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On some optimization problem related to economic equilibrium

by

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Abstract: The paper considers an optimization problem in which the minima of a finite collection of objective functions satisfy some unilateral constraints and are linked together by a certain subdifferential relationship. The governing relations are stated as a variational inequality defined on a nonconvex feasible set. By the reduction to the variational inequality involving nonmonotone multivalued mapping, defined over nonnegative orthant, the existence of solutions is examined. The prototype is the general economic equilibrium problem. The exemplification of the theory for the quadratic multi-objective function is provided.

Keywords: Optimization problem, variational inequalities, duality, equilibrium.

1. Introduction

Consider the problem of finding minimizers $\boldsymbol{x}_j \in \boldsymbol{R}^n_+$ of a finite collection of convex objectives $V_j : \boldsymbol{R}^n_+ \to \boldsymbol{R} \cup \{+\infty\}$, $j = 1, \ldots, m$. The minimizers are assumed to fulfill unilateral constraints of the form $\langle A_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle \leq \phi_j(\boldsymbol{\pi})$, determined via given functions $\phi_j(\cdot)$. A vector $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ should be found together with $\boldsymbol{x}_j$ by means of the postulated subdifferential relation $\sum_{j=1}^m A_j^T \boldsymbol{x}_j \in \partial\Phi_+(\boldsymbol{\pi})$ with $\Phi_+(\cdot)$ being a convex function.

The main feature of the aforementioned problem is that the feasible set of the corresponding variational inequality for the unknowns $\boldsymbol{\pi}$, $\boldsymbol{x}_j$, $j = 1, \ldots, m$, is nonconvex and, hence, the standard theory of variational inequalities (see Kinderlehrer and Stampacchia, 1980, Ekeland and Temam, 1976) cannot be used to obtain solutions. The approach presented here does not include the notion of Pareto optimum nor of its generalizations (see Pallaschke and Rolewicz, 1997, Luc, 1989, Lee et al., 1998, Hadjisavvas and Schaible, 1998 and the references therein) but, roughly speaking, is based on the calculation of objectives' parametrized constrained minima. Some ideas from Naniewicz and Panagiotopoulos (1995) concerning the general treatment of nonmonotone inequality problems are applied.

The aim of this paper is to:

1. Formulate the general existence result for the aforementioned problem involving $\boldsymbol{\pi}$ and $\boldsymbol{x}_j$ as the basic unknowns;

2. Formulate conditions ensuring the existence of positive solutions $\boldsymbol{\pi} > \mathbf{0}$ only;

3. Formulate conditions under which only a trivial solution $\boldsymbol{\pi} = \mathbf{0}$ is available;

4. Discuss the problem of the existence of ideal minima for the vectorial objective $\boldsymbol{V} = (V_1, V_2)$;

5. Provide the explicit form of solutions for quadratic objective function $\boldsymbol{V} = (V_1, \dots, V_m)$.

The organization of the paper is as follows. In Sections 3 and 4 the existence result is obtained by reduction of the problem to the following variational inequality with $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ as the basic unknown:

$$\langle \mathcal{R}(\boldsymbol{\pi}), \boldsymbol{\tau} - \boldsymbol{\pi} \rangle + \Phi_+(\boldsymbol{\tau}) - \Phi_+(\boldsymbol{\pi}) \geq 0, \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}^n_+,$$

involving a certain not necessarily monotone, multivalued, upper semicontinuous mapping $\mathcal{R} : \boldsymbol{R}^n_+ \to 2^{\boldsymbol{R}^n_+}$. In Section 5 the conditions ensuring the existence of solutions mentioned in points 2. and 3. are formulated. Section 6 is devoted to the study of existence of an ideal minimum for a vectorial objective of the form $\boldsymbol{V}(\cdot) = \big(V_1(\cdot), V_2(\cdot)\big)$. In Sections 7 and 8 the case of quadratic objectives is investigated and, in particular, the explicit form of solutions for two quadratic objectives defined on $\boldsymbol{R}^2_+$ is provided.

The motivation for this work comes from mathematical economics (see, e.g., Von Neumann, 1945-46, Nash, 1950, Arrow and Intrilligator, 1982, Arrow and Debreu, 1954, Nagurney, 1999, Nagurney and Siokos, 1997, Panek, 2000 and the references quoted there). Assume that in the economy the budgets of traders are given in terms of financial holdings and the amounts of commodities are supposed to be known. The problem consists of finding a market equilibrium which is understood as a system $(\boldsymbol{\pi}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$, where $\boldsymbol{\pi}$ represents the price vector while $\boldsymbol{x}_j$ is a bundle of commodities corresponding to j's trader, $j = 1, \dots, m$. The vectors $\boldsymbol{\pi}$ and $\boldsymbol{x}_j$ are assumed to maximize the trader's utility function $-V_j$ under the budget restrictions $\langle \boldsymbol{A}_j\boldsymbol{\pi}, \boldsymbol{x}_j \rangle \leq \phi_j(\boldsymbol{\pi})$ and fulfill the market equilibrium conditions expressed by the subdifferential relation $\sum_{j=1}^m \boldsymbol{A}_j^T \boldsymbol{x}_j \in \partial\Phi_+(\boldsymbol{\pi})$. If $\boldsymbol{A}_j = \text{Identity}$, $\Phi_+(\boldsymbol{\tau}) := \langle \boldsymbol{S}, \boldsymbol{\tau} \rangle$, $\phi_j(\boldsymbol{\tau}) := B_j$, $\forall\, \boldsymbol{\tau} \in \boldsymbol{R}^n_+$, where $\boldsymbol{S} > \mathbf{0}$ represents a vector of the total amount of commodities on the market and $B_j > 0$ is the budget of j's trader, then this relation can be expressed equivalently as $\boldsymbol{\pi} \in \partial\, \mathrm{ind}_{<\boldsymbol{S}}(\sum_{j=1}^m \boldsymbol{x}_j)$ and state that the market clears for a commodity if the equilibrium price is positive; otherwise, there may be an excess supply of the commodity in equilibrium, in which case its price will be zero.

Some results in the case of the subdifferential market equilibrium condition of the form $\boldsymbol{\pi} \in \partial \operatorname{ind}_{\leq \boldsymbol{S}}(\sum_{j=1}^{m} \boldsymbol{x}_j)$ can be found in Nagurney and Naniewicz (2000) where the utility functions have been assumed to be strictly concave and differentiable. The case of the equality market equilibrium condition has been studied by making use of the homotopy methods in Eaves (1972), Hirsh and Smale (1979), Smale (1976) (see also Chichilnisky, 1993 and the references quoted there).

2. Statement of the problem

First, the basic notations are presented and then some preliminaries introduced.

