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Equity properties of the Shapley value as a power index

by

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Abstract: We study power indices for simple games which have the following "uniform transfer property": when only one losing coalition in a game becomes winning, worths of all other coalitions remaining unchanged, the index increases equally for all players in that coalition and decreases equally for all players not in that coalition. We show that both for superadditive simple games and for all simple games there is only one such index: the Shapley–Shubik index, the restriction of Shapley value to the class of simple games. Moreover, the proof of this fact does not even require the standard assumption of symmetry of power indices which can be replaced by a weaker equal treatment condition.

Keywords: simple game, power index, uniform transfer, Shapley value.

1. Introduction and prerequisites. Simple games

We prove in this note that in the class of all simple games the Shapley value is the only power index with the following property: when a game is modified in such a way that only one coalition changes its worth, then this modification has the same impact on indices of all players in this coalition and the same impact on indices of all players outside this coalition. This property, which we call *uniform transfer property*, clearly has some equity or fairness flavour, because it requires that the index treat equally all players whose role in the game changes in the same manner.

There are numerous axiomatic characterizations of the Shapley value, starting from the classical one by Shapley (1953); for some of the most interesting see e.g. Young (1985) or recently van den Brink (2002). Most of them, however, deal with the value defined on the class of all characteristic function games and include conditions like linearity or marginal contributions condition which are of little or no use when we work on the smaller class of simple games. In this line of research two results which are most closely related to ours are those by Myerson

(1977 and 1980) on games endowed with structures of communication among players. Myerson shows, in particular, that the Shapley value is the only "allocation rule" (depending both on the game and on communication structure) treating the players "fair" in the following sense: if a new link between two players which have not been linked previously is added to the structure, then those two players gain the same amount. For simple games, the first axiomatization was provided by Dubey (1975) using a *transfer* axiom which, however, while mathematically convenient, has no natural interpretation. Recently, Laruelle and Valenciano (2001) have independently proved a version of our Theorem 3 for a broader class of values (not just power indices), but a much narrower class of games.

A *simple game* is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ denotes the set of players and v is a *characteristic function*—any function defined on the set $\mathcal{N}$ of all *coalitions*, i.e. all subsets of N, taking values in $\{0, 1\}$ and satisfying the following conditions:

(1) $v(\emptyset) = 0, \quad v(N) = 1,$

(2) if $S \subset T$, then $v(S) \leq v(T)$ (*monotonicity*).

The value of v on T, $v(T)$, is often called the *worth* of the coalition T. Simple games form an important subclass of *cooperative (transferable utility) games*, whose characteristic functions map $\mathcal{N}$ to $\mathbf{R}$ and must satisfy only $v(\emptyset) = 0$. The set of all n-person simple games will be denoted by $\mathcal{P}_n$, and the set of all simple games with finite number of players by $\mathcal{P}^* = \bigcup_{n=1}^{\infty} \mathcal{P}_n$.

It is usual to identify a (simple) game with its characteristic function. In a simple game *winning coalitions* are those in the inverse image of 1, and the remaining ones are *losing coalitions*. A player i is *decisive* in a coalition S if and only if S is winning and $S \setminus \{i\}$ is losing. If this last condition holds for all players in S, then the coalition S is *minimal winning*. Let us denote by $W(v)$ and by $MW(v)$ the sets of all winning and of all minimal winning coalitions in v, respectively. Formally,

$W(v) = v^{-1}(1),$

$MW(v) = \{T \in \mathcal{N} : v(T) = 1 \text{ and } (S \subset T, S \neq T \Rightarrow v(S) = 0)\}.$

It is well-known that every simple game is uniquely determined by the set of its minimal winning coalitions.

We shall also denote by $D(j, v)$ the set of all coalitions in which player j is decisive (in the game v). Further on $d(j, v)$ will denote the cardinality of $D(j, v)$, and for any natural number m, $d_m(j, v)$ will denote the number of those coalitions in $D(j, v)$ which consist of exactly m players. The cardinality of a set H will be denoted by $\#H$. For brevity, we shall sometimes omit brackets in one-element sets, writing for instance $T \cup j$ instead of $T \cup \{j\}$.

