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A control algorithm applied to polarization states of light in the nonlinear optical resonator

by

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Abstract: In this paper a control algorithm is presented for a nonlinear system defined by a nonlinear mapping $F : \mathbf{R}^n \mapsto \mathbf{R}^n$ of input states to output states of the system, for the problem of motion in the $\mathbf{R}^n$ space of output states along the given trajectory. The application of this algorithm in nonlinear optics is presented, for the control of light polarization transmitted through the Fabry-Pérot resonator filled with a material with nonlinear optical properties. The calculated trajectory in the space of polarization states of output light leads to the phenomenon of optical bistability, i.e. sudden changes of output polarization induced by small variations of the input parameters, and the dependence of the current state of light polarization on the history of the system.

Keywords: inverse modelling, nonlinear systems, optical bistability.

1. Introduction

In recent years the physicists working in the field of nonlinear optics, Boyd (1992), began to investigate the phenomena connected with vector properties of light, i.e. when polarization of light is important for the results of experiments. Earlier, only phenomena which fall within the scope of the scalar theory were taken into account, when the intensity of light was the only parameter describing the state of the light beam.

Particularly, the field of nonlinear optics includes the phenomenon of optical bistability, Gibbs (1985), in which the light transmitted through the nonlinear optical resonator changes its intensity in a nonlinear way. The phenomenon of hysteresis is observed, when the intensity of light outgoing from the resonator is not a unique function of the intensity of the incident light, but depends on its previous values, and is subject to frequent sudden changes with the small change of the value of input intensity.

The theory of this phenomenon is relatively simple. In order to determine the theoretical dependence of the intensity of the outgoing light x on the intensity of the incoming light y one has to invert the graph of the function $y = F(x)$, which determines the transmittance properties of the resonator. This function is determined by a solution to the appropriate Maxwell equations with nonlinear terms, which describe the propagation of light in the media with nonlinear optical properties, together with the boundary conditions, which determine the reflection of light from the mirrors of the resonator.

In the case, when in the experiment the polarization of light is taken into account, the states of input and output light beams y and x are described not by one real number, but by three so called Stokes parameters, Born and Wolf (1965). Therefore, x and y are three-dimensional real vectors, and we will denote them in the following by $\vec{x}$ and $\vec{y}$. There is no simple mathematical receipt for obtaining the inverse of a function $\vec{y} = F(\vec{x})$, where $\vec{x}, \vec{y} \in \mathbf{R}^3$. Since in the experiment it is convenient to fix the polarization of the incident beam, and only its intensity is varied, the problem is reduced to the determination of the image $\vec{x}(r)$ of a given trajectory $\vec{y}(t)$, with the known nonlinear mapping $F : \vec{x} \mapsto \vec{y}$.

This problem belongs to the field of the so called "inverse modelling" which is widely used e.g. in geophysics, oceanology and climatology, where such values of parameters of a given mathematical model are looked for, which correspond the best to the real situation described by that model. For example, having a certain model of the Earth's climate one tries to find the parameters which correspond in the best manner to the observed values of atmospheric pressure and temperature. In our case, we will seek to determine the whole set of states (trajectory in the phase space of the system). Below, a certain numerical algorithm to solve this problem will be presented.

2. An algorithm for determining the control of the system described by a nonlinear mapping

The input state of the system is described by a vector $\vec{x}$ composed of n real numbers. The output state of the system is described by a vector $\vec{y}$ composed of m real numbers. Below, because of the application considered, we will assume $n = m$. This assumption is not crucial to the working of the algorithm. The output state is uniquely determined by the input state, by means of a certain nonlinear mapping:

$$\vec{y} = F(\vec{x}) . \tag{1}$$

The problem considered is to control the system, i.e in this case to select such an input state, which corresponds to the required output state.

The definition of "control" used in this paper is somewhat different from that of the mathematical control theory. It is motivated by the application of

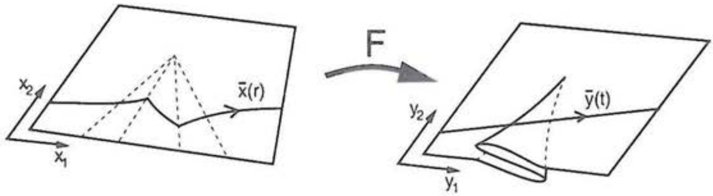

Figure 1. A nonlinear non-monotonic mapping F for a system with two degrees of freedom, and the corresponding trajectories in the spaces of input and output states.

an algorithm which will be presented here, to physical experiments in optics, where one "controls" or "adjusts" the input laser light in order to obtain the required polarization and intensity of the output laser light.

