The top cycle and uncovered solutions for weak tournaments

by

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Summary: In this paper we study axiomatic properties of the top cycle and uncovered solutions for weak tournaments. Subsequently, we establish its connection with the rational choice theory.
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1. Introduction

An abiding problem in choice theory has been the one of characterizing those choice functions which are obtained as a result of some kind of optimisation. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin (1985). More recently, the emphasis has focused on binary relations defined on non-empty subsets of a given set, such that the choice function coincides with the best subset corresponding to a feasible set of alternatives. This problem has been provided with a solution in Lahiri (1999), although the idea of binary relations defined on subsets is a concept which owes its analytical origins to Pattanaik and Xu (1990).

Given a binary relation, the idea of a function which associates with each set a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author. Laslier (1997) provides a very exhaustive survey of the related theory when the given binary relation is reflexive, complete and anti-symmetric.

In this paper we extend the above set of binary relations to include those which are not necessarily anti-symmetric. Such binary relations which are reflex-

as weak tournaments. Given a weak tournament, a solution is a function which associates to each non-empty subset a non-empty collection of elements from the subset, on the basis of the given weak tournament. Lucas (1992) has a discussion of abstract games and related solution concepts. The concept of an abstract game is originally due to von Neumann and Morgenstern and they are very similar to the weak tournaments that we study in this paper. Much of what is discussed in Laslier (1997) and references therein carry through into this framework. An important consequence of both these frameworks is that often a set may fail to have an element which is best with respect to the given binary relation. To circumvent this problem the concept of the top cycle set is introduced, which selects from among the feasible alternatives only those which are best with respect to the transitive closure of the given relation. The top cycle set is always non-empty and in this paper we provide an axiomatic characterization of the top-cycle solution. It is subsequently observed that the top cycle solution is the coarsest solution which satisfies two innocuous assumptions.

An alternative x is said to cover another alternative y if and only if x is preferred to y and for every other third element z if (a) y is at least as good as z, then so is x; (b) if y is preferred to z then so is x. Given any feasible set, its uncovered set is the set of all elements in the feasible set which are not covered by any other element in the same set. The question that naturally arises is the following : Given a choice function, under what condition does a binary relation exist, whose uncovered sets always coincide with the values of the choice function? This problem is answered in this paper, where instead of defining the covering relation globally, we consider the covering relation for each individual feasible set, by simply looking at the restriction of the comparison function to that set. In such a situation that fact that x covers y in a particular feasible set does not imply that x covers y globally. In effect, we are then concerned with what Sen (1997) calls 'menu based' relations. In this paper we also address the problem of axiomatically characterizing the uncovered solution (where 'covering' is now defined as a 'menu-based' concept).

In Peris and Subiza (1999) it is shown that a considerable portion of the theory developed in the context of tournaments, carry through to weak tournaments as well. Our axiomatic characterizations are, however, different from the ones available in Peris and Subiza (1999).

In a final section of this paper we revert to the context of classical rational choice theory. By exploiting the close similarity between a solution and a choice function, we discuss the necessary implications of the results established in the earlier sections of the paper, which apply to choice functions. This leads to a modest extension of the theory that has been summarized in Moulin (1985).

2. Solutions

Let X be a finite, non-empty set and given any non empty subset A of X, let

denotes the set of all non-empty subsets of X. If $A \in [X]$, then $\#(A)$ denotes the number of elements in A. If $A, B \in [X]$, then $A \subset\subset B$, is used to denote "A is a proper subset of B".

A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$; (c) transitive if $\forall x, y, z \in X$, $[(x, y) \in R \& (y, z) \in R$ implies $(x, z) \in R]$; (d) anti-symmetric if $[\forall x, y \in X, (x, y) \in R \& (y, x) \in R$ implies $x = y]$. Given a binary relation R on X and $A \in [X]$, let $R|A = R \cap (A \times A)$. Let Π denote the set of all reflexive and complete binary relations. If $R \in \Pi$, then R is called a weak tournament. Given a binary relation R, let $P(R) = \{(x, y) \in R / (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R / (y, x) \in R\}$. $P(R)$ is called the asymmetric part of R and $I(R)$ is called the symmetric part of R. Given a binary relation R on X and $A \in [X]$, let $G(A, R) = \{x \in A / \forall y \in A : (x, y) \in R\}$. Given $A \in [X]$, let $\Delta(A)$ denote the diagonal of A i.e. $\Delta(A) \equiv \{(x, x) / x \in A\}$.