Players i and j are *interchangeable* in a simple game v when $D(i, v) \cap \mathcal{N}_{-ij} = D(j, v) \cap \mathcal{N}_{-ij}$, where $\mathcal{N}_{-ij}$ is the set of all coalitions containing neither i nor j. This is equivalent to contributing the same to every coalition in $\mathcal{N}_{-ij}$. Player i is a *null player* in a game v if and only if for every coalition S, $v(S \cup i) = v(S)$. For simple games, being a null player is clearly equivalent to not being decisive in any coalition. There is also another simple equivalent condition which will be used in the sequel:

LEMMA 1. *Player i is a null player in a simple game v if and only if $i \notin \bigcup MW(v)$, i.e., if i does not belong to any minimal winning coalition in v.*

We omit the proof, which is simple.

2. Power indices and their essential properties

All terms like "winning", "losing" or "decisive" relate to the fact that simple games are widely used in political theory in analyses of various voting assemblies—legislatives, councils, shareholders in a corporation etc. It is therefore of interest to assess how "strong" the players in a simple game are in comparison one to another. Many different measures of this relative power have been proposed and discussed; they are called *power indices.* Formally, a *power index* is any function $p : \mathcal{P}^* \to \bigcup_{n=1}^{\infty}[0,1]^n$ such that for every n

(1) $p(\mathcal{P}_n) \subset [0,1]^n$ and

(2) for every $v \in \mathcal{P}_n$, $\sum_{j=1}^{n} p_j(v) = 1$.

Here, $p_j(v)$ is the jth coordinate of the vector $p(v) \in [0,1]^n$—the *individual power index* of player j in the game v.

This definition of power indices—in particular, assumption (2)—is disputed by some authors who claim that the indices should not necessarily be normalized; see e.g. Felsenthal and Machover (1998) or Laruelle and Valenciano (1999). In effect, those authors consider simply restrictions of values (usually satisfying some extra conditions) of cooperative games to the class $\mathcal{P}^*$. This approach can sometimes lead to interesting general theorems, but on the other hand it is hardly consistent with the notion of power index as a measure of relative power. In this paper we adhere to the long terminological tradition of requiring the power indices to be normalized, as for instance in Freixas and Gambarelli (1997).

Among numerous power indices, the best-known and most widely applied are the Shapley value (Shapley 1953), the Banzhaf index (Banzhaf 1965), the prenucleolus (Schmeidler 1969) and the Johnston index (Johnston 1978). For a more exhaustive survey, see Freixas and Gambarelli (1997) or Shubik (1985).

The index we shall deal with in this paper is the *Shapley value*, denoted usually by ϕ, which was originally defined for all cooperative games; it is also known as the *Shapley–Shubik index* when restricted to the class of simple games. For any simple game $v \in \mathcal{P}_n$ the Shapley value of v, $\phi(v)$, is defined by the formula

$$\phi_i(v) = \sum_{m=1}^{n} \frac{(m-1)!(n-m)!}{n!} d_m(i,v).$$

A reasonable power index clearly should have some properties justifying its use. For instance, it should equally treat players whose positions in the game are the same, it should not decrease when a player absorbs another non-null player, etc. Unfortunately, even the most commonly accepted indices do not have some of the desirable properties; the most manifest of such violations are frequently named "paradoxes". Some standard properties which are most often expected from power indices include symmetry (anonymity) and null player property.

Symmetry (**S**) : Let v, w be two n-person games. If there exists a permutation Π of the set N such that $w(S) = v(\Pi(S))$ for every coalition $S \subset N$, then $p_i(w) = p_{\Pi(i)}(v)$ for every player $i \in N$.

Null player property (**NP**) : If i is a null player in the game v, then $p_i(v) = 0$.

These two conditions are satisfied by all most common power indices—including, of course, the Shapley value—and are even sometimes included in the definition of a power index (e.g. in Felsenthal and Machover, 1995).

