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Normalization of transfer matrix of linear systems by feedbacks

by

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Abstract: The transfer matrix $T(s) = C\left[I_n s - A\right]^{-1} B$ of a linear system $\dot{x} = Ax + Bu$, $y = Cx$ can be always written in the standard form $T(s) = P(s)/d(s)$, where $P(s)$ is the polynomial matrix and $d(s)$ is the minimal common denominator. The irreducible transfer matrix is called normal if every nonzero second order minor of $P(s)$ is divisible by $d(s)$. It is shown that for an unnormal transfer matrix of the controllable system there exists a state-feedback gain matrix K such that the closed-loop system transfer matrix $T(s) = C\left[I_n s - (A + BK)\right]^{-1} B$ is normal. In the case of the output-feedback the closed-loop transfer matrix $T_c(s) = C\left[I_n s - (A + BFC)\right]^{-1} B$ can be made normal only if the system is controllable and observable.

Keywords: linear system, transfer matrix, normalization, feedback.

1. Introduction

Lampe and Rosenwasser (2000), Rosenwasser and Lampe (2000) investigated the relationships between the time-domain description and the frequency-domain description. They have shown, for example, that if the normal transfer matrix is written in the standard form $T(s) = P(s)/d(s)$ ($d(s)$ is the minimal common denominator), then every second order nonzero minor of $P(s)$ is divisible by $d(s)$. It was shown in Rosenwasser and Lampe (2000) that cyclic matrices are structurally stable, i.e. if the matrix $A \in R^{n\times n}$ is cyclic (the minimal polynomial is equal to the characteristic polynomial) and $A_0 \in R^{n\times n}$ is an arbitrary matrix, there exists a positive number ε_0 such that for all $|\varepsilon| < \varepsilon_0$ the matrix $A + \varepsilon A_0$ is cyclic. Some implications of this approach for the electrical circuits have been discussed in Kaczorek (2001). The main subject of this paper is to show that for an unnormal transfer matrix of the controllable system there exists a state-feedback gain matrix K such that the closed-loop system transfer matrix $T_c(s) = C\left[I_n s - (A+BK)\right]^{-1} B$ is normal. The normalization of transfer matrix by output-feedbacks will be also considered.

2. Preliminaries

Let $R^{m\times n}$ be the set of $m \times n$ real matrices and $R^n := R^{n\times 1}$.

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where $x = x(t) \in R^n$ is the state vector, $u = u(t) \in R^m$ and $y = y(t) \in R^p$ are the input and output vectors, respectively, and $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. It is assumed that $\operatorname{rank} B = m$ and $\operatorname{rank} C = p$.

The transfer matrix of the system (1) is given by

$$T(s) = C\left[I_n s - A\right]^{-1} B \tag{2}$$

which can be written in the standard form

$$T(s) = \frac{P(s)}{d(s)} \tag{3}$$

where $P \in R^{p\times m}[s]$, $R^{p\times m}[s]$ is the set of $p \times m$ polynomial matrices and $d(s)$ is the minimal common denominator of all entries of $T(s)$.

In what follows the following elementary row or column operations will be used, see Gantmacher (1959), Kaczorek (1992–1993).

1. Multiplication of any row (column) by any nonzero number (scalar).
2. Addition of any row (column) multiplied by a polynomial to another row (column).
3. Interchange of any rows (columns).

Using elementary row and column operations we may transform any polynomial matrix $P \in R^{p\times m}[s]$ to its Smith canonical form, see Gantmacher (1959), Kailath (1980)

$$P_S(s) = \operatorname{diag}\left[i_1(s), i_2(s), \ldots, i_r(s), 0, \ldots, 0\right] \in R^{p\times m}[s] \tag{4}$$

where $i_1(s), \ldots, i_r(s)$ are monic invariant polynomials satisfying the divisibility condition $i_{k+1}(s) \mid i_k(s)$, i.e. $i_{k+1}(s)$ is divisible with zero remainder by $i_k(s)$, $k = 1, \ldots, r-1$ and $r = \operatorname{rank} P(s)$.

The invariant polynomials can be determined by the relation

$$i_k(s) = \frac{D_k(s)}{D_{k-1}(s)} \qquad (D_0(s) = 1), \qquad k = 1, ..., r \tag{5}$$

where $D_k(s)$ is the greatest common divisor of all the $k \times k$ minors of $P(s)$.