The following example shows that given $R \in \Pi$ and $A \in [X]$, $G(A, R)$ may be empty:

EXAMPLE 1 *Let* $X = \{x, y, z\}$ *and let* $R = \Delta(X) \cup \{(x, y), (y, z), (z, x)\}$. *Clearly,* $G(X, R)$ *is empty.*

Given $R \in \Pi$, $A \in [X]$, *let* $T(R|A)$ *be a binary relation on* A *defined as follows:* $(x, y) \in T(R|A)$ *if and only if there exists a positive integer* K *and* $x_1, \ldots, x_K$ *in* A *with (i)* $x_1 = x$, $x_K = y$: *(ii)* $(x_i, x_i + 1) \in R \forall i \in \{1, \ldots K - 1\}$. $T(R|A)$ *is called the transitive hull of* R *in* A. *Clearly* $T(R|A)$ *is always transitive.*

Given $R \in \Pi$, $A \in [X]$, $G(A, T(R|A))$ *is called the top cycle set of* R *in* A. *Clearly,* $G(A, T(R|A))$ *is non-empty whenever* $R \in \Pi$ *and* $A \in [X]$.

Let, R *belong to* Π. *An* R-*based solution on* X *is a function* $S : [X] \to [X]$ *such that:*

(i) $\forall A \in [X] : S(A) \subset A$;
(ii) $\forall x, y \in X : x \in S(\{x, y\})$ *if and only if* $(x, y) \in R$.

Thus, in particular, $R = R^S \equiv \cup\{S(\{x, y\}) \times \{x, y\} / x, y \in X\}$

If $\forall A \in [X]$, $G(A, R)$ *is non-empty valued then the associated solution is called the* R-*based best solution on* X. *In further course of the paper, whenever there is no scope for confusion, an* R-*based solution will be simply referred to as a solution.*

The Top Cycle solution denoted $TC : [X] \to [X]$ *is defined as follows:* $\forall A \in [X] : TC(A) = G(A, T(R|A))$.

Given $R \in \Pi$, $A \in [X]$ *and* $x, y \in X$, *we say that* x *covers* y *via* R *in* A *if:*

(i) $x, y \in A$;
(ii) $(x, y) \in P(R)$;
(iii) $\forall z \in A : [(y, z) \in R$ *implies* $(x, z) \in R]$;

Given $R \in \Pi$, let $\widehat{R}(A) = \{(x, y) \in A \times A/x$ covers y via R in $A\}$. Let $UC(A) = \{x \in A/$ if $y \in A$ then $(y, x) \notin \widehat{R}(A)\}$. It is easy to see that $\forall A \in [X]$, $\widehat{R}(A)$ is a transitive binary relation on A. Thus $UC(A) \neq \phi$ whenever $A \in [X]$. Hence, (i) $\forall A \in [X] : UC(A) \subset A$; (ii) $\forall x, y \in X : x \in UC(\{x, y\})$ if and only if $(x, y) \in R$.

The solution $UC : [X] \to [X]$ is called the uncovered solution.

OBSERVATION 1 $\forall A, B \in [X]$ and $x, y \in A : [(x, y) \in \widehat{R}(A)$ and $B \subset A]$ implies $[(x, y) \in \widehat{R}(B)]$.

The proof of this observation follows immediately from the relevant definitions. Given $A \in [X]$ and $x \in X$ let $s(x, A) = \#\{y \in A/(x, y) \in P(R)\} - \#\{y \in A/(y, x) \in P(R)\}$. The Copeland solution $Co : [X] \to [X]$ is defined as follows: $\forall A \in [X] : Co(A) = \{x \in A/\forall y \in A : s(x, A) \geq s(y, A)\}$.

OBSERVATION 2 $\forall A \in [X] : Co(A) \subset UC(A)$.

