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# Adaptive stabilization of undamped flexible structures

by

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**Abstract:** In this paper non-identifier-based adaptive stabilization of undamped flexible structures is considered in the case of collocated input and output operators. The systems have poles and zeros on the imaginary axis. In the case where velocity feedback is available, the adaptive stabilizer is constructed by an adaptive PD-controller (proportional plus derivative controller). In the case where only position feedback is available, the adaptive stabilizer is constructed by an adaptive P-controller for the augmented system which consists of the controlled system and a parallel compensator. Numerical examples are given to illustrate the effectiveness of the proposed controllers.

**Keywords:** adaptive stabilization, undamped flexible structures, parallel compensators.

## 1. Introduction

The advantage of adaptive control is that good control performance can be achieved even in the presence of various uncertainties. Non-identifier-based high-gain adaptive stabilization has been also investigated, Ilchmann (1993). In the design of high-gain adaptive controllers, it is usually required that the system have no unstable zeros. A linear system described by a second-order differential equation without damping term has poles and zeros on the imaginary axis, Williams (1989). The velocity feedback cannot asymptotically stabilize systems which have a pole at the origin.

In this paper we consider non-identifier-based adaptive stabilization of undamped flexible structures which may have a pole at the origin. In the case where velocity feedback is available, the adaptive stabilizer is constructed by an adaptive PD-controller (proportional plus derivative controller). In the case where only position feedback is available, the adaptive stabilizer is constructed by an adaptive P-controller for the augmented system which consists of the controlled system and a parallel compensator. Numerical simulation results show

## 2. Problem statement

We consider a model for the structural dynamics governed by the following second-order differential equation with $m$ inputs and $m$ outputs:

$$M\frac{d^2z(t)}{dt^2} + Az(t) = Bu(t), \tag{1}$$

$$z(0) = z_0, \quad \frac{dz(0)}{dt} = z_1, \tag{2}$$

$$y(t) = B^T z(t), \tag{3}$$

where $z$ is the $n$-dimensional vector of generalized coordinates, $u(t) \in \mathbf{R}^m$ is the $m$-dimensional control input vector, and $y(t) \in \mathbf{R}^m$ is the $m$-dimensional measurement output vector. The mass and stiffness matrices $M$ and $A$ of the structure satisfy that $M = M^T$ is positive definite, and $A = A^T$ is positive semi-definite. We assume that at least one of the initial vectors $z_0$ and $z_1$ is not zero. The system (1), (2) and (3) is used as an $n$-mode model of large flexible space structures, see Gawronski (1996), Joshi (1996), Williams (1989).

Apply the feedback control for the system (1)

$$u(t) = -\delta y(t) \quad \delta > 0, \tag{4}$$

where $\delta$ is the control gain. Then the closed-loop system becomes

$$M\frac{d^2z(t)}{dt^2} + (A + \delta BB^T)z(t) = 0, \tag{5}$$

$$z(0) = z_0, \quad \frac{dz(0)}{dt} = z_1. \tag{6}$$

We assume that for any $\delta > 0$ the matrix $(A + \delta BB^T)$ is positive definite. The closed-loop system (5) and (6) is not asymptotically stable. In order to asymptotically stabilize the system (5) and (6), velocity feedback or an $m$-dimensional parallel compensator such that

$$\frac{d\xi}{dt} = A_c\xi(t) + B_c u(t), \quad \xi(0) = \xi_0 \tag{7}$$

is necessary.

The objective of adaptive stabilization is to construct the control input $u$ such that the closed-loop system will be asymptotically stable without explicit knowledge of $M$, $A$ and $B$.

In the design of high-gain adaptive controllers, it is usually required that the system have no unstable zeros. It should be noted that our system may have poles and zeros on the imaginary axis, Williams (1989) and does not satisfy the condition.

In this paper we shall show that a non-identifier-based adaptive stabilizer can be designed for the system (1), (2) and (3) with collocated actuators and

## 3. Adaptive stabilization in the case where velocity feedback is available

In this section we shall design a non-identifier-based adaptive stabilizer for the system (1), (2) and (3) in the case where velocity $\dot{y}(t) = B^T\dot{z}(t)$ is available.