More precisely, for a given trajectory $\vec{y} = \vec{y}(t)$, we want to determine the corresponding trajectory $\vec{x} = \vec{x}(r)$, such as to fulfill the equation:

$$\vec{y}(t) = F(\vec{x}(r)) , \tag{2}$$

where t and r are the parameters of the respective trajectories (see Fig. 1).

Because the mapping F is in general not single-valued, the inverse mapping F^{-1} may not exist, and the determination of the appropriate trajectory $\vec{x}(r)$, which should satisfy Eq. (2) may not be straightforward, because in such a case we cannot apply $F^{(-1)}$ to $\vec{y}(t)$.

In the one-dimensional case the solution is simple, it is sufficient to invert the graph of the mapping F. As can be seen from Fig. 2, control of the system is affected by the non-monotonic character of the mapping F. It leads to the phenomenon of bistability (multistability), i.e. sudden changes of the output state of the system caused by a small variation of the input state. The current state of the system depends on its history, i.e. for the motion along the trajectory $\vec{y}(t)$ the resultant trajectory $\vec{x}(r)$ is different from the motion in the opposite direction $\vec{y}(-t)$ (see Fig. 2).

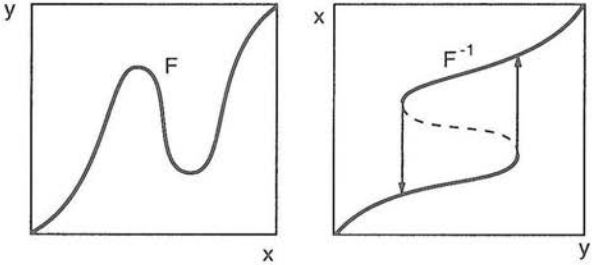

Figure 2. A mapping F describing a system with one degree of freedom and its inverse leading to the bistability of the system.

The proposed algorithm of control of the system given by the maping F proceeds as follows:

1. Create an objective function:
$$\widetilde{f}(\vec{x}) = f(F(\vec{x})) \tag{3}$$
where the function $f(\vec{y}) \geq 0$ determines the deviation of the point $\vec{y}$ from the required trajectory $\vec{y}(t)$.
2. Move in the space of input states $\vec{x}$ in such a way as to minimize the value of the objective function $\widetilde{f}(\vec{x})$.

The approximate position of the next point $\vec{x}^{(0)}$ is determined by extrapolation of positions of previous points, i.e. the curve which approximates the trajectory is extrapolated at length λ, which is the length of the single step along the trajectory $\vec{x}(r)$. To determine the exact placement of the next point $\vec{x}$ the procedure of minimization of the objective funtion $\widetilde{f}(\vec{x})$ is called, with the starting point $\vec{x}^{(0)}$ (see Fig. 3).

At these points, where the mapping F is non-invertible, i.e. when the Jacobian $\det F = |\partial\vec{y}/\partial\vec{x}|$ vanishes, the trajectory $\vec{x}(r)$ may not be differentiable and can sharply change its direction. In this case, the expected place of the next point $\vec{x}^{(0)}$ on the trajectory is shifted by a random displacement vector $\vec{\beta}$ of the length λ, in order to place the next point $\vec{x}$ on the second branch of the trajectory $\vec{x}(r)$ (see Fig. 4).

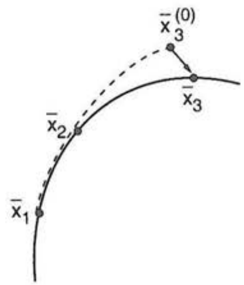

Figure 3. The expected placement $\vec{x}^{(0)}$ of the point x on the trajectory $\vec{x}(r)$ obtained by a polynomial extrapolation.

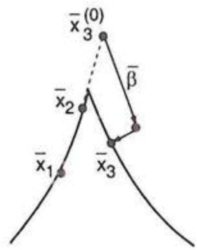

Figure 4. The random displacement $\vec{\beta}$ of the expected placement of a point on the trajectory near the singularity of the mapping.