Proof of Observation 2. Let $A \in [X]$ and $x \in Co(A)$. Towards a contradiction suppose $x \notin UC(A)$. Then, there exists $y \in A$, such that $(y, x) \in \widehat{R}(A)$. But then, $\{z \in A/(x, z) \in P(R)\} \subset \{z \in A/(y, z) \in P(R)\}$ and $\{z \in A/(z, y) \in P(R)\} \subset\subset \{z \in A/(z, x) \in P(R)\}$. Thus, $s(y, A) > s(x, A)$, contradicting $x \in Co(A)$. This proves the observation. ∎

The following proposition is an extension of a result valid for tournaments which is available in Laslier (1997):

PROPOSITION 1 $\forall A \in [X] : Co(A) \subset UC(A) \subset TC(A)$.

The proof is provided in the Appendix.

EXAMPLE 2 Let $X = \{x, y, z\}$ and let $R = \Delta(X) \cup \{(x, y), (y, z), (z, y), (x, z), (z, x)\}$. Now, $Co(X) = \{x\} \subset\subset \{x, z\} = UC(X) \subset\subset X = TC(X)$.

3. Axioms for the Top Cycle Solution

A solution S on X is said to satisfy:
Strong Condorcet (SC): if $\forall A \in [X] : [x \in A]$ and $[\forall y \in A \backslash \{x\} : (x, y) \in P(R)]$ implies $[S(A) = \{x\}]$;
  Expansion Independence (EI): if $\forall A \in [X] : [x \in S(A), y \in A, (y, z) \in R]$ implies $[x \in S(A \cup \{z\})]$;
  Existence of an Inessential Alternative (EIA): if $\forall A \in [X]$ with $\#(A) \geq 2$ and $\forall x \in S(A)$, there exists $y \in A$ (possibly depending on A and x) such that $x \in S(A \backslash \{y\})$.

Proof. It is clear that TC satisfies SC, EI and EIA. Hence let S be any solution that satisfies SC, EI and EIA. Let $A \in [X]$. If $\#(A)$ is one or two there is nothing to prove, since $S(A) = TC(A)$ by definition. Thus, suppose $S(A) = TC(A)$ whenever $\#(A) = 1, \ldots k$. Let $\#(A) = k + 1$. Let $x \in A$. If $\forall y \in A \backslash \{x\} : (x, y) \in P(R)$ then $S(A) = \{x\} = TC(A)$. Hence, suppose $\forall x \in A$ there exists $y \in A \backslash \{x\}$ such that $(y, x) \in R$.

Let $x \in TC(A)$. Since TC satisfies EIA, there exists $z \in A$ such that $x \in TC(A \backslash \{z\})$. By the induction hypothesis $S(A \backslash \{z\}) = TC(A \backslash \{z\})$. If $(x, z) \in R$ then by EI, $x \in S(A)$. If $(x, z) \notin R$, then, since $x \in TC(A) = G(A, T(R|A))$ there exists $w \in A$ such that $(x, w) \in T(R|A)$ and $(w, z) \in R$. Then by EI once again $x \in S(A)$. Hence, $TC(A) \subset S(A)$.

Now, suppose $x \in S(A)$ and towards a contradiction suppose $x \notin TC(A)$. By EIA there exists $z \in A$ such that $x \in S(A \backslash \{z\})$. By the induction hypothesis $S(A \backslash \{z\}) = TC(A \backslash \{z\})$. If $(x, z) \in R$ then by EI applied to TC, $x \in TC(A)$. Hence, suppose $(x, z) \notin R$. Thus, $(z, x) \in P(R)$. Let $y \in TC(A)$. Clearly, $y \neq x$. Suppose $y \neq z$. Thus $y \in A \backslash \{z\}$. Thus $(x, y) \in T(R|A)$ which combined with $y \in TC(A)$ gives us $x \in TC(A)$. Hence, $y = z$. If for some $w \in A \backslash \{x, z\}$ we had $(w, z) \in R$, then since $x \in TC(A \backslash \{z\})$ and $w \in A \backslash \{z\}$ we would get $x \in TC(A)$. Thus, $\forall w \in A : (z, w) \in P(R)$. But then, by SC, $S(A) = \{z\}$, contradicting $x \in S(A)$. Thus $x \in TC(A)$. Hence, $S(A) \subset TC(A)$. Thus, $S(A) = TC(A)$.

By a standard induction argument it now follows that $\forall A \in [X] : S(A) = TC(A)$. ■

A solution S on X is said to satisfy:

Converse Condorcet (CC): if $\forall A \in [X]$ and $x \in A : [\forall y \in A \backslash \{x\} : (y, x) \in P(R)]$ implies $[x \notin S(A)]$;

Weak Existence of an Inessential Alternative $(WEIA)$: if $\forall A \in [X]$ with $\#(A) \geq 4$ and $\forall x \in S(A)$, there exists $y \in A$ (possibly depending on A and x) such that $x \in S(A \backslash \{y\})$.