We shall consider an adaptive PD-controller

$$\begin{cases} u(t) = -K(t)y(t) - k(t)\dot{y}(t), \\ K(t) = \gamma\|y(t)\|^2 + \kappa,\ \gamma > 0,\ \kappa > 0, \\ \dot{k}(t) = r\|\dot{y}(t)\|^2,\ k(0) > 0,\ r > 0, \end{cases} \tag{8}$$

where $\|\cdot\|$ denotes the Euclidean norm and $\gamma$, $\kappa$, $r$ are design parameters. If the adaptive controller (8) is applied to the system (1), (2) and (3), the resulting closed-loop system becomes

$$M\frac{d^2z(t)}{dt^2} + k(t)BB^T\frac{dz(t)}{dt} + [A + K(t)BB^T]z(t) = 0, \tag{9}$$

$$z(0) = z_0, \quad \frac{dz(0)}{dt} = z_1,$$

$$\begin{cases} K(t) = \gamma\|B^T z(t)\|^2 + \kappa, \quad \gamma > 0,\ \kappa > 0, \\ \dot{k}(t) = r\|B^T\dot{z}(t)\|^2, \quad k(0) > 0,\ r > 0. \end{cases} \tag{10}$$

First we define the following energy-like (Liapunov-like) functions

$$E(t) = \frac{1}{2}\dot{z}(t)^T M\dot{z}(t) + \frac{1}{2}z(t)^T Az(t) + \frac{1}{4\gamma}K^2(t), \tag{11}$$

$$W(t) = E(t) + \frac{1}{2r}k^2(t). \tag{12}$$

Along the solution of the system (9) and (10) it holds that

$$\begin{aligned} \dot{E}(t) &= \langle M\ddot{z}(t), \dot{z}(t)\rangle + \langle Az(t), \dot{z}(t)\rangle + \frac{1}{2\gamma}K(t)\dot{K}(t) \\ &= -\langle Az(t) + k(t)BB^T\dot{z}(t) + K(t)BB^T z(t), \dot{z}(t)\rangle \\ &+ \langle Az(t), \dot{z}(t)\rangle + \frac{1}{2\gamma}K(t)\dot{K}(t) \\ &= -k(t)\|B^T\dot{z}(t)\|^2 - K(t)[B^T z(t)]^T[B^T\dot{z}(t)] + K(t)[B^T z(t)]^T[B^T\dot{z}(t)] \\ &= -k(t)\|\dot{y}(t)\|^2. \end{aligned}$$

Then, $E(\cdot) \in L^\infty$, and thus $\dot{z}(\cdot)$, $k(\cdot)$, $K(\cdot) \in L^\infty$. Moreover $z(\cdot) \in L^\infty$ follows from the positive-definiteness of $A + \kappa BB^T$, since $z(\cdot)^T Az(\cdot)$, $B^T z(\cdot) \in L^\infty$.

Along the solution of the system (9) and (10) we have

In our system (9) and (10) it holds that

$$\kappa \leq K(t) < \infty,\ 0 < k(0) < k(t) < \infty,\ y(\cdot) \in L^{\infty} \text{ and } \dot{y}(\cdot) \in L^{\infty}. \tag{13}$$

We have that $W(\infty) = W(0)$. That is,

$$E(\infty) + \frac{1}{2r}k^2(\infty) = E(0) + \frac{1}{2r}k^2(0).$$

Since $E(\infty) \geq 0$, it follows that

$$k(\infty) \leq \sqrt{2rE(0) + k^2(0)}.$$

Next we show that the closed-loop system (9) is asymptotically stable, that is, all solutions of (9) asymptotically converge to zero. To prove this, we use La Salle's invariance principle, La Salle and Lefschetz (1961). According to this principle, all solutions of (9) and (10) asymptotically tend to the maximal invariant subset of the following set

$$S = \{(z, \dot{z}, k) | \dot{E} = 0\}. \tag{14}$$

If $S$ contains only the solution $z = 0$ and $\dot{z} = 0$, it holds that

$$\lim_{t\to\infty} z(t) = 0 \text{ and } \lim_{t\to\infty} \dot{z}(t) = 0. \tag{15}$$