We propose another condition which seems quite plausible for power indices, calling it the *uniform transfer property*. It requires that whenever a simple game is modified in such a way that only one coalition changes its worth—ceteris paribus—then the resulting change of the individual indices should discriminate neither among players in that coalition nor among players outside that coalition. Stating it formally,

DEFINITION. A power index p has the uniform transfer property (**UTP**) if for every two games $v,\ v' \in \mathcal{P}^*$ whose characteristic functions differ only on one coalition

$$(v(T) = 1,\ \ v'(T) = 0,\ \ v(U) = v'(U)\ \ \forall U \neq T)$$

the following equalities hold:

$$p_i(v) - p_i(v') = p_j(v) - p_j(v') > 0 \quad \forall\ i, j \in T,$$
$$p_k(v) - p_k(v') = p_l(v) - p_l(v') < 0 \quad \forall\ k, l \notin T.$$

That is, after the modification of the game as above, all the players in T should gain the same and all the players in $N \setminus T$ should lose the same.

One motivation for **UTP** comes from considering the *transfer property* as formulated by Felsenthal and Machover (1995). An index p has the transfer property if, whenever a game w is obtained from another game v by a *transfer of power* from player i to j, then $p_i(v) > p_i(w)$ (or, equivalently, $p_j(w) > p_j(v)$; the transfer of power from i to j means that $(v(T) > w(T)$ implies $i \in T$ and $j \notin T)$ and $(v(T) < w(T)$ implies $i \notin T$ and $j \in T))$. This very reasonable and so far hardly explored property prevents power indices from exhibiting some of the most outrageous paradoxes. Now, changing the worth of only one coalition, T, from 1 to 0 is exactly a transfer of power from any player in T to any player in $N \setminus T$. Since it is only the worth of T which makes the two games distinct, it seems natural to postulate that the change of any individual's power index should depend only on whether that individual belongs to T or not. This is exactly the uniform transfer property.

The transfer property and the uniform transfer property seem unrelated at first sight. However, we show in the next section that **UTP** is a pretty restrictive condition and that, in particular, it implies the transfer property.

3. Uniqueness of the Shapley value

In this section we aim at characterizing power indices possessing the uniform transfer property first for superadditive simple games and then for all simple games. It will turn out that there is only one index with this property in both classes, namely the Shapley value.

Let us define the following partial order relation $\sqsupset$ on the set $\mathcal{P}_n$:

$$v \sqsupset v' \Leftrightarrow W(v) \supset W(v').$$

We first prove

THEOREM 1. *Let p and p' be two symmetric power indices satisfying the conditions* **NP** *and* **UTP**, *and denote by $\mathcal{Q}_{p,p'}$ the set of all simple games on which the indices p and p' differ. Then*
(i) every minimal game (according to the relation $\sqsupset$) in the set $\mathcal{Q}_{p,p'}$ has exactly two minimal winning coalitions, S , S' and $S' = N \setminus S$,
(ii) every maximal game (according to the relation $\sqsupset$) in the set $\mathcal{Q}_{p,p'}$ has exactly two maximal losing coalitions, S , S' and $S' = N \setminus S$.

Proof. Let w be a $\sqsupset$-minimal game in the set $\mathcal{Q}_{p,p'}$ and let S be some minimal winning coalition in the game w. Clearly, $S \neq N$ (when N is the only winning coalition, all symmetric indices take the same value on w). Therefore, there exists a game y such that $W(y) = W(w) \setminus \{S\}$. Since w is minimal, $p(y) = p'(y)$ and so, denoting by δ the difference $p - p'$, we have for every player i

$$\delta_i(w) = p_i(w) - p'_i(w) = (p_i(w) - p_i(y)) + (p'_i(y) - p'_i(w)),$$

and because of the **UTP** property of both p and p', each of two differences above takes the same value for all $i \in S$ and the same value for all $j \notin S$. Denote the common value of $\delta_i(w)$ for all $i \in S$ (of $\delta_j(w)$ for all $j \notin S$) by $\delta_S(w)$ ($\delta_{N \setminus S}(w)$). Now, three cases are possible:

(a) S is the *only* minimal winning coalition in w. Then, all players not belonging to S are null players in w (by Lemma 1, since they do not belong to any minimal winning coalition), and all players from S are non-null and interchangeable. Since both p and p' have the properties **S** and **NP**, we have $p_i(w) = p'_i(w) = 0$ for $i \notin S$ and $p_j(w) = p'_j(w) = 1/s$ for all $j \in S$ (where $s = \#S$), so $p(w) = p'(w)$, which is a contradiction.

