algorithm becomes in this case:

$$\begin{cases} \text{Given } u_0 \in L^2(\Omega), \\ \text{find } u_{n+1} \in H^2(\Omega) \cap H_0^1(\Omega) : \\ \frac{1}{\lambda} u_{n+1} - \Delta u_{n+1} = j'_n(\cdot, u_n) + \frac{1}{\lambda} u_n + k, \end{cases} \quad (\text{ALG1})$$

**THEOREM 6.1** *The sequence  $(u_n)_n$  defined by the algorithm (ALG1) admits a subsequence  $(u_{s(n)})_n$  which converges to some critical point  $u$  of  $f$ . Moreover,  $u$  is a solution of:*

$$\begin{cases} -\Delta u(x) \in \beta(x, u(x)) + k(x), & \text{in } \Omega \\ u \in H^2(\Omega) \cap H_0^1(\Omega), \end{cases} \quad (\text{P})$$

where  $\beta(x, \cdot)$  is the maximal monotone graph in  $\mathbf{R}$  such that  $0 \in \beta(x, 0)$  and  $\partial j(x, r) = \beta(x, r)$ .

**Proof.** We assert that Theorem 6.1 is a direct consequence of Theorem 4.5. Indeed, let us prove that the conditions of Theorem 4.5 are satisfied.

1. Because  $j_n(x, r)$  is nonnegative,  $h_n(u)$  is non-negative.
2. The sequence  $(h_n^*)_n$  is strongly diagonal-coercive. In fact, following Brézis (1992), we have:

$$h_n^*(p_n) = \int_{\Omega} j_n^*(x, p_n(x)) \, dx. \quad (9)$$

Because  $(j_n(x, \cdot))_n$  increases to  $j(x, \cdot)$ , the hypothesis  $(7)_2$  implies that:

$$h_n^*(p) \geq d \int_{\Omega} p^2 \, dx, \quad (10)$$

where  $d$  is a nonnegative constant. We then have, thanks to (9) and (10):

$$\begin{aligned} \frac{h_n^*(p_n)}{\|p_n\|_{L^2}^2} &= \frac{\int_{\Omega} j_n^*(x, p_n(x)) \, dx}{\sqrt{\int_{\Omega} (p_n)^2 \, dx}} \geq d \sqrt{\int_{\Omega} (p_n)^2 \, dx} \\ \Rightarrow \lim_{\|p_n\|_{L^2} \rightarrow \infty} \frac{h_n^*(p_n)}{\|p_n\|_{L^2}^2} &= +\infty. \end{aligned}$$

3. Because the sequence  $(h_n)_n$  is increasing, then it Mosco-converges to  $h = \sup_n h_n$ , Attouch (1989).
4. Let us prove that the sequence  $(u_n)_n$  admits a subsequence  $(u_{s(n)})_n$  such that the sequences  $(u_{s(n)})_n$  and  $(u_{s(n)+1})_n$  converge in the strong topology of  $L^2(\Omega)$ . According to (ALG1),  $u_{n+1}$  satisfies:

$$\begin{cases} u_{n+1} \in H^2(\Omega) \cap H_0^1(\Omega) \\ \frac{1}{\lambda} u_{n+1} - \Delta u_{n+1} = \frac{1}{\lambda} u_n + j_n'(\cdot, u_n) + k, \end{cases}$$

in the distributional sense. We multiply the preceding equation by  $u_{n+1} - u_n$ , integrate in  $\Omega$  and get:

$$\begin{aligned} \frac{1}{\lambda} \int_{\Omega} (u_{n+1} - u_n)^2 \, dx + \int_{\Omega} |\nabla u_{n+1}|^2 \, dx - \int_{\Omega} \nabla u_{n+1} \cdot \nabla u_n \, dx \\ - \int_{\Omega} j_n'(\cdot, u_n) (u_{n+1} - u_n) \, dx = \int_{\Omega} k (u_{n+1} - u_n) \, dx \end{aligned}$$