The characteristic polynomial $\varphi(s) = \det[I_n s - A]$ of the matrix $A \in R^{n\times n}$ and its minimal polynomial $\Psi(s)$ are related by Gantmacher (1959).

$$\Psi(s) = \frac{\varphi(s)}{D_{n-1}(s)}. \tag{6}$$

From (4)–(6) it follows that $\Psi(s) = \varphi(s)$ if and only if

$$D_1(s) = D_2(s) = \cdots = D_{n-1}(s) = 1. \tag{7}$$

A matrix $A \in R^{n\times n}$ satisfying (7) (or equivalently $\Psi(s) = \varphi(s)$) is called cyclic.

In what follows the following terminology will be used, see Lampe and Rosenwasser (2000), Rosenwasser and Lampe (2000).

A full rank polynomial matrix $A(s) \in R^{m\times n}[s]$, $\operatorname{rank} A(s) = \min(m,n)$ is called latent if at least one of its invariant polynomials is non-unity and it is called alatent if all its invariant polynomials are equal to 1.

A pair $A(s) \in R^{n\times n}[s]$, $B(s) \in R^{n\times m}[s]$ is called irreducible if the corresponding augmented matrix

$$[A(s), B(s)] \in R^{n\times(n+m)}[s]$$

is alatent or equivalently if

$$\operatorname{rank}[A(s), B(s)] = n \quad \text{for all } s \in \mathbf{C} \text{ (the field of complex numbers)} \tag{8}$$

A polynomial matrix is called simple if only one of its invariant polynomials is not equal to 1. A matrix $A \in R^{n\times n}$ is called cyclic if the polynomial matrix $[I_n s - A]$ is simple.

Matrices $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ satisfying (2) are called a realisation of a given $T(s) \in R^{p\times m}(s)$. The realisation is called minimal if the matrix A has minimal dimension amongst all realisation of $T(s)$. The realisation is minimal if and only if the pair (A, B) is controllable and the pair (A, C) is observable, see Kailath (1980), Kaczorek (1992–1993). A minimal realisation with a cyclic matrix A is called simple (Lampe and Rosenwasser, 2000, Rosenwasser and Lampe, 2000). The irreducible matrix (3) with $\min(m,p) > 1$ is called normal (Lampe and Rosenwasser, 2000, Rosenwasser and Lampe, 2000) if every nonzero second order minor of the polynomial matrix $P(s)$ is divisible by the polynomial $d(s)$.

The rational matrix (3) is called irreducible if $P(s_i) \neq 0$ for all $i = 1, ..., q$ where s_i is the root of the equation $d(s) = 0$.

The following theorem is a slight modification of Theorem 1 from Lampe and Rosenwasser (2000).

THEOREM 1. *Let $A(s) \in R^{n\times n}[s]$ be nonsigular. Then the matrix*

$$A^{-1}(s) = \frac{\operatorname{adj} A(s)}{\det A(s)} \tag{9}$$

is irreducible if and only if the greatest common divisor of all entries of the adjoint matrix $\operatorname{adj} A(s)$ is equal to 1 or, equivalently, the matrix $A(s)$ is simple.

Proof (compare with Lampe and Rosenwasser, 2000). It is well-known, Gantmacher (1959), that

$$\det A(s) = D_{n-1}(s)\varphi_A(s) \tag{10}$$

where $\varphi_A(s)$ is the minimal polynomial of $A(s)$.

If $\mathrm{adj}A(s) = D_{n-1}(s)\widehat{A}(s)$ then from (9) and (10) we have

$$A^{-1}(s) = \frac{\widehat{A}(s)}{\varphi_A(s)}$$

and the matrix (9) is reducible. The matrix (9) is irreducible if $D_{n-1}(s) = 1$ or equivalently the matrix $A(s)$ is simple. ∎

It can be shown that every second order nonzero minor of the polynomial matrix $P(s)$ is divisible by $d(s)$ if and only if $q(s) = d(s)$ where $q(s)$ is the Millan polynomial of $T(s)$.

In Lampe and Rosenwasser (2000), Rosenwasser and Lampe (2000) the following two theorems have been proved.

THEOREM 2. *The transfer matrix (3) is irreducible if and only if its realisation (A, B, C) is simple.*

THEOREM 3. *The transfer matrix (3) admits a simple realisation (A, B, C) if and only if the matrix $T(s)$ is normal.*

3. Normalization of transfer matrix by feedbacks

3.1. State-feedbacks

Consider the system (1) with the state-feedback

$$u = v + Kx \tag{11}$$

where $v \in R^m$ and $K \in R^{m\times n}$ is a gain matrix.

Substitution of (11) into (1a) yields

$$\dot{x} = (A + BK)x + Bv. \tag{12}$$

The transfer matrix of the closed-loop system is given by

$$T_c(s) = C\left[I_n s - (A + BK)\right]^{-1} B. \tag{13}$$

The problem of normalization of transfer matrix by state-feedbacks can be stated as follows. Given the system (1) with A not cyclic and the pair (A, C) unobservable, find a gain matrix K such that the closed-loop transfer matrix (13) is normal.