By $\boldsymbol{R}^n$ we denote the Euclidean vector space of all vectors $\boldsymbol{x} = [x_1, \dots, x_n]$, $x_i \in \boldsymbol{R}$, $i = 1, \dots, n$, equipped with the inner product $\langle \cdot, \cdot \rangle : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ defined as

$$\langle \boldsymbol{\pi}, \boldsymbol{x} \rangle = \sum_{i=1}^{n} x_i p_i, \qquad \boldsymbol{x} = [x_1, \dots, x_n],\ \boldsymbol{\pi} = [p_1, \dots, p_n] \in \boldsymbol{R}^n.$$

By $\boldsymbol{R}^{n \times n}$ we denote all $n \times n$ real valued matrices. Moreover, the following notations will be used:

$$\begin{aligned}
&\boldsymbol{R}_+ = \{\alpha \in \boldsymbol{R} \colon \alpha \geq 0\} \\
&\boldsymbol{R}^n_+ = \{\boldsymbol{x} = [x_1, \dots, x_n] \in \boldsymbol{R}^n \colon x_i \geq 0,\ \forall i = 1, \dots, n\}, \\
&\boldsymbol{R}^{n \times n}_+ = \{\boldsymbol{A} = (A_{ik}) \in \boldsymbol{R}^{n \times n} \colon A_{ik} \geq 0,\ \forall i, k = 1, \dots, n\}, \\
&\boldsymbol{R}^n_- = \{\boldsymbol{x} = [x_1, \dots, x_n] \in \boldsymbol{R}^n \colon x_i \leq 0,\ \forall i = 1, \dots, n\}, \\
&\boldsymbol{\pi} = [p_1, \dots, p_n] > \boldsymbol{0} \iff p_i > 0,\ \forall i = 1, \dots, n.
\end{aligned}$$

Throughout the paper it will be assumed that

$$V_j : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}, \qquad j = 1, \dots, m, \tag{1}$$

are convex, proper and lower semicontinuous functions;

$$\phi_j : \boldsymbol{R}^n_+ \to \boldsymbol{R}_+ \text{ with } \phi_j(\boldsymbol{\tau}) > 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{R}^n_+, \qquad j = 1, \dots, m, \tag{2}$$

are continuous functions with positive values;

$$\boldsymbol{A}_j \in \boldsymbol{R}^{n \times n}_+, \qquad \boldsymbol{A}_j \geq \boldsymbol{0}, \qquad \operatorname{Ker} \boldsymbol{A}_j = \{\boldsymbol{0}\}, \qquad j = 1, \dots, m, \tag{3}$$

where $\operatorname{Ker} \boldsymbol{A}_j = \{\boldsymbol{\tau} \in \boldsymbol{R}^n_+ \colon \boldsymbol{A}_j \boldsymbol{\tau} = \boldsymbol{0}\}$. Further, assume

$$\Phi : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}, \qquad \operatorname{Dom} \Phi \cap \operatorname{Int}(\boldsymbol{R}^n_+) \neq \emptyset, \tag{4}$$

to be convex, proper and lower semicontinuous function. Denote by $\partial\Phi\colon \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ its subdifferential. Recall that for a convex function $\varphi : H \to \boldsymbol{R} \cup \{+\infty\}$, H being a Hilbert space, the subdifferential $\partial\varphi\colon H \to 2^H$ is defined by

$$\partial\varphi(u) = \{w \in H\colon \varphi(v) - \varphi(u) \geq \langle w,\, v - u\rangle,\ \forall v \in H\},$$

provided that $\varphi(u) < +\infty$ and $\partial\varphi(u) = \emptyset$, otherwise.

Now we are in a position to formulate the problem to be studied.

Problem (P): Find $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ and $\boldsymbol{x}_j \in \boldsymbol{R}^n_+, j = 1, \ldots, m$, which satisfy for each $j = 1, \ldots, m$ the conditions:

$$V_j(\boldsymbol{x}_j) = \min\left\{V_j(\boldsymbol{x})\colon \langle \boldsymbol{A}_j\boldsymbol{\pi},\, \boldsymbol{x}\rangle \leq \phi_j(\boldsymbol{\pi}) \text{ and } \boldsymbol{x} \in \boldsymbol{R}^n_+\right\}, \tag{$(PM)_j$}$$

$$\Big\langle -\sum_{j=1}^{m} \boldsymbol{A}_j^T\boldsymbol{x}_j,\, \boldsymbol{\tau} - \boldsymbol{\pi}\Big\rangle + \Phi(\boldsymbol{\tau}) - \Phi(\boldsymbol{\pi}) \geq 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{R}^n_+. \tag{PE}$$

The symbol $\boldsymbol{A}_j^T$ is used to denote the transpose of $\boldsymbol{A}_j \in \boldsymbol{R}^{n\times n}$.

3. Minimization problem $(PM)_j$

Throughout this section let us fix $j \in \{1, \ldots, m\}$ and $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ with $\boldsymbol{\pi} \neq \boldsymbol{0}$.

In order to reformulate the problem $(PM)_j$ we introduce $\overline{V}_j : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ by setting

$$\overline{V}_j := V_j + \operatorname{ind}_{\boldsymbol{R}^n_+}, \tag{5}$$

where $\operatorname{ind}_{\boldsymbol{R}^n_+}(\cdot)$ is the indicator function of $\boldsymbol{R}^n_+$, i.e.

$$\operatorname{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \boldsymbol{R}^n_+ \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, define a linear operator $A_{j\boldsymbol{\pi}} : \boldsymbol{R}^n \to \boldsymbol{R}$ by

$$A_{j\boldsymbol{\pi}}\boldsymbol{x} := \langle \boldsymbol{A}_j\boldsymbol{\pi},\, \boldsymbol{x}\rangle, \qquad \boldsymbol{x} \in \boldsymbol{R}^n. \tag{6}$$

If $\operatorname{ind}_{\leq\phi_j(\boldsymbol{\pi})}(\cdot)$ denotes the indicator function of $\{t \in \boldsymbol{R}\colon t \leq \phi_j(\boldsymbol{\pi})\}$, i.e.

$$\operatorname{ind}_{\leq\phi_j(\boldsymbol{\pi})}(t) = \begin{cases} 0 & \text{if } t \leq \phi_j(\boldsymbol{\pi}) \\ +\infty & \text{otherwise,} \end{cases}$$

then by $\partial\operatorname{ind}_{\leq\phi_j(\boldsymbol{\pi})} : \boldsymbol{R} \to 2^{\boldsymbol{R}}$ will be denoted its subdifferential in the sense of convex analysis (Ekeland and Temam, 1976).