(b) There exists some other minimal winning coalition T in w different from S and from $N \setminus S$. Then at least one of the sets T, $N \setminus T$—denote it by U—has non-empty intersections with both S and $N \setminus S$. (Actually, when we assume w to be superadditive, it is T that must intersect both S and $N \setminus S$).

Therefore, by **UTP** of p and p', $\delta_U(w) = \delta_S(w)$ and $\delta_U(w) = \delta_{N\setminus S}(w)$, and hence for all $i, j \in N$ we obtain $\delta_i(w) = \delta_j(w) = \delta_N(w)$. But this is possible only when $\delta_N(w) = 0$, because otherwise $\sum_{i\in N} p_i(w) \neq \sum_{i\in N} p'_i(w)$, which is incompatible with the notion of an index of power. Thus, again $p(w) = p'(w)$.

(c) There are exactly two minimal winning coalitions in w: S and $N \setminus S$. Since this is the only remaining possibility, we have proved (i).

The proof of (ii) goes exactly the same way, except that we start from the game in which all nonempty coalitions are winning and on which all symmetric indices take the same value, then take a $\sqsupset$-maximal game z in $\mathcal{Q}_{p,p'}$ and make some maximal losing coalition in z winning. ■

An immediate corollary of Theorem 1 is the characterization of all power indices satisfying symmetry, null player condition and **UTP** on the class $\mathcal{SP}^*$ of all superadditive simple games. A game is *superadditive* if the sum of worths of any two disjoint coalitions does not exceed the worth of their union. For simple games, superadditivity is equivalent to the condition that every two winning coalitions intersect.

THEOREM 2. *The Shapley value is the only symmetric power index on* $\mathcal{SP}^*$ *satisfying the* **NP** *and* **UTP** *conditions.*

Proof. It is obvious that the Shapley value satisfies symmetry and **NP**, and it is straightforward to check that it also satisfies **UTP**. (Actually, this is true for all simple games). To prove the converse, just apply Theorem 1 (i) to ϕ and to any index $p \neq \phi$ and recall that there cannot be two disjoint winning coalitions in a superadditive simple game. Thus the set $\mathcal{Q}_{\phi,p} \cap \mathcal{SP}^*$ has no minimal element, so it must be empty and we have $p = \phi$ on the whole class of superadditive simple games. ■

REMARK 1. Laruelle and Valenciano (2001) have proved a result (Theorems 2 (i) and 6 (i)) which is slightly more general than Theorem 2. They relax the **NP** condition, requiring only that the individual indices of all null players in all games be equal and smaller than all other individual indices, and assume the indices of players to sum up to the same number (not necessarily unity) in all games. Assuming **UTP** (which they call "symmetric gain-loss") they obtain that each symmetric "index" on $\mathcal{P}_n$ with those properties is of the form $\alpha \cdot \phi + \beta$, where β is the unit n-vector multiplied by the index of a null player, and α is a positive constant. However, our proof of Theorem 2 (including the proof of Theorem 1), being significantly simpler than that by Laruelle and Valenciano, easily carries through also to their case.

We are also able to show that Theorem 2 generalizes to all simple games. However, the proof of this fact is harder and makes use of some additional lemmata.