Since TC satisfies EIA it also satisfies $WIEA$. In fact, we can now prove the following:

THEOREM 2 *Let S be any solution on X which satisfies SC, CC and $WEIA$. Then, $\forall A \in [X] : S(A) \subset TC(A)$.*

Proof. Step 1: Let S be any solution on X which satisfies SC and CC. Then, $\forall A \in [X]$ with $\#(A) \geq 3 : S(A) \subset TC(A)$.

Proof of Step 1. For $\#(A) \geq 2$ there is nothing to prove since by the definition of a solution all of them agree on such sets. Hence, suppose $\#(A) = 3$. Let $A = \{x, y, z\}$ with $x \neq y \neq z \neq x$. Suppose, without loss of generality that $x \in S(A)$. If $(x, y), (x, z) \in R$, then $x \in TC(A)$. Thus, suppose without loss of generality that $(y, x) \in P(R)$. If $(z, x) \in P(R)$ then by CC, $x \notin S(A)$, contradicting what we have assumed. Hence, (x, z) must belong to R. If $(z, y) \in R$ then again

$x \in TC(A)$. If $(y, z) \in P(R)$, then by SC, $S(A) = \{y\}$, contradicting $x \in S(A)$. Thus $S(A) \subset TC(A)$.

Step 2: Let S be any solution on X such that $\forall A \in [X]$ with $\#(A) \geq 3$: $S(A) \subset TC(A)$. Suppose S satisfies SC, $WEIA$. Then, $\forall A \in [X]$: $S(A) \subset TC(A)$.

Proof of Step 2. Suppose that $\forall A \in [X]$ with $3 \leq \#(A) \leq m$: $S(A) \subset TC(A)$. Let $\#(A) = m + 1$. Thus $\#(A) \geq 4$. Let $x \in S(A)$. By $WEIA$, there exists $y \in A$ such that $x \in S(A \backslash \{y\})$. By the induction hypothesis $S(A \backslash \{y\}) \subset TC(A \backslash \{y\})$. Thus, $x \in TC(A \backslash \{y\})$. If $(x, y) \in R$, then clearly $x \in TC(A)$. Suppose $(y, x) \in P(R)$. If $\forall z \in A \backslash \{y\} : (y, z) \in P(R)$, then by SC, S(A)=\{y\}, contradicting $x \in S(A)$. Hence, there exists $z \in A \backslash \{x, y\}$ such $(z, y) \in R$. Since, $x \in TC(A \backslash \{y\})$ and $z \in A \backslash \{y\}$, $(z, y) \in R$ implies $x \in TC(A)$. Thus $S(A) \subset TC(A)$. Step 2 combined with Step 1 and a standard induction argument proves the theorem. ∎

In fact, the above proof reveals the following:

THEOREM 3 *Let S be any solution on X which satisfies SC and EIA. Then, $\forall A \in [X]$: $S(A) \subset TC(A)$.*

CC is not required once we replace $WEIA$ by EIA, since then the induction argument can begin from $\#(A) \geq 2$.

4. The uncovered solution

A solution S on X is said to satisfy:

Expansion (E): if $\forall A, B \in [X] : S(A) \cap S(B) \subset S(A \cup B)$.

It is easy to see that both TC and UC satisfy E:

(i) Let $A, B \in [X]$ and suppose $x \in UC(A) \cap UC(B)$. Towards a contradiction suppose that $x \notin UC(A \cup B)$. Hence there exists $y \in A \cup B$, such that y covers x via R in $A \cup B$. Without loss of generality suppose $y \in A$. By Observation 1, y covers x via R in A. This contradicts $x \in UC(A)$. Thus, UC satisfies E.

(ii) Let $A, B \in [X]$ and suppose $x \in TC(A) \cap TC(B)$. Towards a contradiction suppose that $x \notin TC(A \cup B)$. Hence there exists $y \in A \cup B$, such that $(x, y) \notin T(R | A \cup B)$. Without loss of generality suppose $y \in A$. Thus $(x, y) \notin T(R | A \cup B)$ implies that $(x, y) \notin T(R | A)$. This contradicts $x \in TC(A)$. Thus, TC satisfies E.