Now we are ready to reformulate $(PM)_j$ as follows:

Problem $(\overline{\boldsymbol{P}}_j)$ $\quad \overline{v}_j := \inf\{\overline{V}_j(\boldsymbol{x}) + \operatorname{ind}_{\leq\phi_j(\boldsymbol{\pi})}(A_{j\boldsymbol{\pi}}\boldsymbol{x})\colon \boldsymbol{x} \in \boldsymbol{R}^n\}$.

Following the Fenchel duality theory (see Aubin, 1993) the dual problem of $(\overline{P}_j)$ can be formulated. For this purpose let $A^\star_{j\pi} : \boldsymbol{R} \to \boldsymbol{R}^n$ denote the transpose of $A_{j\pi}$, which takes the form

$$A^\star_{j\pi}\alpha = \alpha A_j \pi, \qquad \alpha \in \boldsymbol{R}. \tag{7}$$

We also let $\overline{V}^\star_j : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ denote the conjugate of $\overline{V}_j$, defined by

$$\overline{V}^\star_j(\mu) := \sup_{x \in \boldsymbol{R}^n} \left\{ \langle \mu, x \rangle - \overline{V}_j(x) \right\}, \qquad \mu \in \boldsymbol{R}^n, \tag{8}$$

which under the hypothesis $\partial \overline{V}_j = \partial V_j + \partial \operatorname{ind}_{\boldsymbol{R}^n_+}$, $\partial \operatorname{ind}_{\boldsymbol{R}^n_+}(\cdot)$ being the subdifferential of $\operatorname{ind}_{\boldsymbol{R}^n_+}(\cdot)$, has the property (see Aubin, 1993)

$$\partial \overline{V}^\star_j = \left(\partial V_j + \partial \operatorname{ind}_{\boldsymbol{R}^n_+} \right)^{-1}. \tag{9}$$

From now on, this hypothesis will be assumed to hold throughout the paper.

According to the Fenchel theory the dual problem of $(\overline{P}_j)$ reads:

$$\underline{v}_j := \inf \left\{ \overline{V}^\star_j(-A^\star_{j\pi}\alpha) + \operatorname{ind}^\star_{\leq \phi_j(\pi)}(\alpha) \colon \alpha \in \boldsymbol{R} \right\},$$

where

$$\operatorname{ind}^\star_{\leq \phi_j(\pi)}(\alpha) = \sup_{t \leq \phi_j(\pi)} \{\alpha t\} = \alpha \phi_j(\pi) + \operatorname{ind}_{\boldsymbol{R}_+}(\alpha), \qquad \alpha \in \boldsymbol{R}, \tag{10}$$

is the conjugate of $\operatorname{ind}_{\leq \phi_j(\pi)}(\cdot)$, $\boldsymbol{R}_+ = \{t \in \boldsymbol{R} \colon t \geq 0\}$. Using (7) and (10) the dual of $(\overline{P}_j)$ can be written equivalently as

Problem $(\underline{P}_j)$ $\qquad \underline{v}_j := \inf \left\{ \overline{V}^\star_j(-\alpha A_j \pi) + \alpha \phi_j(\pi) \colon \alpha \in \boldsymbol{R}_+ \right\}.$

From the Fenchel theorem (see Aubin, 1993) it follows that

$$\overline{v}_j + \underline{v}_j \geq 0. \tag{11}$$

To formulate the next result based on the Fenchel theorem let us introduce the notations: for any set K the symbol "$\operatorname{Int} K$" stands for the interior of K and "$\operatorname{Dom} U$" is the effective domain of U.

PROPOSITION 1 *Assume that in the algebraic sense it holds that*

$$0 \in \operatorname{Int} \left\{ A_{j\pi} \operatorname{Dom} \overline{V}_j - \{t \in \boldsymbol{R} \colon t \leq \phi_j(\pi)\} \right\}. \tag{12}$$

Then

$$\overline{v}_j + \underline{v}_j = 0 \tag{13}$$

and there exists $\alpha_j \in \boldsymbol{R}_+$ such that

$$\overline{V}_j^\star(-A_{j\boldsymbol{\pi}}^\star \alpha_j) + \alpha_j \phi_j(\boldsymbol{\pi}) = \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}) + \alpha_j \phi_j(\boldsymbol{\pi}) = \underline{v}_j \tag{14}$$

(α_j is a solution of $(\underline{P}_j)$). If additionally,

$$\operatorname{Dom} \partial \overline{V}_j^\star \supset \boldsymbol{R}_-^n, \tag{15}$$

then there exists $\boldsymbol{x}_j \in \boldsymbol{R}_+^n$ with $\langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle - \phi_j(\boldsymbol{\pi}) = A_{j\boldsymbol{\pi}} \boldsymbol{x}_j - \phi_j(\boldsymbol{\pi}) \leq 0$, such that

$$V_j(\boldsymbol{x}_j) = \overline{v}_j \tag{16}$$

($\boldsymbol{x}_j$ is a solution of $(\overline{P}_j)$). Moreover,

$$-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi} \in \partial \overline{V}_j(\boldsymbol{x}_j) \tag{17}$$

$$\alpha_j \in \partial \operatorname{ind}_{\leq \phi_j(\boldsymbol{\pi})}\big(\langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle\big). \tag{18}$$

REMARK Notice that if $\boldsymbol{A}_j \geq \boldsymbol{0}$, $\operatorname{Ker} \boldsymbol{A}_j = \{\boldsymbol{0}\}$ and $\operatorname{Dom} V_j \supset \boldsymbol{R}_+^n$, then the hypothesis (12) is fulfilled for any $\boldsymbol{\pi} \in \boldsymbol{R}_+^n \setminus \{\boldsymbol{0}\}$.

COROLLARY 1 *Under the hypotheses (12) and (15) the following compatibility conditions hold*

$$\begin{aligned} &V_j(\boldsymbol{x}_j) + \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}) = -\alpha_j \langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle \\ &\alpha_j\big(\langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle - \phi_j(\boldsymbol{\pi})\big) = 0. \end{aligned} \tag{19}$$

COROLLARY 2 *Under the hypotheses (12) and (15), α_j satisfies the variational inequality*

$$\big\langle \boldsymbol{A}_j \boldsymbol{\pi}, -\partial \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}) \big\rangle (t - \alpha_j) + \phi_j(\boldsymbol{\pi})(t - \alpha_j) \geq 0, \quad \forall t \geq 0. \tag{20}$$