To this end, let us introduce some more notation. Denote for any nonempty coalition $U \subset N$:

$$\begin{aligned} \Xi_U^+ &= \{v \in \mathcal{P}_n \colon U \in MW(v),\ N \setminus U \in W(v)\}, \\ \Xi_U^- &= \{v \in \mathcal{P}_n \colon U \in MW(v),\ N \setminus U \notin W(v)\}; \end{aligned}$$

and for any game $v \in \Xi_U^+ \cup \Xi_U^-$ (i.e., such that $U \in MW(v)$)

v_{-U} = the game obtained from v by changing the worth of U:

$W(v_{-U}) = W(v) \setminus \{U\}$.

LEMMA 2. *If y, z are two games in Ξ_S^+ or in Ξ_S^- and the power index p satisfies* **UTP**, *then for every $k \in N$*

$$p_k(y_{-S}) - p_k(y) = p_k(z_{-S}) - p_k(z).$$

Proof. Fix a player $i \in S$ and for any game $v \in \Xi_S^+ \cup \Xi_S^-$ denote by $\epsilon(v, S)$ the difference $p_i(v) - p_i(v_{-S})$. By the definition of **UTP**,

$$\begin{aligned} p_k(v_{-S}) &= p_k(v) - \epsilon(v, S) && \text{for } k \in S, \\ p_k(v_{-S}) &= p_k(v) + \frac{s\epsilon(v, S)}{n-s} && \text{for } k \notin S \end{aligned}$$

(where $s = \#S$), and

$$\begin{aligned} p_k(y_{-S}) &= p_k(y) - \epsilon(y, S) && \text{for } k \in S, \\ p_k(y_{-S}) &= p_k(y) + \frac{s\epsilon(y, S)}{n-s} && \text{for } k \notin S. \end{aligned}$$

We need to show that $\epsilon(v, S) = \epsilon(y, S)$ for any $y, v \in \Xi_S^+$ and for any $y, v \in \Xi_S^-$. We present the proof for $y, v \in \Xi_S^+$; the proof for $y, v \in \Xi_S^-$ is analogous.

When the characteristic functions of v and y differ only on one coalition T distinct from S, it is clear that T must be minimal winning in v or in y. We may assume without loss of generality that $T \in MW(y)$; then $v(T) = 0$, $y(T) = 1$, i.e., $v = y_{-T}$. Now the game v_{-S} is derived from both v and y_{-S} by making exactly one minimal winning coalition—respectively, S and T—losing. Thus, $v_{-S} = (y_{-T})_{-S} = (y_{-S})_{-T}$ and so

$$\begin{aligned} p_k(v_{-S}) &= p_k(y_{-S}) - \epsilon(y_{-S}, T) && \text{for } k \in T, \\ p_k(v_{-S}) &= p_k(y_{-S}) + \frac{t\epsilon(y_{-S}, T)}{n-t} && \text{for } k \notin T, \end{aligned}$$

and since v is derived from y in the same way, also

$$\begin{aligned} p_k(v) &= p_k(y) - \epsilon(y, T) && \text{for } k \in T, \\ p_k(v) &= p_k(y) + \frac{t\epsilon(y, T)}{n-t} && \text{for } k \notin T \end{aligned}$$

where $t = \#T$ and $\epsilon(w,T) = p_j(w) - p_j(w_{-T})$ for any game w with $T \in MW(w)$, $j \in T$. Using the four above equations, we can compute $p(v_{-S})$ in two ways to obtain

$$p_k(v_{-S}) - p_k(y) =$$

$$-\epsilon(v,S) - \epsilon(y,T) = -\epsilon(y_{-S},T) - \epsilon(y,S) \qquad \text{when } k \in S \cap T, \tag{1}$$

$$-\epsilon(v,S) + \frac{t\epsilon(y,T)}{n-t} = \frac{t\epsilon(y_{-S},T)}{n-t} - \epsilon(y,S) \qquad \text{when } k \in S \setminus T, \tag{2}$$

$$\frac{s\epsilon(v,S)}{n-s} - \epsilon(y,T) = -\epsilon(y_{-S},T) + \frac{s\epsilon(y,S)}{n-s} \qquad \text{when } k \in T \setminus S, \tag{3}$$

$$\frac{s\epsilon(v,S)}{n-s} + \frac{t\epsilon(y,T)}{n-t} = \frac{t\epsilon(y_{-S},T)}{n-t} + \frac{s\epsilon(y,S)}{n-s} \qquad \text{when } k \notin (S \cup T). \tag{4}$$