From $\dot{E} = 0$ it results in $\dot{y} = B^T\dot{z} = 0$. Then $K(t)$ is a positive constant $K_c$. This implies that

$$M\frac{d^2 z(t)}{dt^2} + (A + K_c BB^T)z(t) = 0,\quad z(0) = z_0,\quad \frac{dz(0)}{dt} = z_1, \tag{16}$$

$$\zeta(t) = B^T\dot{z}(t) = 0 \text{ for all } t \geq 0. \tag{17}$$

If the system

$$\begin{cases} M\dfrac{d^2 z(t)}{dt^2} + (A + K_c BB^T)z(t) = 0, \\ \zeta(t) = B^T\dot{z}(t) \end{cases} \tag{18}$$

is observable, Kaczorek (1992), Sinha (1984), we have that $z_0 = 0$ and $z_1 = 0$, which yields $z(t) = 0$ and $dz(t)/dt = 0$. Then, $S$ contains only the solution $z = 0$ and $\dot{z} = 0$. The closed-loop system (9) is asymptotically stable. Using Lemma 1 and Lemma 2 given below, we arrive at the following theorem:

THEOREM 1 *Suppose that the system (1), (2) and (3) is observable. Then the adaptive PD-controller (8) asymptotically stabilizes the system (1), (2) and (3).*

We can demonstrate the following lemmas:

LEMMA 1 *Suppose that $M > 0$ and $A \geq 0$. Then the system (18) is observable,* Proof. We can reformulate the system (18) as a set of first order equations. The system is equivalent to

$$\frac{d}{dt}\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -M^{-1}(A+K_cBB^T) & 0 \end{bmatrix}\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \tag{19}$$

$$\zeta(t) = [0\ B^T][z(t), \dot{z}(t)]^T.$$

The system (18) is observable, if for any complex number $s$ the equations

$$\begin{cases} sw_1 - w_2 = 0, \\ M^{-1}(A+K_cBB^T)w_1 + sw_2 = 0, \\ B^T w_2 = 0 \end{cases} \tag{20}$$

have no nonzero solution $w = \{w_1, w_2\}$, Kaczorek (1992), Sinha (1984). When $s \neq 0$, we have $B^T w_1 = 0$ and (20) becomes

$$\begin{cases} sw_1 - w_2 = 0, \\ M^{-1}Aw_1 + sw_2 = 0, \\ B^T w_1 = 0. \end{cases} \tag{21}$$

If the system (1), (2) and (3) is observable, (21) has no nonzero solution $w$ for any complex number $s$.

Next, consider the case where $s = 0$. From (20) it follows that $w_2 = 0$ and

$$M^{-1}(A+K_cBB^T)w_1 = 0.$$

Since $M > 0$, we have

$$(A+K_cBB^T)w_1 = 0.$$

From this, we obtain that

$$w_1{}^T A w_1 + K_c\|B^T w_1\|^2 = 0,$$

which implies that $B^T w_1 = 0$, since $A \geq 0$. Again we have (21). Therefore if the system (1), (2) and (3) is observable, (20) has no nonzero solution $w$ for any complex number $s$. The system (18) is observable. We have proved the lemma.

LEMMA 2 *Suppose that $M > 0$ and $A \geq 0$. Then $A + \delta BB^T$ is positive definite for any $\delta > 0$, if the system (1), (2) and (3) is observable.*

Proof. Suppose that $A + \delta BB^T$ is not positive definite for some $\delta > 0$. When $s = 0$ in (21), we have $w_2 = 0$ and

$$(A+\delta BB^T)w_1 = 0,$$

since $M > 0$. There exists a nonzero $w_1$. This provides the contradiction for

Finally, we show that the adaptive D-controller (only the velocity feedback) such that $u(t) = -k(t)\dot{y}(t)$ cannot asymptotically stabilize the system when $A$ has an eigenvalue 0. It follows from the fact that the system

$$\begin{cases} M\dfrac{d^2z(t)}{dt^2} + Az(t) = 0, \\ \zeta(t) = B^T\dot{z}(t) \end{cases} \tag{22}$$

is not observable when $A$ has an eigenvalue 0. We can reformulate the system as a set of first order equations. The system is equivalent to