Proof. From (17) and (18) it follows that

$$\boldsymbol{x}_j \in \partial \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}) \quad \text{and} \quad \langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \rangle \in \partial \operatorname{ind}_{\leq \phi_j(\boldsymbol{\pi})}^\star(\alpha_j) \tag{21}$$

which, thanks to (10), leads easily to (20), as desired. ∎

LEMMA 1 *If $\alpha_j \in \boldsymbol{R}_+$ satisfies (20) and $\partial \overline{V}_j = \partial V_j + \partial \operatorname{ind}_{\boldsymbol{R}_+^n}$ then*

$$\begin{aligned} &\alpha_j\big(\langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_{j0} \rangle - \phi_j(\boldsymbol{\pi})\big) \\ &\in \big\langle \partial V_j(\boldsymbol{x}_j) - \partial V_j(\boldsymbol{x}_{j0}), \boldsymbol{x}_j - \boldsymbol{x}_{j0} \big\rangle - \langle \boldsymbol{\lambda}_j, \boldsymbol{x}_{j0} \rangle - \langle \boldsymbol{\lambda}_{j0}, \boldsymbol{x}_j \rangle, \end{aligned} \tag{22}$$

where

$$\begin{aligned} &\boldsymbol{x}_j \in \partial \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}), \qquad \boldsymbol{x}_{j0} \in \partial \overline{V}_j^\star(\boldsymbol{0}) \\ &\boldsymbol{\lambda}_j \in \partial \operatorname{ind}_{\boldsymbol{R}_+^n}(\boldsymbol{x}_j), \qquad \boldsymbol{\lambda}_{j0} \in \partial \operatorname{ind}_{\boldsymbol{R}_+^n}(\boldsymbol{x}_{j0}). \end{aligned}$$

Proof. Let $\boldsymbol{x}_j \in \partial \overline{V}_j^\star(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi})$ and $\boldsymbol{x}_{j0} \in \partial \overline{V}_j^\star(\mathbf{0})$. Then, by the well known result of the subdifferential calculus we get $-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi} \in \partial \overline{V}_j(\boldsymbol{x}_j)$ and $\mathbf{0} \in \partial \overline{V}_j(\boldsymbol{x}_{j0})$. Since $\partial \overline{V}_j = \partial V_j + \partial \operatorname{ind}_{\boldsymbol{R}^n_+}$, there exist $\boldsymbol{\lambda}_j \in \partial \operatorname{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}_j)$ and $\boldsymbol{\lambda}_{j0} \in \partial \operatorname{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}_{j0})$ such that

$$-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi} \in \partial V_j(\boldsymbol{x}_j) + \boldsymbol{\lambda}_j, \qquad \mathbf{0} \in \partial V_j(\boldsymbol{x}_{j0}) + \boldsymbol{\lambda}_{j0}. \tag{23}$$

Hence we get

$$\begin{aligned} &\left\langle -\alpha_j \boldsymbol{A}_j \boldsymbol{\pi},\, \boldsymbol{x}_j - \boldsymbol{x}_{j0} \right\rangle \in \left\langle \partial V_j(\boldsymbol{x}_j) - \partial V_j(\boldsymbol{x}_{j0}),\, \boldsymbol{x}_j - \boldsymbol{x}_{j0} \right\rangle \\ &+\left\langle \boldsymbol{\lambda}_j - \boldsymbol{\lambda}_{j0},\, \boldsymbol{x}_j - \boldsymbol{x}_{j0} \right\rangle, \end{aligned}$$

which thanks to (19) and the compatibility conditions $\boldsymbol{\lambda}_j \bullet \boldsymbol{x}_j = \mathbf{0}$ and $\boldsymbol{\lambda}_{j0} \bullet \boldsymbol{x}_{j0} = \mathbf{0}$ ([1]) leads to (22). The proof is complete. ■

PROPOSITION 2 *Let the hypotheses of Proposition 1 be satisfied. Moreover suppose that there exists a constant $M_j > 0$ such that*

$$\begin{aligned} E_j &:= \left\{ \boldsymbol{x} \in \boldsymbol{R}^n_+ \colon \min\left\{ \left\langle \partial V_j(\boldsymbol{x}),\, \boldsymbol{x} \right\rangle \right\} \le 0 \right\} \\ &\subset \{ \boldsymbol{y} \in \boldsymbol{R}^n_+ \colon |y| \le M_j \}, \qquad M_j > 0. \end{aligned} \tag{24}$$

Then the set $\Lambda_j(\boldsymbol{\pi})$ of all solutions of (20) is nonempty, convex, closed and bounded.

Proof. The existence of solutions has been already established in Proposition 1, so $\Lambda_j(\boldsymbol{\pi}) \neq \emptyset$. Furthermore, $\Lambda_j(\boldsymbol{\pi})$ as the set of all solutions of variational ineqquality (20) involving maximal monotone mapping $G_j(t) := \left\langle \boldsymbol{A}_j \boldsymbol{\pi},\, -\partial \overline{V}_j^\star(-t \boldsymbol{A}_j \boldsymbol{\pi}) \right\rangle$ is convex and closed (see Ekeland and Temam, 1976). For the boundedness recall that $\left\langle \boldsymbol{A}_j \boldsymbol{\pi},\, \boldsymbol{x}_j \right\rangle \le \phi_j(\boldsymbol{\pi})$ and $\boldsymbol{x}_j \in \partial \overline{V}_j^\star\left(-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi}\right)$, so that

$$-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi} \in \partial V_j(\boldsymbol{x}_j) + \boldsymbol{\lambda}_j, \qquad \boldsymbol{\lambda}_j \in \partial \operatorname{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}_j), \qquad \boldsymbol{\lambda}_j \bullet \boldsymbol{x}_j = \mathbf{0}, \tag{25}$$

and consequently

$$-\alpha_j \left\langle \boldsymbol{A}_j \boldsymbol{\pi}, \boldsymbol{x}_j \right\rangle \in \left\langle \partial V_j(\boldsymbol{x}_j), \boldsymbol{x}_j \right\rangle.$$

Since $-\alpha_j \left\langle \boldsymbol{A}_j \boldsymbol{\pi},\, \boldsymbol{x}_j \right\rangle \le 0$, due to (24), the boundedness of $\{\boldsymbol{x}_j\}$ follows. When combined with

$$\min\left\{ \left\langle \partial V_j(\boldsymbol{x}_j), \boldsymbol{x}_j \right\rangle \right\} \ge V_j(\boldsymbol{x}_j) - V_j(0),$$

and lower semicontinuity of V_j this implies the existence of $m_j \ge 0$ such that

$$\min\left\{ \left\langle \partial V_j(\boldsymbol{x}_j),\, \boldsymbol{x}_j \right\rangle \right\} \ge V_j(\boldsymbol{x}_j) - V_j(0) \ge -m_j.$$

[1] For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$ the notation $\boldsymbol{x} \bullet \boldsymbol{y} = [x_1 y_1, \ldots, x_n y_n]$ is used.