Moreover, $T \setminus S \neq \emptyset$ (since $S, T \in MW(y)$) and $T \neq N \setminus S$ (because $v \in \Xi_S^+$). Therefore at least one of the sets $S \cap T$, $N \setminus (S \cup T)$ must be nonempty. This guarantees that (3) holds non-vacuously and so does (1) or (4). Subtracting (1) from (3) yields $\epsilon(y,S) = \epsilon(v,S)$ directly, and subtracting (4) from (3) gives $\epsilon(y,T) = \epsilon(y_{-S},T)$ and so $\epsilon(y,S) = \epsilon(v,S)$.

When z and y differ on more than one coalition, we repeatedly apply the above argument for pairs of "neighbouring" games. Since every game $y \in \Xi_S^+$ can be obtained from any other $z \in \Xi_S^+$ by successive adding or removing one minimal winning coalition (different from S and from $N \setminus S$), we can prove $\epsilon(y,S) = \epsilon(z,S)$ for any $y, z \in \Xi_S^+$. ■

By applying once again the definition of **UTP**, Lemma 2 gives the following direct

COROLLARY. *For every coalition $S \in \mathcal{N}$ there exist two numbers $\epsilon^+(S)$ and $\epsilon^-(S)$ such that $\epsilon(v,S) = \epsilon^+(S)$ for every $v \in \Xi_S^+$, $\epsilon(v,S) = \epsilon^-(S)$ for every $v \in \Xi_S^-$.*

LEMMA 3. *When $\#S = s$ such that $n/2 \leq s < n$,*

$$\epsilon^-(S) = \frac{(s-1)!(n-s)!}{n!} \qquad \text{and} \qquad \epsilon^+(N \setminus S) = \frac{(n-s-1)!s!}{n!}.$$

Proof. Let us consider the game v^{s+1} with $W(v^{s+1}) = \{T\colon \#T > s\}$ and the game z with $W(z) = \{S\} \cup \{T\colon \#T > s\}$. Obviously $v^{s+1} = z_{-S}$. When $s \geq n/2$, both these games clearly are superadditive, so by Theorem 2 $p(v^{s+1}) = \phi(v^{s+1})$ and $p(z) = \phi(z)$. Since obviously $v^{s+1} \in \Xi_S^-$, we have

$$\epsilon^-(S) = p_i(z) - p_i(v^{s+1}) = \phi_i(z) - \phi_i(v^{s+1}) = \frac{(s-1)!(n-s)!}{n!}$$

for any player $i \in S$. For the second equality, consider the games v^{n-s} with $W(v^{n-s}) = \{T\colon \#T \geq n-s\}$ and $y = (v^{n-s})_{-N\setminus S}$ with $W(y) = \{T \neq$

$N \setminus S\colon\ \#T \geq n-s\}$. Clearly, for $s \geq n/2 \quad v^{n-s} \in \Xi^{+}_{N\setminus S}$. Moreover, since every game $\underline{w}_T$ having exactly two maximal losing coalitions, T and $N \setminus T$ (where $\emptyset \neq T \neq N$), has many $(n-s)$-element losing coalitions, it cannot satisfy $\underline{w}_T \sqsupset y$. Thus, by Theorem 1 (ii), for every maximal element $\underline{w}$ of $\mathcal{Q}_{p,\phi}$ $\underline{w} \not\sqsupset y$, so neither y nor v^{n-s} can belong to $\mathcal{Q}_{p,\phi}$. Therefore $p(v^{n-s}) = \phi(v^{n-s})$ and $p(y) = \phi(y)$ and so (for any $i \in N \setminus S$)

$$\epsilon^{+}(N \setminus S) = p_i(v^{n-s}) - p_i(y) = \phi_i(v^{n-s}) - \phi_i(y) = \frac{(n-s-1)!\,s!}{n!}. \qquad \blacksquare$$

THEOREM 3. *The Shapley value is the only symmetric power index on $\mathcal{P}^*$ satisfying the* **NP** *and* **UTP** *conditions.*

Proof. As before, let p be an index with the required properties different from ϕ. We know from theorem 1 (i) that any $\sqsupset$-minimal game for which $p \neq \phi$ must have exactly two minimal winning coalitions, S and $N \setminus S$. Denote this game by $\overline{w}_S$.