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -M^{-1}A & 0 \end{bmatrix}\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \\ &y(t) = [0\ B^T][z(t),\ \dot{z}(t)]^T. \end{aligned} \tag{23}$$

If $A$ has an eigenvalue 0, the equations

$$\begin{cases} -sw_1 + w_2 = 0, \\ -M^{-1}Aw_1 - sw_2 = 0, \\ B^Tw_2 = 0 \end{cases} \tag{24}$$

have a nonzero solution $w = \{w_1, w_2\}$ for $s = 0$. In fact $w_1 = \phi_0$ ($\phi_0$ is the corresponding eigenvector of $A$ to the eigenvalue 0) and $w_2 = 0$ satisfies (21) for $s = 0$. Then, the system (19) is not observable, Kaczorek (1992), Sinha (1984). Therefore, when $A$ has an eigenvalue 0, the proportional action (that is, position feedback) is necessary to stabilize the system (1), (2) and (3).

## 4. Adaptive stabilization in the case where velocity feedback is not available

In this section we shall design an adaptive stabilizer for the system (1),(2) and (3) in the case where velocity $\dot{y}(t) = B^T\dot{z}(t)$ is not available.

We first introduce an $m$-dimensional parallel compensator

$$\frac{d\xi(t)}{dt} = -A_c\xi(t) + B_cu(t), \quad \xi(0) = 0, \tag{25}$$

where $\xi(t)$ is the $m$-dimensional compensator state vector, $A_c$ and $B_c$ are $m$-dimensional diagonal matrices such that $A_c = diag[\alpha_i], B_c = diag[\beta_i]$, $\alpha_i \geq 0$, $\beta_i > 0$ $(i = 1, \ldots, m)$.

For the augmented system (1) and (25) we apply an adaptive controller

$$\begin{cases} u(t) = -K(t)[y(t) + \xi(t)] \\ \qquad = -K(t)y_\xi(t), \end{cases} \tag{26}$$

where $\gamma$ and $\kappa$ are design parameters. The resulting closed-loop system becomes

$$
\begin{aligned}
&M\frac{d^2z(t)}{dt^2} + [A + K(t)BB^T]z(t) + K(t)B\xi(t) = 0,\\
&z(0) = z_0, \quad \frac{dz(0)}{dt} = z_1,
\end{aligned} \tag{27}
$$

$$
\frac{d\xi(t)}{dt} = -[A_c + B_cK(t)]\xi(t) - B_cK(t)B^Tz(t), \quad \xi(0) = 0. \tag{28}
$$

We define the following energy-like (Liapunov-like) function

$$
E(t) = \frac{1}{2}\dot{z}(t)^TM\dot{z}(t) + \frac{1}{2}z(t)^TAz(t) + \frac{1}{2}\xi(t)^TP\xi(t) + \frac{1}{4\gamma}K^2(t), \tag{29}
$$

where $P$ is an $m$-dimensional diagonal matrix such that $P = diag[p_i]$, $p_i > 0$ $(i = 1, \ldots, m)$ which will be determined later.

Along the solution of the system (27) and (28) it holds that

$$
\begin{aligned}
\dot{E}(t) &= -\langle Az(t) + K(t)BB^Tz(t), \dot{z}(t)\rangle - K(t)\xi(t)^TB^T\dot{z}(t)\\
&+ z(t)^TA\dot{z}(t) + \xi(t)^TP\dot{\xi}(t) + \frac{1}{2\gamma}K(t)\dot{K}(t)\\
&= -K(t)y(t)^T\dot{y}(t) - K(t)\xi(t)^T\dot{y}(t) + \xi(t)^TP\dot{\xi}(t) + K(t)y_\xi(t)^T\dot{y}_\xi(t)\\
&= -K(t)y(t)^T\dot{y}(t) - K(t)\xi(t)^T\dot{y}(t) + \xi(t)^TP\dot{\xi}(t)\\
&+ K(t)y(t)^T\dot{y}(t) + K(t)\xi(t)^T\dot{y}(t) + K(t)y(t)^T\dot{\xi}(t) + K(t)\xi(t)^T\dot{\xi}(t)\\
&= \xi(t)^T[P + K(t)I_m]\dot{\xi}(t) + K(t)y(t)^T\dot{\xi}(t)\\
&= -\xi(t)^T[P + K(t)I_m][A_c + K(t)B_c]\xi(t)\\
&- K(t)\xi(t)^T[A_c + PB_c + 2K(t)B_c]y(t) - K^2(t)y(t)^TB_cy(t).
\end{aligned}
$$