3. Bistable changes of polarization of light transmitted through the nonlinear resonator

The nonlinear system that we consider is the beam of a laser light transmitted through the optical resonator (cavity) filled with the medium with nonlinear optical properties, Petykiewicz (1992). The state of the system is the polarization state of light, i.e. the vector of the electric field $\vec{E}$ of the light wave. The state of light which is completely polarized is described by three parameters:

two real Cartesian components E_x and E_y of the electric field vector $\vec{E}$, which is perpendicular to the direction of propagation along the z axis, and the phase shift $e^{i\phi}$ between them. Two complex Cartesian components E_x and E_y of the electric field are often utilized, in which case their common phase factor has no physical meaning. Then, the so called matrix of coherence $\mathcal{J}$ is introduced, which is defined by Born (see Born and Wolf, 1965):

$$\mathcal{J}_{ij} = E_i E_j^* \qquad \text{where } i, j = x, y \tag{4}$$

and then three Stokes parameters follow, which describe the polarization of light in the most convenient way:

$$\begin{aligned} s_1 &= \frac{1}{4}(\mathcal{J}_{xy} + \mathcal{J}_{yx}) \\ s_2 &= \frac{i}{4}(\mathcal{J}_{xy} - \mathcal{J}_{yx}) \\ s_3 &= \frac{1}{4}(\mathcal{J}_{xx} - \mathcal{J}_{yy}). \end{aligned} \tag{5}$$

The fourth Stokes parameter $s_0 = (\mathcal{J}_{xx} + \mathcal{J}_{yx})/4$ depends on the other three: $s_0 = \sqrt{s_1^2 + s_2^2 + s_3^2}$, and is equal to the intensity of light.

The input and output states of our system are hence described by vectors of three polarization parameters $\vec{s} = (s_1, s_2, s_3)$ from the space $\mathbf{R}^3$.

We assume that the light beam passes through the resonating cavity, i.e through the system of two half-reflecting mirrors and is subject to partial reflection (see Fig. 5). As a result of multiple reflections, there arises a system

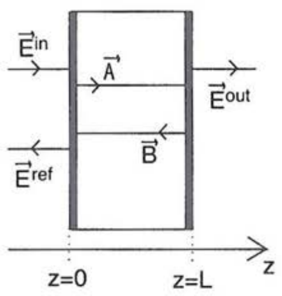

Figure 5. The nonlinear optical resonance cavity with the system of light waves running through it.

of two running waves in the cavity, one of them, with the electric field vector $\vec{A}(z)$, is propagating in the direction of the z axis, and the second with the electric field vector $\vec{B}(z)$, propagates in the opposite direction. The boundary conditions on the boundary planes of the cavity of the length L can be written as follows (Petykiewicz, 1992):

$$\vec{E}^{\,\mathrm{ref}} = r\vec{E}^{\,\mathrm{in}} \qquad \vec{A}(0) = t\vec{E}^{\,\mathrm{in}} + r\vec{B}(0) \qquad \text{for } z = 0 \tag{6}$$

$$\vec{E}^{\,\rm out} = t\vec{A}(L) \qquad \vec{B}(L) = r\vec{A}(L) \qquad \text{for } \; z = L\,, \tag{7}$$

where r and t denote the respective reflection and transmission amplitude coefficients of the mirrors, $\vec{E}^{\,\rm in}$, $\vec{E}^{\,\rm out}$ and $\vec{E}^{\,\rm ref}$ denote the electric field vectors of the incident, the transmitted and the reflected wave, respectively.

Let us note that it follows from the boundary conditions (6) and (7) that the output state, i.e vector $\vec{E}^{\,\rm out}$, uniquely determines the input state, i.e vector $E^{\rm in}$. In the opposite direction there is no such unique mapping, because to determine the output state from the input state it would require to know the reflected wave $\vec{E}^{\,\rm ref}$, which is usually not measured in the experiment.

In the experiment, the polarization state of the input laser beam is usually fixed and its intensity is varied, and the changes of polarization and intensity of the output wave are registered. Let us note that due to the peculiar character of the boundary conditions (6) and (7), we will use the above described control algorithm to change the output state according to the required trajectory (fixed polarization) at input.