Now, according to (19) we get either $\alpha_j = 0$ or $\langle \boldsymbol{A}_j\boldsymbol{\pi}, \boldsymbol{x}_j\rangle = \phi_j(\boldsymbol{\pi})$, therefore we finally conclude that

$$\alpha_j \le \frac{m_j}{\phi_j(\boldsymbol{\pi})}.$$

The proof is complete. ∎

It has been proved in Proposition 2 that to any $\boldsymbol{\pi} \in \boldsymbol{R}^n_+ \setminus \{\boldsymbol{0}\}$ one can assign the set $\Lambda_j(\boldsymbol{\pi})$ of all solutions of variational inequality (20), which is nonempty, closed, convex and bounded. Now observe that if $\boldsymbol{\pi} \in \boldsymbol{R}^n_+ \setminus \{\boldsymbol{0}\}$ is small enough then $\langle \boldsymbol{A}_j\boldsymbol{\pi}, \boldsymbol{x}_{j0}\rangle < \phi_j(\boldsymbol{\pi})$ for $\boldsymbol{x}_{j0} \in \partial\overline{V}^\star_j(\boldsymbol{0})$ which, by (22), yields $\Lambda_j(\boldsymbol{\pi}) = \{\boldsymbol{0}\}$. Since $\phi(\boldsymbol{0}) > 0$, we obtain easily that $\Lambda_j(\boldsymbol{0}) = \{0\}$ and, hence, the continuity of $\Lambda_j(\cdot)$ at $\boldsymbol{0}$ follows. It turns out that $\Lambda_j(\cdot)$ treated as a multivalued mapping from $\boldsymbol{R}^n_+$ into $2^{\boldsymbol{R}_+}$ is upper semicontinuous.

PROPOSITION 3 *Assume that the following hypotheses hold:*

$$\operatorname{Dom} V_j \supset \boldsymbol{R}^n_+, \tag{26}$$

$$\operatorname{Dom} \partial\overline{V}^\star_j \supset \boldsymbol{R}^n_-, \tag{27}$$

$$\big\{\boldsymbol{x} \in \boldsymbol{R}^n_+ \colon \min\{\langle \partial V_j(\boldsymbol{x}), \boldsymbol{x}\rangle\} \le 0\big\} \ \textit{is bounded.} \tag{28}$$

Then $\Lambda_j : \boldsymbol{R}^n_+ \to 2^{\boldsymbol{R}_+}$ is an upper semicontinuous mapping from $\boldsymbol{R}^n_+$ into $2^{\boldsymbol{R}_+}$ with nonempty, closed, convex and bounded values.

Proof. It has been already proved in Proposition 2 than $\Lambda_j(\cdot)$ has nonempty, closed, convex and bounded values. Thus it remains to show its upper semicontinuity. For this purpose assume that $\{\boldsymbol{\pi}_k\} \subset \boldsymbol{R}^n_+$ and $\alpha_k \in \Lambda_j(\boldsymbol{\pi}_k)$ are such that $\boldsymbol{\pi}_k \to \boldsymbol{\pi}^\star$ and $\alpha_k \to \alpha^\star$ for some $\boldsymbol{\pi}^\star$ and $\alpha^\star \in \boldsymbol{R}_+$, respectively. Our aim now is to show that $\alpha^\star \in \Lambda_j(\boldsymbol{\pi}^*)$.
From (17) it follows that

$$-\alpha_k \boldsymbol{A}_j\boldsymbol{\pi}_k \in \partial V_j(\boldsymbol{x}_k) + \boldsymbol{\lambda}_k, \qquad \boldsymbol{\lambda}_k \in \partial \operatorname{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}_k),$$

which implies

$$-\alpha_k\langle \boldsymbol{A}_j\boldsymbol{\pi}_k, \boldsymbol{x}_k\rangle \in \langle \partial V_j(\boldsymbol{x}_k), \boldsymbol{x}_k\rangle.$$

But the left hand side of this relation is nonpositive. Therefore, by the hypothesis (28), the boundedness of $\{\boldsymbol{x}_k\}$ results. Consequently, one can suppose that $\boldsymbol{x}_k \to \boldsymbol{x}^\star$ for some $\boldsymbol{x}^\star \in \boldsymbol{R}^n_+$ (by passing to a subsequence, if necessary). According to (17) and (18) we get

$$-\alpha_k \boldsymbol{A}_j\boldsymbol{\pi}_k \in \partial\overline{V}^\star_j(\boldsymbol{x}_k)$$
$$\alpha_k \in \partial \operatorname{ind}_{\le\phi_j(\boldsymbol{\pi}_k)}\big(\langle \boldsymbol{A}_j\boldsymbol{\pi}_k, \boldsymbol{x}_k\rangle\big),$$

or equivalently

$$-\alpha_k A_j \pi_k \in \partial \overline{V}_j^\star(x_k)$$
$$\langle A_j \pi_k, x_k \rangle - \phi_j(\pi_k) \in \partial \operatorname{ind}_{\geq 0}(\alpha_k),$$

which allows passing to the limit as $k \to \infty$. By the continuity of $\phi_j(\cdot)$, the maximal monotonicity of $\partial \overline{V}_j^\star(\cdot)$ and $\partial \operatorname{ind}_{\geq 0}(\cdot)$, we get

$$-\alpha^\star A_j \pi^\star \in \partial \overline{V}_j^\star(x^\star)$$
$$\langle A_j \pi^\star, x^\star \rangle - \phi_j(\pi^\star) \in \partial \operatorname{ind}_{\geq 0}(\alpha^\star).$$

But the last inclusion can be written equivalently as

$$\alpha^\star \in \partial \operatorname{ind}_{\leq \phi_j(\pi^\star)}\big(\langle A_j \pi^\star, x^\star \rangle\big),$$

from which we deduce that $\alpha^\star \in \Lambda_j(\pi^\star)$, as desired. The proof is complete. ■

The results of Proposition 1 and Proposition 3 can be summarized as follows.