Here we take $p_i = \alpha_i/\beta_i$ $(i = 1, \ldots, m)$, that is, $P = A_cB_c^{-1}$. Then we obtain

$$
\begin{aligned}
\dot{E}(t) &= -\xi(t)^T[P + K(t)I_m]^2B_c\xi(t)\\
&- 2K(t)\xi(t)^T[P + K(t)I_m]B_cy(t) - K^2(t)y(t)^TB_cy(t)\\
&= -\{[P + K(t)I_m]\xi(t) + K(t)y(t)\}^T\\
&\times B_c\{[P + K(t)I_m]\xi(t) + K(t)y(t)\} \leq 0.
\end{aligned} \tag{30}
$$

This implies that $\dot{z}(\cdot)$, $z(\cdot)$, $\xi(\cdot)$ and $K(\cdot)$ are in $L^\infty$.

To prove that $z(t)$ and $\dot{z}(t)$ asymptotically converge to zero, we use La Salle's invariance principle, La Salle and Lefschetz (1961).

From $\dot{E} = 0$ it results in

$$
y(t) = -\frac{1}{K(t)}[P + K(t)I_m]\xi(t). \tag{31}
$$

Then

$$
u(t) = -K(t)\xi(t) + [P + K(t)I_m]\xi(t) = P\xi(t) \tag{32}
$$

$$
\frac{d\xi(t)}{dt} = -(A_c + PB_c)\xi(t) = 0, \quad \xi(0) = 0 \tag{33}
$$

This implies that $\xi(t) = 0$ and $u(t) = 0$. In this case we obtain

$$\begin{cases} M\dfrac{d^2z(t)}{dt^2} + Az(t) = 0, \quad z(0) = z_0, \quad \dfrac{dz(0)}{dt} = z_1, \\ y(t) = B^T z(t) = 0 \quad \text{for all } t \geq 0. \end{cases} \tag{34}$$

If the system

$$\begin{cases} M\dfrac{d^2z(t)}{dt^2} + Az(t) = 0, \quad z(0) = z_0, \quad \dfrac{dz(0)}{dt} = z_1, \\ y(t) = B^T z(t) \end{cases} \tag{35}$$

is observable, we have that $z_0 = 0$ and $z_1 = 0$, which yields $z(t) = 0$ and $dz(t)/dt = 0$.

We have obtained the following theorem:

THEOREM 2 *Suppose that the system (1), (2) and (3) is observable. Then the adaptive controller (26) asymptotically stabilizes the system (1),(2) and (3).*

## 5. Numerical example

In this section we give simulation results to illustrate our theory. We design the proposed controllers for a simple flexible structure – a three-mass system. The system is with masses $m_1, m_2$, and $m_3$ and stiffness $k_1, k_2, k_3, k_4$ such that

$$\begin{cases} m_1\ddot{q}_1(t) = -k_1 q_1(t) + k_2[q_2(t) - q_1(t)], \\ m_2\ddot{q}_2(t) = -k_2[q_2(t) - q_1(t)] + k_3[q_3(t) - q_2(t)], \\ m_3\ddot{q}_3(t) = -k_3[q_3(t) - q_2(t)] - k_4 q_3(t) + u(t), \\ y(t) = q_3(t), \end{cases} \tag{36}$$

where $q_1, q_2$ and $q_3$ are the displacements of the masses $m_1, m_2$, and $m_3$, respectively. The control input $u(t)$ acts at mass No. 3 and the output $y(t)$ is the displacement of mass No. 3.