Moulin (1986) has established the following:

PROPOSITION 2 *Let S be any solution satisfying SC and E. If $[\forall A \in [X]$ with $\#(A) = 3$ we have $UC(A) \subset S(A)]$ then $[\forall A \in [X] : UC(A) \subset S(A)]$.*

Contraction (*Con*): if $\forall A \in [X]$ with $\#(A) \geq 4$, $[x \in S(A)]$ implies [there exists a positive integer $K \geq 2$ and sets $A_1, \ldots, A_K \in [A] \backslash \{A\}$ such that $(i) \cup \{A_k / k = 1, \ldots K\} = A$; (ii) $x \in \cap \{S(A_k)/k = 1, \ldots K\}]$.

Dutta and Laslier (1999) establish that UC satisfies *Con*. However, TC does not as the following example reveals:

EXAMPLE 3 *Let* $X = \{x, y, z, w\}$ *where* x, y, z, w *are all distinct. Let* $R = \Delta(X) \cup \{(x, y), (z, x), (w, x), (y, z), (w, y), (z, w)\}$. *Clearly,* $x \in TC(X)$. *Let* $A \in [X] \backslash \{X\}$, *with* $\#(A) \geq 2$. *Suppose that* $y \notin A$. *Then,* $x \notin TC(A)$. *Hence,* $x \in TC(A)$ *and* $\#(A) \geq 2$ *implies* $y \in A$. *Suppose* $x, y \in A \cap B$ *where* $A, B \in [X] \backslash \{X\}$, $A \neq B$, $A \not\subset B \not\subset A$. *Without loss of generality suppose that* $A = \{x, y, z\}$ *and* $B = \{x, y, w\}$. *Then,* $x \notin TC(B)$. *Thus,* TC *does not satisfy Con.*

A solution S on X is said to satisfy:

Tie Splitting (*TS*): if $\forall A, B \in [X]$ with $A \cap B = \phi : [A \times B \subset I(R)$ implies $S(A \cup B) = S(A) \cup S(B)]$;

Strong Type 1 Property (*ST1P*): if $\forall x, y, z \in X$; $[(y, x) \in P(R), (x, z) \in P(R), (z, y) \in R]$ implies $S(\{x, y, z\}) = \{x, y, z\}$.

Note: Let S be any solution satisfying E. If S satisfies $ST1P$ then $[\forall A \in [X]$ with $\#(A) = 3$ we have $UC(A) \subset S(A)]$.

PROPOSITION 3 *Let* S *be a solution on* X *such that* $S(A) = UC(A) \forall A \in [X]$. *Then,* S *satisfies* $SC, CC, TS, ST1P, E$ *and* Con.

Proof. We have already seen that UC satisfies E, and $SC, CC, TS, ST1P$ being easy to verify let us show that S satisfies Con. Let $A \in [X]$ with $\#(A) \geq 4$ and $x \in S(A)$. Thus, $y \in A$, $y \neq x$ implies either $[(x, y) \in R]$ or [there exists $z_y \in A$ with either $((x, z_y) \in R$ and $(y, z_y) \notin R)$ or $((x, z_y) \in P(R)$ and $(y, z_y) \notin P(R))]$. Let $A_0 = \{y \in A/(x, y) \in R\}$. Clearly, $A_0 \neq \phi$, since $x \in A_0$. Further, since there does not exist $y \in A_0$, such that y covers x via R in A_0, $x \in S(A_0)$.

Case 1. $A_0 = A$. Since $\#(A) \geq 4$, there exists $\overline{y} \in A \backslash \{x\}$ such that $A \backslash \{x, \overline{y}\} \neq \phi$. Let $A_1 = \{x, \overline{y}\}$ and $A_2 = A - \{\overline{y}\}$. Clearly, $A_1 \subset\subset A$, $A_2 \subset\subset A$ and $A_1 \cup A_2 = A$. Further, $x \in S(A_1) \cap S(A_2)$.