THEOREM 1 *Assume that for $j = 1, \dots, m$ the hypotheses below hold:*

$$\operatorname{Dom} V_j \supset \boldsymbol{R}_+^n, \tag{29}$$
$$\operatorname{Dom} \partial \overline{V}_j^\star \supset \boldsymbol{R}_-^n, \tag{30}$$
$$E_j = \{x \in \boldsymbol{R}_+^n \colon \min\{\langle \partial V_j(x), x \rangle\} \leq 0\}$$
$$\subset \{y \in \boldsymbol{R}_+^n \colon |y| \leq M_j\}, \qquad M_j > 0, \tag{31}$$

Then, for any $\pi \in \boldsymbol{R}_+^n$ the optimization problem: Find $x_j \in \boldsymbol{R}_+^n$ such that

$$V_j(x_j) = \min\{V_j(y) \colon \forall y \in \boldsymbol{R}_+^n \text{ with } \langle A_j \pi, y \rangle \leq \phi_j(\pi)\} \tag{32}$$

has at least one solution. Moreover, there exists $\alpha_j \in \Lambda_j(\pi)$, $\Lambda_j(\pi)$ being the set of all solutions of the variational inequality

$$\langle A_j \pi, -\partial \overline{V}_j^\star(-\alpha_j A_j \pi) \rangle (t - \alpha_j) + \phi_j(\pi)(t - \alpha_j) \geq 0, \quad \forall t \geq 0, \tag{33}$$

with the property that

$$x_j \in \partial \overline{V}_j^\star(-\alpha_j A_j \pi). \tag{34}$$

Additionally, $\Lambda_j : \boldsymbol{R}_+^n \to 2^{\boldsymbol{R}_+}$ has nonempty, closed, convex and bounded values and it is upper semicontinuous from $\boldsymbol{R}_+^n$ into $2^{\boldsymbol{R}_+}$.

4. Problem (PE)

Let us recall that $\Phi : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ has been assumed to be a convex, lower semicontinuous function with $\operatorname{Dom}\Phi \cap \operatorname{Int}(\boldsymbol{R}^n_+) \neq \emptyset$ and the problem (PE) consists in finding $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ such that

$$\Big\langle -\sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{x}_j,\, \boldsymbol{\tau} - \boldsymbol{\pi} \Big\rangle + \Phi(\boldsymbol{\tau}) - \Phi(\boldsymbol{\pi}) \geq 0, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}^n_+. \tag{35}$$

Taking into account (34) we can introduce a multivalued mapping $\mathcal{R} : \boldsymbol{R}^n_+ \to 2^{\boldsymbol{R}^n_+}$ by setting

$$\mathcal{R}(\boldsymbol{\pi}) := -\sum_{j=1}^{m} \boldsymbol{A}_j^T \partial \overline{V}_j^{\star}\big(-\Lambda_j(\boldsymbol{\pi})\boldsymbol{A}_j\boldsymbol{\pi}\big), \qquad \boldsymbol{\pi} \in \boldsymbol{R}^n_+, \tag{36}$$

which leads to the consideration of the following variational inequality with multivalued operator $\mathcal{R}$: Find $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ such that

$$\big\langle \mathcal{R}(\boldsymbol{\pi}),\, \boldsymbol{\tau} - \boldsymbol{\pi} \big\rangle + \Phi(\boldsymbol{\tau}) - \Phi(\boldsymbol{\pi}) \geq 0, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}^n_+. \tag{37}$$

By a solution of (37) we mean each $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ for which there exists $\boldsymbol{X} \in \mathcal{R}(\boldsymbol{\pi})$ with the property that

$$\big\langle \boldsymbol{X},\, \boldsymbol{\tau} - \boldsymbol{\pi} \big\rangle + \Phi(\boldsymbol{\tau}) - \Phi(\boldsymbol{\pi}) \geq 0, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}^n_+. \tag{38}$$

PROPOSITION 4 *Under the hypotheses of Theorem 1, $\mathcal{R}$ given by (36) is a multivalued, upper semicontinuous mapping from $\boldsymbol{R}^n_+$ into $2^{\boldsymbol{R}^n_+}$ with nonempty, convex, closed and bounded values.*

Proof. According to Theorem 1, $\Lambda_j : \boldsymbol{R}^n_+ \to 2^{\boldsymbol{R}_+}$ is an upper semicontinuous mapping with nonempty, closed, convex and bounded values. Further, as maximal monotone $\partial \overline{V}_j^{\star} : \boldsymbol{R}^n_+ \to 2^{\boldsymbol{R}^n_+}$ has closed, convex and, by the hypothesis (30), nonempty values. Thus we easily deduce that values of $\mathcal{R}$ are nonempty and closed.

For the convexity we show that $\partial \overline{V}_j^{\star}(-\alpha_1 \boldsymbol{A}_j\boldsymbol{\pi}) = \partial \overline{V}_j^{\star}(-\alpha_2 \boldsymbol{A}_j\boldsymbol{\pi})$ for any $\alpha_1, \alpha_2 \in \Lambda_j(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$. To this end, assume that $\widetilde{\boldsymbol{x}}_1, \widetilde{\boldsymbol{x}}_2 \in \partial \overline{V}_j^{\star}\big(-\Lambda_j(\boldsymbol{\pi})\boldsymbol{A}_j\boldsymbol{\pi}\big)$. There exist $\alpha_1, \alpha_2 \in \Lambda_j(\boldsymbol{\pi})$ with the property that

$$\widetilde{\boldsymbol{x}}_1 \in \partial \overline{V}_j^{\star}(-\alpha_1 \boldsymbol{A}_j\boldsymbol{\pi}) \qquad \text{and} \quad \widetilde{\boldsymbol{x}}_2 \in \partial \overline{V}_j^{\star}(-\alpha_2 \boldsymbol{A}_j\boldsymbol{\pi}).$$

For $t \in [0,1]$ set $\widetilde{\boldsymbol{x}}_t = t\widetilde{\boldsymbol{x}}_1 + (1-t)\widetilde{\boldsymbol{x}}_2$. Since $\widetilde{\boldsymbol{x}}_1$ and $\widetilde{\boldsymbol{x}}_2$ are solutions of (32), we have

$$\begin{aligned}
&V_j(\widetilde{\boldsymbol{x}}_1) = V_j(\widetilde{\boldsymbol{x}}_2) = V_j(\widetilde{\boldsymbol{x}}_t),\\
&V_j(\widetilde{\boldsymbol{x}}_k) + \overline{V}_j^{\star}(-\alpha_k \boldsymbol{A}_j\boldsymbol{\pi}) = -\alpha_k \langle \boldsymbol{A}_j\boldsymbol{\pi}, \widetilde{\boldsymbol{x}}_k\rangle,\\
&\alpha_k\big(\langle \boldsymbol{A}_j\boldsymbol{\pi},\, \widetilde{\boldsymbol{x}}_k\rangle - \phi_j(\boldsymbol{\pi})\big) = 0, \qquad k = 1,2.
\end{aligned}$$