The system parameters are follows: $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.6$, $k_4 = 0.3$, $q_1(0) = 0$, $\dot{q}_1(0) = 0$, $q_2(0) = 0$, $\dot{q}_2(0) = 0$, $q_3(0) = 0.5$, $\dot{q}_3(0) = 0$. In this case the mass matrix $M$ is positive definite and the stiffness matrix $A$ is positive semi-definite. It can be easily shown that the system (36) is observable. We can design the proposed controllers to the system.

In the case where velocity feedback is available, tuning the parameters $\gamma$, $\kappa$, $r$ and $k(0)$, we designed an adaptive PD-controller

$$\begin{cases} u(t) = -K(t)y(t) - k(t)\dot{y}(t), \\ K(t) = \|y(t)\|^2 + 20, \end{cases} \tag{37}$$

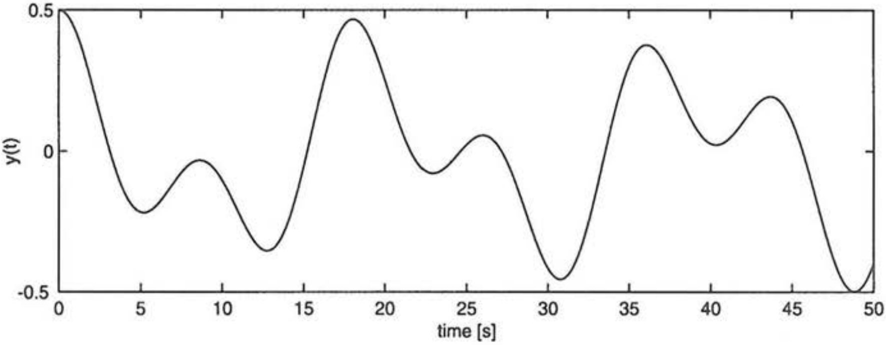


Figure 1. Open-loop output response

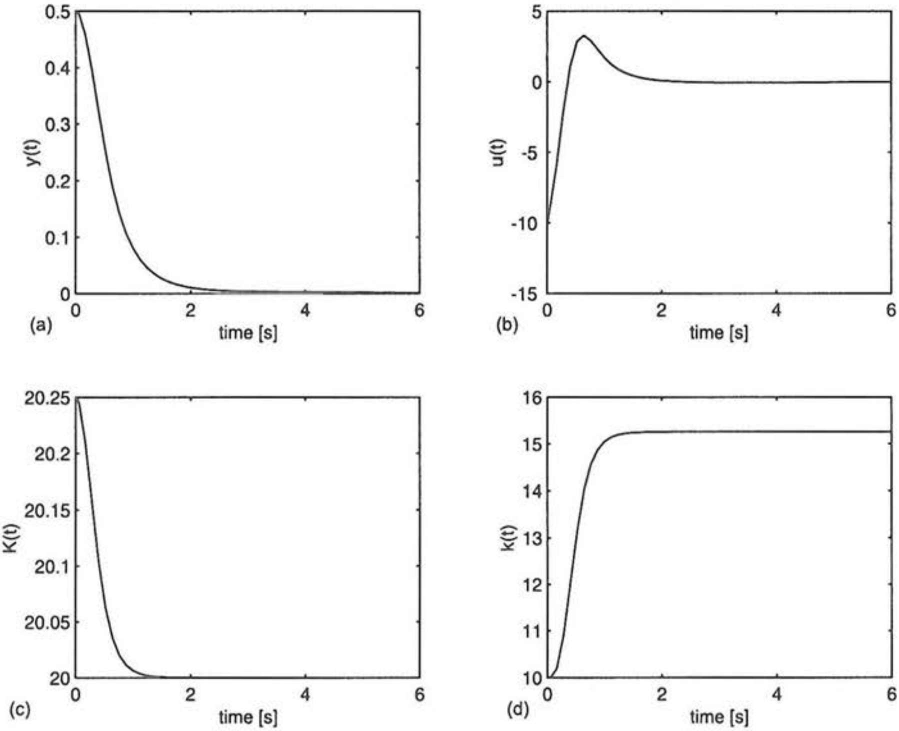


Figure 2. The case of PD-controller. (a) Controlled output $y(t)$. (b) Control input $u(t)$. (c) P-controller gain $K(t)$. (d) D-controller gain $K(t)$.