Applying Young's inequality and the convexity of  $j_n$ , we obtain:

$$\begin{aligned} & \frac{1}{\lambda} \int_{\Omega} (u_{n+1} - u_n)^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u_{n+1}|^2 dx - \int_{\Omega} k u_{n+1} dx \\ & \leq \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} j_n(x, u_n) dx \\ & \quad + \int_{\Omega} j_n(x, u_{n+1}) dx - \int_{\Omega} k u_n dx. \end{aligned} \quad (11)$$

Moreover, the sequence  $(j_n(x, \cdot))_n$  is increasing. Let us define:

$$\sigma_n = \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} j_n(\cdot, u_n) dx - \int_{\Omega} k u_n dx.$$

We deduce from (11) that the sequence  $(\sigma_n)_n$  is decreasing. According to the hypothesis  $(7)_1$  and Young's inequality, we have:

$$\begin{aligned} \sigma_n & \geq \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \|c\|_{L^\infty} \int_{\Omega} (u_n)^2 dx - c(\varepsilon) \int_{\Omega} k^2 dx \\ & \quad - \varepsilon \int_{\Omega} (u_n)^2 dx. \end{aligned} \quad (12)$$

Applying Rellich's Lemma, Brézis (1992), and because  $\|c\|_{L^\infty} < \lambda_1$ , we can choose  $\varepsilon > 0$  in (12) such that:

$$\sigma_n \geq -c(\varepsilon) \int_{\Omega} k^2 dx.$$

Adding up, term to term, the quantities given in (11), we obtain:

$$\begin{aligned} & \sum_{n=0}^{+\infty} \frac{1}{\lambda} \int_{\Omega} (u_{n+1} - u_n)^2 dx \leq \sigma_0 - \lim_{n \rightarrow \infty} \sigma_n \\ & \Rightarrow \lim_{n \rightarrow \infty} \int_{\Omega} (u_{n+1} - u_n)^2 dx = 0. \end{aligned}$$

Let us now prove that the sequence  $(u_n)_n$  is bounded in  $H_0^1(\Omega)$ . Because the sequence  $(\sigma_n)_n$  is decreasing, we have, for every  $n \geq 1$ :  $\sigma_n \leq \sigma_1$ , which implies:

$$\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} j_n(x, u_n) dx - \int_{\Omega} k u_n dx \leq \sigma_1.$$

The hypothesis  $(7)_1$  and Young's inequality imply:

$$\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx \leq \frac{\varepsilon + \|c\|_{L^\infty}}{2\lambda_1} \int_{\Omega} u_n^2 dx + c(\varepsilon) \int_{\Omega} k^2 dx + \sigma_1. \quad (13)$$

Applying again Rellich's Lemma and choosing  $\varepsilon > 0$  such that:  $\alpha = \lambda_1 - \varepsilon - \|c\|_{L^\infty} > 0$ , we obtain, thanks to (13):

$$\int_{\Omega} |\nabla u_n|^2 dx < \frac{2\lambda_1}{c(\varepsilon)} \left( \int_{\Omega} k^2 dx + \sigma_1 \right)$$

Let  $(u_{s(n)})_n$  be any subsequence of  $(u_n)_n$  strongly converging to some  $u$  in  $L^2(\Omega)$ . According to (13),  $(u_{s(n)+1})_n$  also converges to  $u$  in the strong topology of  $L^2(\Omega)$ . Consequently, the hypothesis 4. is verified. According to Theorem 4.5,  $u$  is a critical point of  $f$ . There exists  $p$  in  $\partial g(u) \cap \partial h(u)$ , and because:  $\partial g(u) = \{-\Delta u - k\}$  and  $\partial h(u) = \beta(\cdot, u)$ , we get:

$$\begin{cases} -\Delta u - k \in \beta(\cdot, u) \\ u \in H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

$u$  is thus a solution of (P). ■

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