Firstly, we consider the case $\alpha_1 = 0$. Then $\widetilde{x}_1 \in \partial\overline{V}_j^\star(\mathbf{0})$ and the foregoing relations imply $V_j(\widetilde{x}_2) + \overline{V}_j^\star(\mathbf{0}) = 0$. Hence $\widetilde{x}_2 \in \partial\overline{V}_j^\star(\mathbf{0})$ which by convexity of $\partial\overline{V}_j^\star(\mathbf{0})$ yields $\widetilde{x}_t \in \partial\overline{V}_j^\star(\mathbf{0})$. Secondly, we suppose that $\alpha_1, \alpha_2 \neq 0$. Taking into account that $-\alpha_k A_j \pi \in \partial\overline{V}_j(\widetilde{x}_k)$, $k = 1, 2$, we obtain

$$\langle \alpha_1 A_j \pi,\, y - \widetilde{x}_1 \rangle + V_j(y) - V_j(\widetilde{x}_1) \geq 0, \quad \forall y \in R_+^n,$$
$$\langle \alpha_2 A_j \pi,\, y - \widetilde{x}_2 \rangle + V_j(y) - V_j(\widetilde{x}_2) \geq 0, \quad \forall y \in R_+^n.$$

Since in such case, $\langle A_j\pi, \widetilde{x}_k \rangle - \phi_j(\pi) = 0$, $k = 1, 2$, we are allowed to conclude that $\langle A_j\pi, \widetilde{x}_t \rangle - \phi_j(\pi) = 0$, and consequently

$$\langle \alpha_1 A_j \pi,\, y - \widetilde{x}_t \rangle + V_j(y) - V_j(\widetilde{x}_t) \geq 0, \quad \forall y \in R_+^n,$$
$$\langle \alpha_2 A_j \pi,\, y - \widetilde{x}_t \rangle + V_j(y) - V_j(\widetilde{x}_t) \geq 0, \quad \forall y \in R_+^n.$$

Hence, by adding these inequalities multiplied by $t^\star$ and $1 - t^\star$, $t^\star \in [0, 1]$, respectively, we arrive at

$$\langle \alpha_{t^\star} A_j \pi,\, y - \widetilde{x}_t \rangle + V_j(y) - V_j(\widetilde{x}_t) \geq 0, \quad \forall y \in R_+^n,$$
$$\alpha_{t^\star} := t^\star \alpha_1 + (1 - t^\star)\alpha_2,$$

from which we deduce easily that $\widetilde{x}_t \in \partial\overline{V}_j^\star(-\alpha_{t^\star} A_j \pi)$ for any $t, t^\star \in [0, 1]$. Thus, $\partial\overline{V}_j^\star(-\alpha A_j \pi)$ does not depend on $\alpha \in \Lambda_j(\pi)$. Thanks to the maximal monotonicity of $\partial\overline{V}_j^\star$, the convexity and closedness of $\partial\overline{V}_j^\star(\tau)$ results for any $\tau \in R_+^n$, with the same forwarded to $\mathcal{R}(\pi)$.
The boundedness is a consequence of (31). Indeed, if $x_j \in \partial\overline{V}_j^\star(-\Lambda_j(\pi) A_j \pi)$, $\pi \in R_+^n$, then $-\alpha_j A_j \pi \in \partial\overline{V}_j(x_j)$, $\alpha_j \in \Lambda_j(\pi)$, ensuring that $\langle \partial V_j(x_j), x_j \rangle \ni -\alpha_j \phi_j(\pi) \leq 0$. This, by (31), leads to $|x_j| \leq M_j$. Since $-\sum_{j=1}^m A_j^T x_j \in \mathcal{R}(\pi)$, the boundedness of $\mathcal{R}$ follows. The proof is complete. ■

THEOREM 2 *Assume that the hypotheses (29)–(31) hold and suppose that for some* $M > 0$,

$$\Big\{\tau \in R_+^n \colon \Phi(\tau) \leq \sum_{j=1}^m \phi_j(\tau) + \Phi(\mathbf{0})\Big\} \subset \{\tau \in R_+^n \colon |\tau| \leq M\}. \tag{39}$$

Then the problem: Find $\pi \in R_+^n$ *such as to satisfy the variational inequality*

$$\langle \mathcal{R}(\pi),\, \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in R_+^n, \tag{40}$$

has at least one solution.

Proof. Let $\mathcal{B}_{2M} = \{\tau \in R_+^n \colon |\tau| \leq 2M\}$. Consider the following problem: Find $\pi \in \mathcal{B}_{2M}$ satisfying the variational inequality

$$\langle \mathcal{R}(\pi),\, \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in \mathcal{B}_{2M}, \tag{41}$$

Since $\mathcal{B}_{2M}$ is compact, $\mathcal{R}$ is upper semicontinuous from $\boldsymbol{R}^n_+$ into $2^{\boldsymbol{R}_+}$ and has nonempty, closed, convex and bounded values, while $\Phi : \boldsymbol{R}^n_+ \to \boldsymbol{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous, the existence of $\boldsymbol{\pi} \in \mathcal{B}_{2M}$ fulfilling (41) can be derived by the use of the classical results concerning variational inequalities with multivalued mappings (see Browder, 1968, Browder and Hess, 1972). Substituting $\boldsymbol{\tau} = \boldsymbol{0}$ into (41) yields

$$\Big\langle -\sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{x}_j,\, -\boldsymbol{\pi} \Big\rangle + \Phi(\boldsymbol{0}) \geq \Phi(\boldsymbol{\pi})$$

for some $\boldsymbol{x}_j \in \partial \overline{V}_j^\star\big(-\Lambda_j(\boldsymbol{\pi})\boldsymbol{A}_j\boldsymbol{\pi}\big)$. In view of

$$\Big\langle \sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{x}_j,\, \boldsymbol{\pi} \Big\rangle = \sum_{j=1}^{m} \langle \boldsymbol{x}_j,\, \boldsymbol{A}_j\boldsymbol{\pi} \rangle \leq \sum_{j=1}^{m} \phi_j(\boldsymbol{\pi}),$$

we are led to the conclusion that

$$\Phi(\boldsymbol{\pi}) \leq \sum_{j=1}^{m} \phi_j(\boldsymbol{\pi}) + \Phi(\boldsymbol{0}),$$

which, thanks to (39), gives $|\boldsymbol{\pi}| \leq M$. Accordingly, having in mind the validity of (41) for any $\boldsymbol{\tau} \in \mathcal{B}_{2M}$, we easily deduce (40). The proof is complete. ■

Theorems 1 and 2 allow to draw the conclusion concerning our basic problem.