We assume that the optical cavity is filled with a medium with nonlinear optical properties. Then, the evolution of the polarization state of the light transmitted through such a cavity is described by a set of nonlinear ordinary differential equations, derived from Maxwell's equations in the customary approximations of nonlinear optics. For an example, we will consider a cavity filled with the magnetic superlattice with the antiferromagnetic spiral structure. In this case the nonlinear differential equations describing propagation of light in the cavity are the following, Wierzbicki and Kociński (1999):

$$\begin{aligned} \frac{dA_1}{dz} &= -i\,\Gamma_1 A_1^* A_2^2 \\ \frac{dA_2}{dz} &= -i\,\Gamma_2 A_2^* A_1^2 \\ \frac{dB_1}{dz} &= \frac{dB_2}{dz} = 0\,, \end{aligned} \tag{8}$$

where A_1, A_2, B_1 and B_2 are the so called circular components of the electric field vectors $\vec{A}$ and $\vec{B}$ of two waves running in the cavity:

$$\begin{aligned} A_{x,y} &= A_1 \pm iA_2 \\ B_{x,y} &= B_1 \pm iB_2\,, \end{aligned} \tag{9}$$

and Γ_1 and Γ_2 are material constants. The medium considered here is a kind of an "optical valve", because only the wave running forward is the subject of nonlinear interaction. This peculiar phenomenon is caused be the spiral structure of this medium.

The nonlinear mapping $F : \mathbf{R}^3 \mapsto \mathbf{R}^3$ from Eq. (1) is determined in the following steps:

1. A vector of Stokes parameters $\vec{s}^{\,\rm out}$ of output light polarization is given.

2. It is transformed to two circular components $\vec{E}_1^{\rm out}$ and $\vec{E}_2^{\rm out}$.
3. From boundary conditions (7) we determine the circular components $A_1(L)$, $A_2(L)$, $B_1(L)$ and $B_2(L)$ of waves running inside the resonator, at the second mirror (for $z = L$).
4. Numerical integration of Eqs. (8) is carried out, from the point $z = L$ backwards to the point $z = 0$, and thus the values $A_1(0)$, $A_2(0)$, $B_1(0)$ and $B_2(0)$ inside the resonator at the first mirror are determined.
5. From the boundary conditions (6) we determine the circular components $E_1^{\rm in}$ and $E_2^{\rm in}$ of the wave incident at the resonator.
6. From them we calculate the vector of Stokes parameters $\vec{s}^{\rm in}$ of the incident beam polarization.

To obtain the description of a real experimental situation, we fix the polarization of the incident beam, e.g. we choose the linear polarization, where the electric field vector vibrates in a fixed plane. For the xz plane: $E_y = 0$, and the Stokes parameters of the incident beam according to Eqs. (4) and (5) should fulfill the following conditions:

$$s_2 = s_3 = 0 \qquad s_1 = s_0 \text{ are free to change}. \tag{10}$$

Therefore, the function f from Eq. (3), which determines the deviation from the given trajectory will be:

$$f(\vec{s}^{\rm in}) = s_2^2 + s_3^2\,. \tag{11}$$

To minimize the objective function $f(F(\vec{s}^{\rm out}))$ from Eq. (3) the subspace searching method for unconstrained minimization was utilized, Rowan (1990), due to its simplicity, available source code, and because other methods require knowledge of the gradient of the minimized function. In the case considered, the minized function is determined from a set of nonlinear differential equations coupled with the algebraic boundary conditions, which prohibits an efficient way of calculating the derivatives of such a function. Numerical integration of Eqs. (8) was carried out by the Runge–Kutta method of the fifth order, for the parameters $\Gamma_1 = 1.1$, $\Gamma_2 = 0.9$.

The extrapolation scheme, utilized to determine the starting point for minimization, was Lagrange's polynomial extrapolation with 4 points, Press et al. (1996). For the multidimensional case here, i.e. for the polynomial interpolation of a curve in N dimensions, the independent variable is the length of the trajectory, which was approximated by lengths of intervals between consecutive points on the trajectory. The interpolated values were coordinates of points, and for N-dimensional space N interpolating polynomials are required.

Fig. 6 presents the numerically calculated trajectory of the system in the space of three output Stokes parameters, which is the result of the nonlinear mapping of the input trajectory of fixed polarization from Eq. (10). Figs. 7 and 8 present the dependence of polarization parameters of the ouput light on the input light parameter s_0, i.e. on the intensity of the incident light.