THEOREM 4. *Let the matrix of (1) be not cyclic and the pair (A, C) be unobservable. Then there exists a gain matrix K such that the transfer matrix (13) is normal if and only if the pair (A, B) is controllable.*

Proof. NECESSITY. It is well-known, Kaczorek (1992–1993), Kailath (1980), that the pair $(A + BK, B)$ is controllable if and only if the pair (A, B) is controllable. If the pair (A, B) is uncontrollable then by Theorem 3 the transfer matrix (13) is not normal. Thus if the pair (A, B) is uncontrollable then there does not exist K such that the transfer matrix (13) is normal.
SUFFICIENCY. If the pair (A, B) is controllable then there exists a non-singular matrix $T \in R^{n\times n}$ such that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \multicolumn{3}{c}{\dotfill} \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \qquad \bar{B} = TB = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix},$$

$$A_{ij} \in R^{d_i \times d_j}, \qquad B_i \in R^{d_i \times m} \tag{14a}$$

where

$$A_{ij} = \begin{cases} \begin{bmatrix} 0 \vdots I_{d_i-1} \\ \cdots\cdots \\ -a_i \end{bmatrix} & \text{for } i = j, \\ \begin{bmatrix} 0 \\ \cdots \\ -a_{ij} \end{bmatrix} & \text{for } i \neq j, \end{cases} \qquad B_i = \begin{bmatrix} 0 \\ \cdots \\ b_i \end{bmatrix},$$

$$a_{ij} = \begin{bmatrix} a_0^{ij} & a_1^{ij} & \cdots & a_{d_j-1}^{ij} \end{bmatrix}, \qquad b_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & b_{i,i+1} & \cdots & b_{im} \end{bmatrix} \tag{14b}$$

and $d_1, ..., d_m$ are the controllability indexes satisfying $\sum_{i=1}^{m_1} d_i = n$.
Let

$$\widehat{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1m} \\ 0 & 1 & \cdots & b_{2m} \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 \end{bmatrix}^{-1} \tag{15}$$

and

$$K = \begin{bmatrix} K_1 \\ k \end{bmatrix}, \qquad K_1 \in R^{(m-1)\times n}, \qquad k = [k_1 \;\; k_2 \;\; \cdots \;\; k_n] \in R^{1\times n}. \tag{16}$$

Using (14a) and (15) it is easy to verify that

$$\widetilde{B} = \bar{B}\widehat{B} = \operatorname{diag}[\tilde{b}_1, \ldots, \tilde{b}_m], \qquad \tilde{b}_i = [0 \;\cdots\; 0 \;\; 1]^T \in R^{d_i} \tag{17}$$

(T denotes the transpose).

Define

$$\bar{K} = \widehat{B}^{-1} K T^{-1} = \begin{bmatrix} -a_{n_1} + e_{n_1+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -a_{n_{m-1}} + e_{n_{m-1}+1} \\ -a_{n_m} - k \end{bmatrix} \tag{18}$$

where $n_i = \sum_{k=1}^{i} d_k$, a_{n_i} is the n_ith row of the matrix $\bar{A}$, e_i is the ith row of I_n and k is defined by (16).

Using (15), (17) and (18) it is easy to verify that

$$A_c = T(A+BK)T^{-1} = \bar{A} + \bar{B}KT^{-1} = \bar{A} + \bar{B}\widehat{B}\widehat{B}^{-1}KT^{-1} = \bar{A} + \widetilde{B}\bar{K}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ k_1 & k_2 & k_3 & \cdots & k_n \end{bmatrix}. \tag{19}$$

The matrix (19) is cyclic and k will be used to make the pair (A_c, C) observable. In Appendix (Lemma A.1) it is shown that if (A_c) has the Frobenius canonical form (19) then it is always possible to choose its entries $k_1, \ldots, k_n$ so that the pair (A_c, C) is observable. By Theorem 3 if the matrix A_c is cyclic, the pair (A_c, B) is controllable and the pair (A_c, C) is observable then the transfer matrix (13) is normal. ■

REMARK. In general case there exist many different gain matrices K normalizing the transfer matrix.

If the pair (A, B) is controllable then the gain matrix can be found by the use of the following procedure.

PROCEDURE

STEP 1. Compute a non-singular matrix T transforming the pair (A, B) to the canonical form (14) and $\bar{A}$, $\bar{B}$, $\widehat{B}$, $\widetilde{B}$.