Assume first that $s \geq n/2$. The game $\overline{w}_S$ is obtained from the game $\widetilde{w}_S$, in which winning coalitions are exactly proper supersets of S and of $N \setminus S$, by adding two minimal winning coalitions, S and $N \setminus S$. Thus for each k in S

$$\begin{aligned} &p_k(\widetilde{w}_S) - p_k(\overline{w}_S) = -\epsilon^{+}(S) + \frac{n-s}{s} \cdot \epsilon^{-}(N \setminus S) \\ &= \frac{n-s}{s} \cdot \epsilon^{+}(N \setminus S) - \epsilon^{-}(S) \end{aligned}$$

and so, by Lemma 3,

$$p_k(\widetilde{w}_S) - p_k(\overline{w}_S) = \frac{n-s}{s} \cdot \frac{(n-s-1)!\,s!}{n!} - \frac{(s-1)!\,(n-s)!}{n!} = 0.$$

But $\overline{w}_S$ is minimal in $\mathcal{Q}_{\phi,p}$, so the indices p and ϕ must coincide on the game $\widetilde{w}_S$ and—by the above equality—also on the game $\overline{w}_S$.

Thus for $\#S \geq n/2$ the game $\overline{w}_S$ cannot belong to $\mathcal{Q}_{\phi,p}$, which also obviously implies the analogous statement for $\#S < n/2$. By combining this with Theorem 1 (i) we obtain that the set $\mathcal{Q}_{\phi,p}$ has no minimal element, and therefore it must be empty. $\blacksquare$

REMARK 2. It is worthwhile to notice that the signs of differences in the definition of **UTP**, introduced there in order to stress the intuitive link between **UTP** and transfer property, have not been used anywhere in this section. The null player property and symmetry combined with equations in **UTP** are sufficient to assure the desired signs of $p_k(v) - p_k(v')$.

4. Replacing symmetry by equal treatment

While assuming symmetry ("anonymity") is standard when working with power indices, it is also interesting to investigate whether "full" symmetry is necessary for the results or a weaker condition of *equal treatment* is sufficient, as it has been shown in quite general settings for values (Malawski 2002). In view of the proofs of theorems in the preceding section, the answer turns out to be quite simple in our case. It suffices to observe that in all proofs only equal treatment property—i.e., indices of interchangeable players being equal—has been used instead of symmetry. Actually, the earlier version of Lemma 3 (Malawski 1999) has been strenghtened in this paper to eliminate the only use of symmetry in the old proof of Theorem 3—the case $s = n/2$, for which symmetry (but not equal treatment) directly implies the equality $p(\overline{w}_S) = \phi(\overline{w}_S)$.

We can therefore re-state Theorems 2 and 3 in the following form:

DEFINITION. A power index p has the equal treatment property (**ET**) if the indices of interchangeable players in any game are equal, i.e., if

$$(\forall S \text{ s.t. } i, j \notin S \;\; v(S \cup i) = v(S \cup j)) \;\; \Rightarrow \;\; p_i(v) = p_j(v).$$

THEOREM 4. *The Shapley value is the only power index on $\mathcal{SP}^*$ and on $\mathcal{P}^*$ satisfying the conditions* **ET**, **NP** *and* **UTP**.

The assumptions of this theorem combine two natural aspects of "equity" for a power index: **ET** requires that the index treat equally players who play the same roles in a game (interchangeable players), and **UTP** postulates that it react equally to changes which affect the players' roles in the same way. Therefore, in our opinion, Theorem 4 offers a particularly strong support for the Shapley value as a power index.

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