Case 2. $A_0 \subset\subset A$. In this case, let $A_1 = A_0$ and for $y \in A \backslash A_1$, let $A_y = \{x, y, z_y\}$. Since $\#(A) \geq 4$, $A_y \subset\subset A$ whenever $y \in A \backslash A_1$. Further, $\forall y \in A \backslash A_1 : x \in S(A_y)$. Also, $A_1 \cup (\bigcup_{y \in A \backslash A_1} A_y) = A$. Hence, S satisfies Con. ∎

LEMMA 1 *If* $\#(X) \leq 3$ *and* S *is a solution on* X *which satisfies* SC, TS *and* $ST1P$, *then* S *is the uncovered solution.*

Proof. Let S and X be as in the statement of the lemma. If $\#(X) = 1$ or 2, there is nothing to prove since $S(A) = UC(A) \forall A \in [X]$ by the definition

$S(A) = UC(A)$. Thus suppose $A = X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. If there exists $a \in X : (a, b) \in P(R) \forall b \in X$, then $S(X) = \{a\} = UC(X)$, by SC of both S and UC. Hence, suppose that $\forall a \in X$, there exists $b \in X \backslash \{a\} : (b, a) \in R$.

Case 1. $I(R) = X$. Then by TS of C and UC, $S(X) = UC(X) = X$.
Thus, without loss of generality suppose that $(x, y) \in P(R)$. Hence, by what has been mentioned before, in Case 1, $(z, x) \in R$.

Case 2. $(z, x), (y, z) \in P(R)$.
By $ST1P$, $S(X) = \{x, y, z\} = UC(X)$.

Case 3. $(z, x) \in P(R)$, $(y, z) \in I(R)$.
By $ST1P$, $S(X) = \{x, y, z\} = UC(X)$.

Case 4. $(z, x) \in I(R)$, $(y, z) \in P(R)$.
By $ST1P$, $S(X) = \{x, y, z\} = UC(X)$.

Case 5. $(z, x) \in I(R)$, $(y, z) \in I(R)$.
Thus, $\{z\} \times \{x, y\} \subset I(R)$. By TS, $S(X) = S(\{z\}) \cup S(\{x, y\}) = \{x, z\} = UC(X)$. This proves Lemma 1. ∎

A look at the proof of Lemma 1 reveals that we have essentially proved the following:

LEMMA 2 *Let S be a solution on X which satisfies SC, TS and $ST1P$. Then $\forall A \in [X]$ with $\#(A) \leq 3$, $S(A) = UC(A)$.*

The above observation follows by noting that $UC(A)$ depends on the restriction of R to A only.

Note. If in Lemma 1 (or, for that matter, in Lemma 2), we replace SC by CC and E we do not get the desired result as the following example reveals:

EXAMPLE 4 *Let $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. Let $S(X) = \{x, y\}$, where $R = \Delta(X) \cup \{(y, x), (y, z), (x, z)\}$. S satisfies CC, E, TS and $ST1P$, the last two properties being satisfied vacuously. However, $UC(X) = \{y\} \neq S(X)$. Note that S does not satisfy SC, since $(y, x), (y, z) \in P(R)$ and yet $S(X) \neq \{y\}$.*

In Dutta and Laslier (1999) we find the following property for a solution S on X:

Type One Property (T1P): $\forall x, y, z \in X : [(y, x) \in P(R), (x, z) \in P(R), (z, x) \in I(R)]$ implies $S(\{x, y, z\}) = \{x, y, z\}$.
Clearly, $T1P$ is weaker than $ST1P$. In fact, if we replace $ST1P$ by $T1P$ in Lemma 1 (or Lemma 2), we do not get the desired result as the following example reveals.

EXAMPLE 5 *Let $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. Let $S(X) = \{x\}$, where $R = \Delta(X) \cup \{(x, y), (y, z), (z, x)\}$. Clearly, S satisfies SC, TS, E, CC and $T1P$ (all vacuously). However, S violates $ST1P$, which under the present situation*

We are now equipped to prove the following theorem:

THEOREM 4 *A solution S on X is the uncovered solution if and only if S satisfies SC, TS, ST1P, E and Con.*

Proof. Proposition 3 tells us that the uncovered solution satisfies all the properties mentioned in the theorem. Hence, let S be a solution on X satisfying SC, TS, $ST1P$ and Con. Let $R \in \Pi$. By Lemma 2, $S(A) = UC(A) \forall A \in [X]$ with $\#(A) \leq 3$. Suppose $S(A) = UC(A) \forall A \in [X]$ with $\#(A) = 1, \ldots, m$, and let $B \in [X]$ with $\#(B) = m + 1$. Let $x \in S(B)$. Suppose $m + 1 \geq 4$, for otherwise there is nothing to prove. Hence, by Con there exists a positive integer K and non-empty proper subsets $B_1, \ldots, B_K$ such that $B = \bigcup_{i=1}^{K}$ and $x \in \bigcap_{i=1}^{K} S(B_i)$. Clearly $\#(B_i) \leq m$ whenever $i \in \{1, \ldots, K\}$.