STEP 2. Using (18) compute $\bar{K}$ and

$$K = \widehat{B}\bar{K}T \tag{20}$$

with unknown row k.

STEP 3. Choose the row vector k so that the pair (A_c, C) is observable.

STEP 4. To find the desired K substitute k (found in Step 3) into (20).

EXAMPLE 1. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0. \tag{21}$$

It is easy to check that the matrix A is not cyclic, the pair (A, B) is controllable and the pair (A, C) is unobservable.

We are looking for gain matrix $K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}$ such that the closed-loop transfer matrix (13) of the system is normal.

Using the Procedure we obtain

STEP 1. The matrices (21) have already the canonical form (14) and

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right], \qquad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 2 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$$\widehat{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \widetilde{B} = \bar{B}\widehat{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{22}$$

STEP 2. Using (18) and (22) we compute

$$\bar{K} = \begin{bmatrix} -a_2 + e_3 \\ -a_4 - k \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 1 \\ -k_1 & -k_2 & -k_3 & 2 - k_4 \end{bmatrix}$$

and

$$K = \widehat{B}\bar{K}T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 1 \\ -k_1 & -k_2 & -k_3 & 2 - k_4 \end{bmatrix}$$
$$= \begin{bmatrix} 2k_1 & 2 + 2k_2 & 1 + 2k_3 & 2k_4 - 3 \\ -k_1 & -k_2 & -k_3 & 2 - k_4 \end{bmatrix}. \tag{23}$$

STEP 3. The pair $(\bar{A}_c, C)$ with $\bar{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}$ is observable for $k_1 \neq 0$ and arbitrary k_2, k_3, k_4, since

$$\operatorname{rank} \begin{bmatrix} C \\ C\bar{A}_c \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = 4$$

for $k_1 \neq 0$ and arbitrary k_2, k_3, k_4.

STEP 4. The desired gain matrix has the form (23) with $k_1 \neq 0$ and arbitrary k_2, k_3, k_4.

3.2. Output-feedbacks

Now let us consider the system (1) with the output-feedback

$$u = v + Fy \tag{24}$$

where $F \in R^{m \times p}$ is a gain matrix.

From (1) and (24) we have

$$\dot{x} = (A + BFC)x + Bv. \tag{25}$$

The transfer matrix of the closed-loop system is given by

$$T_c(s) = C\left[I_n s - (A + BFC)\right]^{-1} B. \tag{26}$$

The problem of normalization of transfer matrix by output-feedbacks can be stated as follows. Given the system (1) with A not cyclic, the pair (A, B) controllable and the pair (A, C) observable, find a gain matrix F such that the closed-loop transfer matrix (26) is normal.

Note that if the pair (A, C) is unobservable then the pair $(A + BFC, C)$ is also unobservable and the closed-loop transfer matrix (26) is not normal for any gain matrix F. Thus, the normalization problem of transfer matrix by output-feedbacks has a solution only if the pair (A, C) is observable. If, additionally, the pair (A, B) is controllable the normalization problem is reduced to finding a gain matrix such that the closed-loop system matrix $\widehat{A}_c = A + BFC$ is cyclic. Let $K = FC$. Then, using the approach given in the proof of Theorem 4 we may find K given by (20) such that the matrix $\widehat{A}_c = A + BK$ is cyclic. By the Kronecker–Capelli theorem equation $K = FC$ has a solution F for given C and K if and only if

$$\operatorname{rank} C = \operatorname{rank} \begin{bmatrix} C \\ K \end{bmatrix}. \tag{27}$$

Therefore, the following theorem has been proved.

THEOREM 5. *Let the pair (A, B) be controllable, the pair (A, C) be observable and the matrix A of (1) be not cyclic. Then there exists a gain matrix F such that the transfer matrix (26) is normal if and only if the condition (27) is satisfied.*

If (27) holds, then by applying suitable elementary column operations to $K = FC$ we obtain

$$[K_1\ 0] = F[C_1\ 0], \qquad K_1 \in R^{m \times p}, \qquad C_1 \in R^{p \times p} \tag{28}$$

and $\det C_1 \neq 0$ since C by assumption has full row rank. From (28) we have

$$F = K_1 C_1^{-1}. \tag{29}$$

EXAMPLE 2. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 1 & 1 \end{bmatrix}. \tag{30}$$

It is easy to verify that the pair (A, B) is controllable, the pair (A, C) is observable and the matrix A is not cyclic. We are looking for a gain matrix $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ such that the closed-loop transfer matrix (26) of the system is normal.