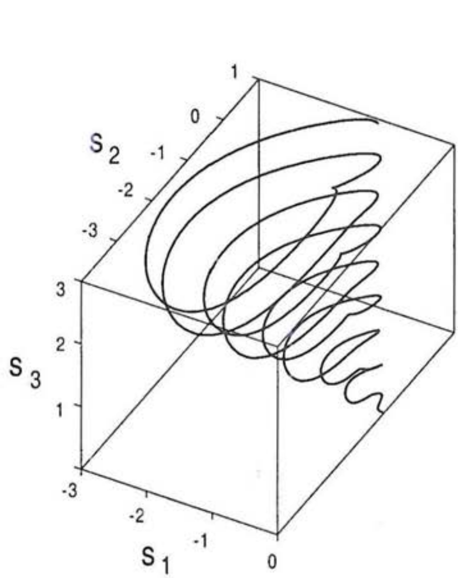

Figure 6. The trajectory of the system in the space of three Stokes parameters s_1, s_2 and s_3 of the transmitted wave, with the assumption that the linearly polarized light is incident on the cavity, with the polarization plane xz.

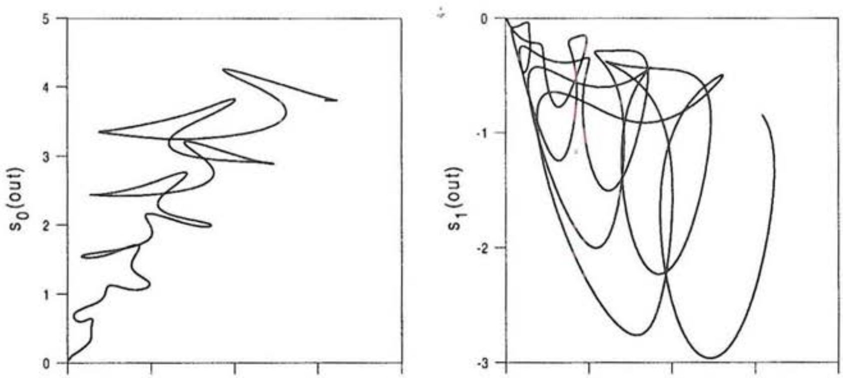

Figure 7. The dependencies of the intensity and s_1 polarization parameter of the output light on the intensity of the linearly polarized light at input.

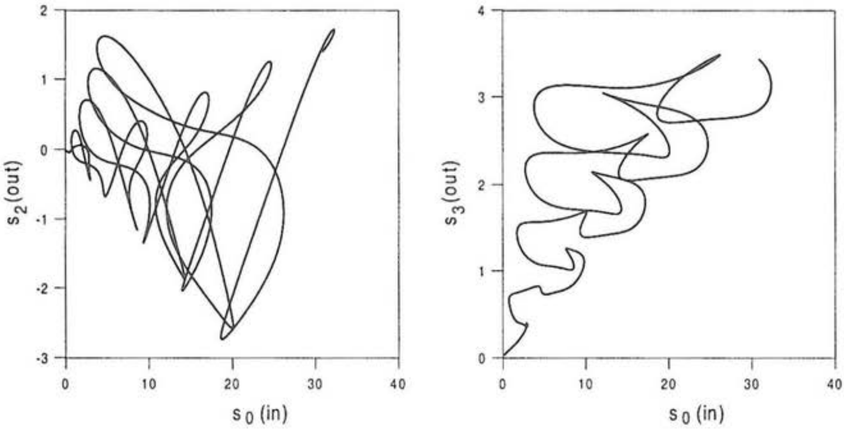

Figure 8. The dependencies of s_2 and s_3 polarization parameters of the output light on the intensity of the linearly polarized light at input.

The figures presented do not yet correspond to the real experiment, in which the intensity s_0 of incident light is a monotonic function of time (it is increased and consecutively decreased). For certain values of the input light intensities, for which the trajectory in the space of output polarization parameters reverses its direction, there appears a sudden jump of values of the output light parameters on the neigbouring branch of the trajectory. Moreover, this jump occurs for a different value of input light intensity, when the intensity is increased or decreased, respectively (see Fig. 9). The above described phenomenon is called "optical polarization bistability".