Simulation results are shown in Figs. 2a,b,c,d. For comparison purposes, the results show the effectiveness of the proposed controller (37).

In the case where velocity feedback is not available, we introduce the one-dimensional parallel compensator

$$\frac{d\xi(t)}{dt} = -0.3\xi(t) + 0.1u(t), \quad \xi(0) = 0. \tag{38}$$

For the augmented system (36) and (38), tuning the parameters $\gamma$, $\kappa$, we designed the controller

$$\begin{cases} u(t) = -K(t)[y(t) + \xi(t)] \\ \qquad = -K(t)y_\xi(t), \\ K(t) = 100\|y_\xi(t)\|^2 + 30. \end{cases} \tag{39}$$

Simulation results are shown in Figs. 3a,b,c. These simulation results show the effectiveness of the proposed controller (39).

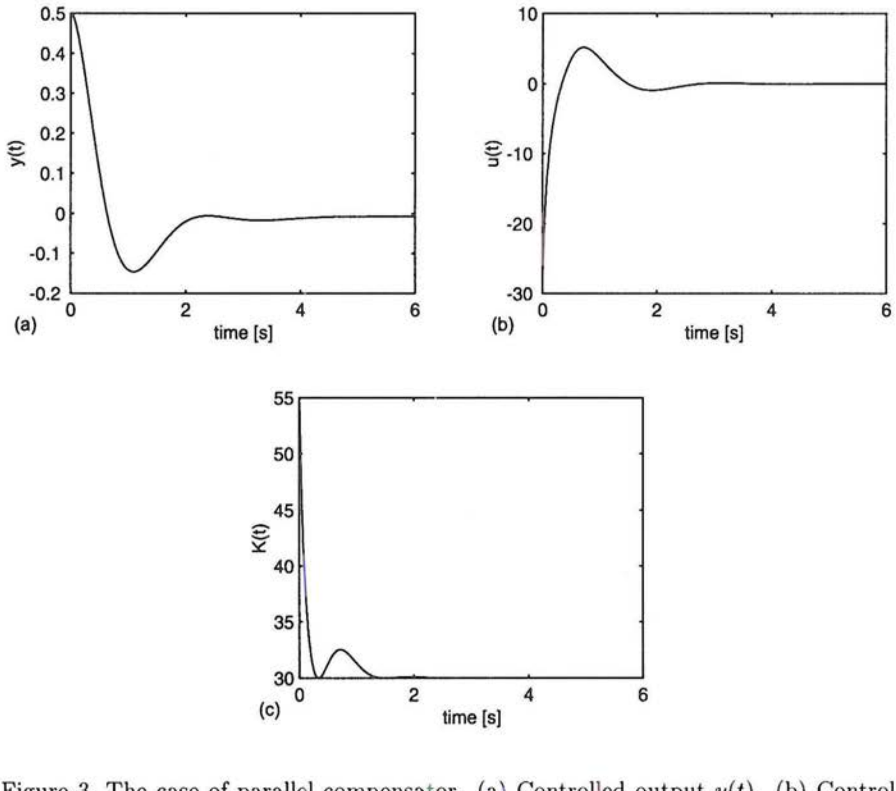


Figure 3. The case of parallel compensator. (a) Controlled output $y(t)$. (b) Control

## 6. Conclusion

We have constructed adaptive stabilizers for undamped flexible structures in the case of collocated input and output operators. The systems have poles and zeros on the imaginary axis. In the case where velocity feedback is available, the adaptive stabilizer has been constructed by an adaptive PD-controller (proportional plus derivative controller). In the case where only position feedback is available, the adaptive stabilizer has been constructed by an adaptive P-controller for the augmented system which consists of the controlled system and a parallel compensator. Numerical simulation results have showed the effectiveness of the proposed controllers.

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