In the same way as in Example 1 we may compute the state-feedback gain matrix and from (23) for $k_1 = 1$, $k_2 = 0$, $k_3 = -\frac{1}{2}$, $k_4 = 2$ we obtain

$$K = \begin{bmatrix} 2 & 2 & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \end{bmatrix}. \tag{31}$$

In this case the condition (27) is satisfied since

$$\operatorname{rank} C = \operatorname{rank} \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 1 & 2 & 1 & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} C \\ K \end{bmatrix}$$

$$= \operatorname{rank} \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ -1 & 0 & 1/2 & 0 \end{bmatrix} = 2.$$

By applying elementary column operations to the matrix

$$\begin{bmatrix} C \\ K \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 1 & 1 \\ \hline 2 & 2 & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \end{array}\right]$$

we obtain

$$\begin{bmatrix} C_1 & 0 \\ K_1 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \end{array}\right].$$

From (29) we obtain the desired matrix

$$F = K_1 C_1^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}.$$

4. Concluding remarks

It has been shown that for an unnormal transfer matrix of the controllable system (1) there exists the state-feedback (11) such that the closed-loop transfer matrix (13) is normal. In the general case the solution to the problem is not unique. A procedure for computation of the state-feedback gain matrix has been given and illustrated by a numerical example.

The necessary and sufficient conditions have been also established for normalization of the transfer matrix of the system (1) by output-feedbacks.

With minor modifications the considerations can be also applied to discrete-time linear systems. An extension of these considerations for singular linear systems will be presented in a next paper. An open problem is an extension of these considerations for standard and singular two-dimensional linear systems, Kaczorek (1992–1993).

I wish to thank very much Professors Lampe and Rosenwasser for their comments and fruitful discussions.

Appendix

LEMMA A.1. *If the matrix A has the Frobenius canonical form*

$$A = \begin{bmatrix} 0 \vdots I_{n-1} \\ \cdots\cdots\cdots \\ -k \end{bmatrix} \in R^{n\times n}, \qquad k = [k_1 \; k_2 \; \cdots \; k_n] \tag{A.1}$$

then for any matrix $C \in R^{p\times n}$ it is possible to choose the row vector k of (A.1) so that the pair (A, C) is observable.

Proof. It is well-known, Kaczorek (1992–1993), Kailath (1980), that the pair (A, C) is observable if and only if $\operatorname{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = n$ for all $s \in \mathbf{C}$. Using elementary row and column operations it is always possible to reduce the matrix

$$\operatorname{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & \cdots & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & s & -1 \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} & s+k_n \\ c_{11} & c_{12} & c_{13} & \cdots & c_{1n-1} & c_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{p1} & c_{p2} & c_{p3} & \cdots & c_{pn-1} & c_{pn} \end{bmatrix} \tag{A.2}$$

to the form

$$\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ p_0(s) & 0 & 0 & \cdots & 0 \\ p_1(s) & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_p(s) & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{A.3}$$

where

$$p_0(s) = s^n + k_n s^{n-1} + \cdots + k_2 s + k_1,$$
$$p_i(s) = c_{in} s^{n-1} + \cdots + c_{i2} s + c_{i1}, \qquad i = 1, ..., p.$$

Note that we may perform elementary row operations on (A.3) and choose $k_1, \ldots, k_n$ so that

$$\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad a \neq 0. \tag{A.4}$$

The matrix (A.4) for $a \neq 0$ has full column rank and the pair (A, C) is observable. ■

Note that there exist many different k such that the pair (A_c, C) is observable.

EXAMPLE. Given the unobservable pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{A.5}$$

find $k = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$ such that the pair (A_c, C) is observable, where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 & k_2 - 1 & k_3 - 2 & k_4 - 3 \end{bmatrix}.$$

Using elementary column and row operations we reduce the matrix

$$\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & \cdots & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \cdots & s & -1 \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} & s+k_n \\ c_{11} & c_{12} & c_{13} & \cdots & c_{1n-1} & c_{1n} \\ \hdotsfor{6} \\ c_{p1} & c_{p2} & c_{p3} & \cdots & c_{pn-1} & c_{pn} \end{bmatrix} \tag{A.6}$$

to the form

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ p_0(s) & 0 & 0 & 0 \\ s^3 & 0 & 0 & 0 \\ s^2 & 0 & 0 & 0 \end{bmatrix},$$

$$p_0(s) = s^4 + (3-k_4)s^3 + (2-k_3)s^2 + (1-k_2)s - k_1. \tag{A.7}$$

Next using elementary row operations we can reduce (A.7) to the form

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ (1-k_2)s - k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $k_1 \neq 0$, $k_2 = 1$ and k_3, k_4 arbitrary the matrix (A.8) has full column rank and the pair (A_c, C) is observable.

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