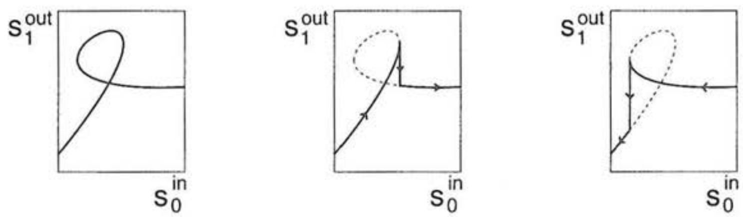

Figure 9. A part of a theoretical dependence of the output polarization of light outgoing from the cavity on the input light intensity, and the real dependencies as should be observed in the experiment, when the intensity of input light is increased and decreased, respectively.

Figs. 10 and 11 present the dependence of Stokes parameters of the light wave transmitted through the cavity on the input light intensity, taking into account the jumps between branches of the trajectory. These figures exhibit characteristic properties of the optical polarization bistability.

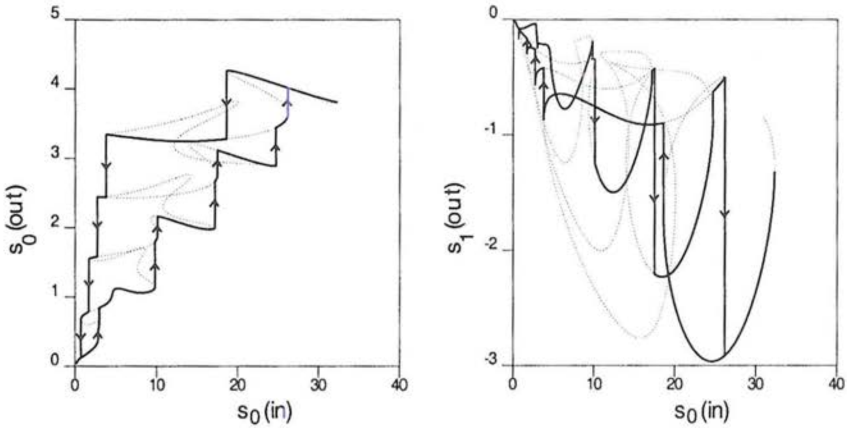

Figure 10. The dependencies of intensity and s_1 polarization parameters of output light on the intensity of input light, taking into account the jumps between the branches of the trajectory in the phase space.

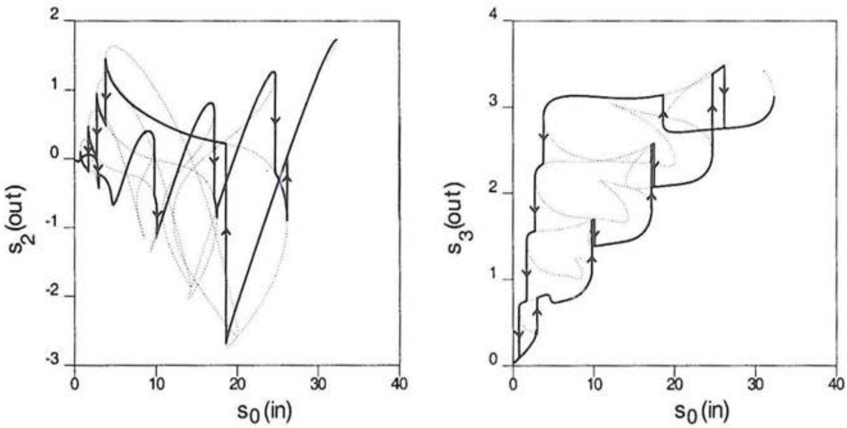

Figure 11. The dependencies of s_2 and s_3 polarization parameters of output light on the intensity of input light, taking into account the jumps between the branches of the trajectory in the phase space.

4. Summary

The control algorithm presented in this paper seems sufficiently general to find an application in problems of inverse modelling of systems described by a nonlinear mapping (1), which do not necessarily belong to the field of nonlinear optics. The phenomenon of bistable changes of values of argument $\vec{x}$ is a general property of nonlinear mappings (1), when the trajectory in the phase space of the system cuts through the so called critical surfaces, in the terminology of theory of catastrophes, i.e. surfaces of the vanishing Jacobian determinant of the mapping, Gilmore (1981).

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