By our induction hypothesis, $S(B_i) = UC(B_i) \forall i \in \{1, \ldots, K\}$. Thus, $x \in \bigcap_{i=1}^{K} UC(B_i)$, and by E, $x \in UC(B)$. Thus, $S(B) \subset UC(B)$. By an exactly similar argument with the roles of S an UC interchanged, we get $UC(B) \subset S(B)$. By a standard induction argument, the theorem is established. ∎

Note. The above theorem is not valid without E or Con.

EXAMPLE 6 *Let $X = \{x, y, z, w\}$ where all of them are distinct. Let $S(X) = \{x\}$, $S(A) = A$ if $\#(A) = 3$, where $R = \Delta(X) \cup \{(x, y), (y, z), (z, w), (w, x), (x, z), (z, x), (y, w), (w, y)\}$. S satisfies SC, $ST1P$, TS (vacuously). Further, let $A_1 = \{x, y\}$ and $A_2 = \{x, z, w\}$. $x \in S(X)$ and $x \in S(A_1) \cap S(A_2)$. Further, $A_1 \cup A_2 = X$, with $A_1 \subset\subset X$ and $A_2 \subset\subset X$. Thus, S satisfies Con. However, $UC(X) = X \neq \{x\} = S(X)$. Observe that S does not satisfy E, since $y \in S(\{x, y, z\}) \cap S(\{y, z, w\})$ but $y \notin S(X)$.*

EXAMPLE 7 *Let X be as above. Let $S(X) = \{x, y\}$, $S(A) = \{x\}$ if $x \in A$, $S(A) = A$ if $x \notin A$ where $R = \Delta(X) \cup (\{x\} \times X) \cup (\{y, z, w\} \times \{y, z, w\})$. Clearly, S satisfies SC, $ST1P$ (vacuously), TS and E. But S does not satisfy $Con : y \in S(X)$. If we take any finite number of non-empty proper subsets of X whose union is X, at least one must contain 'x' and thus its choice set cannot contain 'y'.*

5. Choice functions and extensions of rational choice theory

In this section we discuss the implications of the analysis reported in earlier sections of this paper, in the context of classical rational choice theory.

A choice function (on X) is a function $C : [X] \rightarrow [X]$ such that: $\forall A \in [X] : C(A) \subset A$. Given a choice function C, the binary relation revealed by C denoted R^C is defined as follows: $R^C \equiv \{(x, y)/x \in C(\{x, y\})\}$. Clearly, $R^C \in \Pi$. A choice function C is said to be a top-cycle choice function if

uncovered choice function if $\forall A \in [X] : C(A) = \{x \in A/ \text{ if } y \in A \text{ then } (y,x) \notin \widehat{R}^C(A)\}$, where $\widehat{R}^C(A) = \{(x,y) \in A \times A/x \text{ covers } y \text{ via } R^C \text{ in } A\}$.

A choice function C is said to satisfy:

Strong Condorcet (SC): if $\forall A \in [X] : [x \in A]$ and $[\forall y \in A\backslash\{x\} : (x,y) \in P(R^C)]$ implies $[C(A) = \{x\}]$;

Expansion Independence (EI): if $\forall A \in [X] : [x \in C(A), \; y \in A, \; (y,z) \in R^C]$ implies $[x \in C(A \cup \{z\})]$;

Existence of an Inessential Alternative (EIA): if $\forall A \in [X]$ with $\#(A) \geq 2$ and $\forall x \in C(A)$, there exists $y \in A$ (possibly depending on A and x) such that $x \in C(A\backslash\{y\})$.

As a consequence of the analysis reported earlier it follows that:

THEOREM 5 *A choice function C is a top cycle choice function if and only if C satisfies SC, EI and EIA.*

A choice function C is said to satisfy:

Converse Condorcet (CC): if $\forall A \in [X]$ and $x \in A : [\forall y \in A\backslash\{x\} : (y,x) \in P(R^C)]$ implies $[x \notin C(A)]$;

Weak Existence of an Inessential Alternative $(WEIA)$: if $\forall A \in [X]$ with $\#(A) \geq 4$ and $\forall x \in S(A)$, there exists $y \in A$ (possibly depending on A and x) such that $x \in C(A\backslash\{y\})$.

Since TC satisfies EIA it also saisfies $WIEA$. In fact we can now prove the following:

THEOREM 6 *Let C be any choice function which satisfies SC, CC and WEIA. Then, $\forall A \in [X] : C(A) \subset G(A, T(R^C|A))$.*

THEOREM 7 *Let C be any choice function which satisfies SC and EIA. Then, $\forall A \in [X] : C(A) \subset G(A, T(R^C|A))$.*

CC is not required once we replace $WEIA$ by EIA.

A choice function C is said to satisfy:
Expansion (E): if $\forall A, B \in [X] : C(A) \cap C(B) \subset C(A \cup B)$.

The relevant proposition in Moulin (1986) now translates to the following:

PROPOSITION 4 *Let C be any choice function satisfying SC and E. If $[\forall A \in [X]$ with $\#(A) = 3$ we have $\{x \in A/ \text{ if } y \in A \text{ then } (y,x) \notin \widehat{R}^C(A)\} \subset C(A)]$ then $[\forall A \in [X] : \{x \in A/ \text{ if } y \in A \text{ then } (y,x) \notin \widehat{R}^C(A)\} \subset C(A)]$.*

A choice function C is said to satisfy:
Contraction (Con): If $\forall A \in [X]$ with $\#(A) \geq 4$, $[x \in C(A)]$ implies [there exists a positive integer $K \geq 2$ and sets $A_1, \ldots, A_K \in [A]\backslash\{A\}$ such that $(i) \cup$

A choice function C is said to satisfy:

Tie Splitting (TS): if $\forall A, B \in [X]$ with $A \cap B = \phi : [A \times B \subset I(R^C)$ implies $C(A \cup B) = C(A) \cup C(B)]$;

Strong Type 1 Property $(ST1P)$: if $\forall x, y, z \in X : [(y, x) \in P(R^C), (x, z) \in P(R^C), (z, y) \in R^C]$ implies $C(\{x, y, z\}) = \{x, y, z\}$.

We now have the following theorem:

THEOREM 8 *A choice function C is an uncovered choice function if and only if C satisfies SC, TS, ST1P, E and Con.*

The analysis reported in this section reveals the close similarity between two distinct approaches to rational choice theory. In the first approach given a binary relation which reflects choice between pairs of elements, we try to axiomatically characterize solutions in terms of the given binary relation. In the second approach the binary relation that we consider is the one revealed by the choice set for pairs of elements, and we try to axiomatically characterize choice functions, in terms of the revealed binary relation.

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Appendix

Here we provide a concise proof of Proposition 1. Let us prove the following proposition (and then Proposition 1 follows as a consequence of this and Observation 2 preceding the statement of Proposition 1):

PROPOSITION. $\forall A \in [X] : UC(A) \subset TC(A)$.

Proof. Clearly the proposition holds for $\#(A) = 1$ or 2. Hence assume that the proposition holds for $\#(A) = 1, \ldots, K$ and now let $\#(A) = K + 1$. Suppose $x \in UC(A)$. If $[\forall y \in A \backslash \{x\} : (x, y) \in R]$ then $x \in TC(A)$. Hence, suppose that there exists $y \in A \backslash \{x\}$ such that $(y, x) \in P(R)$. Clearly, $x \in UC(A \backslash \{y\})$. By the induction hypothesis $x \in TC(A \backslash \{y\})$. If $[\forall z \in A \backslash \{y\} : (y, z) \in P(R)]$, then $UC(A) = \{y\}$, contradicting $x \in UC(A)$. Hence, there exists $z \in A \backslash \{y\} : (z, y) \in R$. Since $z \in A \backslash \{y\}$ and $x \in TC(A \backslash \{y\})$ clearly, $(x, z) \in T(R|A)$. Thus $(x, y) \notin T(R|A)$. Thus, $TC(A)$. The proposition now follows by induction on the cardinality of A. ∎