THEOREM 3 *Assume that the hypotheses (29)-(31) and (39) hold. Then there exists at least one* $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ *and* $\boldsymbol{x}_j \in \boldsymbol{R}^n_+$, $j = 1, \dots, m$, *such that*

$$V_j(\boldsymbol{x}_j) = \min\{V_j(\boldsymbol{y})\colon \boldsymbol{y} \in \boldsymbol{R}^n_+ \text{ and } \langle \boldsymbol{A}_j\boldsymbol{\pi},\, \boldsymbol{y} \rangle \leq \phi_j(\boldsymbol{\pi})\}, \qquad (PM)_j$$

$$\Big\langle -\sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{x}_j,\, \boldsymbol{\tau} - \boldsymbol{\pi} \Big\rangle + \Phi(\boldsymbol{\tau}) - \Phi(\boldsymbol{\pi}) \geq 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{R}^n_+. \qquad (PE)$$

Furthermore, the mapping $\Lambda_j(\cdot)$, *which to any* $\boldsymbol{\tau} \in \boldsymbol{R}^n_+$ *assigns all solutions* $\alpha \in \boldsymbol{R}_+$ *of the variational inequality*

$$\big\langle \boldsymbol{\tau}, -\partial \overline{V}_j^\star(-\alpha \boldsymbol{A}_j \boldsymbol{\tau}) \big\rangle (t - \alpha) + \phi_j(\boldsymbol{\pi})(t - \alpha) \geq 0,\ \forall t \geq 0,\ j = 1, \dots, m, \qquad (42)$$

has nonempty, closed, convex and bounded values, is upper semicontinuous from $\boldsymbol{R}^n_+$ *into* $2^{\boldsymbol{R}_+}$, *and has the property that*

$$\boldsymbol{x}_j \in \partial \overline{V}_i^\star\big(-\Lambda_j(\boldsymbol{\pi})\boldsymbol{A}_j\boldsymbol{\pi}\big), \qquad j = 1, \dots, m. \qquad (43)$$

Additionally, π satisfies the variational inequality

$$\langle \mathcal{R}(\pi), \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in R^n_+, \tag{44}$$

where

$$\mathcal{R}(\pi) := -\sum_{j=1}^{m} A_j^T \partial \overline{V}_j^\star\big(-\Lambda_j(\pi) A_j \pi\big), \qquad \pi \in R^n_+, \tag{45}$$

is a multivalued, upper semicontinuous mapping with nonempty, convex, closed and bounded values.

5. Special cases

From Theorem 3 it follows that solutions $x_j, \pi \in R^n_+$ of $(PM)_j$–(PE), $j = 1, \dots, m$, fulfill the condition

$$\sum_{j=1}^{m} A_j^T x_j \in \partial(\Phi + \mathrm{ind}_{R^n_+})(\pi) = \partial\Phi(\pi) + \partial\, \mathrm{ind}_{R^n_+}(\pi), \tag{46}$$

$(\mathrm{Int}(R^n_+) \cap \mathrm{Dom}\,\Phi \neq \emptyset$, (see Rockafellar, 1970, Ekeland and Temam, 1976). Thus, in particular, if $\pi = [p_1, \dots, p_n] \in R^n_+$ has only positive coordinates, i.e. $p_i > 0$ for all $j = 1, \dots, m$, $(\pi > \mathbf{0})$, then instead of (46) we have

$$\sum_{j=1}^{m} A_j^T x_j \in \partial\Phi(\pi), \tag{47}$$

because $\partial\, \mathrm{ind}_{R^n_+}(\pi) = \{\mathbf{0}\}$.
Similarly, if $0 \notin \Lambda_j(\pi)$ for each $j \in \{1, \dots, m\}$, then instead of the inequality constraints in $(PM)_j$, we obtain the equalities

$$\langle A_j \pi, x_j \rangle = \phi_j(\pi), \qquad j = 1, \dots, m. \tag{48}$$

Now we formulate conditions under which (47) and (48) are available. For this purpose notice that from (46) it follows that

$$\pi \in \partial \Phi_+^\star\Big(\sum_{j=1}^{m} A_j^T x_j\Big)$$

where $\Phi_+^\star : R^n \to R \cup \{+\infty\}$ is the conjugate of $\Phi + \mathrm{ind}_{R^n_+}$. Hence, we get

$$\Phi_+^\star\Big(\sum_{j=1}^{m} A_j^T x_j\Big) \leq \Phi_+^\star(\mathbf{0}) + \sum_{j=1}^{m} \langle A_j \pi, x_j \rangle \leq \Phi_+^\star(\mathbf{0}) + \sum_{j=1}^{m} \phi_j(\pi)$$
$$\leq \Phi_+^\star(\mathbf{0}) + \sup_{|\tau| \leq M} \sum_{j=1}^{m} \phi_j(\tau) := M_0 < +\infty.$$

This allows the formulation of the following result.

COROLLARY 3 *Assume all the hypotheses of Theorem 3. Moreover, let $\boldsymbol{\pi}, \boldsymbol{x}_j \in \boldsymbol{R}^n_+$ be solutions of $(PM)_j$–(PE), $j = 1, \dots, m$. Then the following conditions are satisfied:*
(i) If for any $\boldsymbol{y} \in \boldsymbol{R}^n_+$ with $\boldsymbol{y} = \sum_{j=1}^m \boldsymbol{A}_j^T \boldsymbol{y}_j$, $\boldsymbol{y}_j \in E_j$, $j = 1, \dots, m$, and $\Phi^\star_+(\boldsymbol{y}) \le M_0$, we have

$$\forall i \in \{1, \dots, n\} \quad \exists j \in \{1, \dots, m\}$$
$$\text{such that } \left(\boldsymbol{A}_j^{-1} \partial V_j(\boldsymbol{y}_j)\right)_i \cap \left(\boldsymbol{A}_j^{-1}(\boldsymbol{R}^n_+)\right)_i = \emptyset, \tag{49}$$

then $\boldsymbol{\pi} > \mathbf{0}$ and instead of (PE) we get the stronger (47) (the symbol $\boldsymbol{A}_j^{-1}$ denotes the inverse of $\boldsymbol{A}_j$).
(ii) If for any $\boldsymbol{y} \in \boldsymbol{R}^n_+$ with $\boldsymbol{y} = \sum_{j=1}^m \boldsymbol{A}_j^T \boldsymbol{y}_j$, $\boldsymbol{y}_j \in E_j$, $j = 1, \dots, m$, and $\Phi^\star_+(\boldsymbol{y}) \le M_0$, we have

$$\forall j \in \{1, \dots, m\} \quad \exists i \in \{1, \dots, n\} \text{ such that } \left(\partial V_j(\boldsymbol{y}_j)\right)_i \cap \boldsymbol{R}_+ = \emptyset, \tag{50}$$

then (48) holds, i.e. the inequality constraints in $(PM)_j$ become, in fact, the equalities.

Proof. We shall have established the assertion if we show that (i) implies that $0 \notin \Lambda_j(\boldsymbol{\pi})$ for each $j = 1, \dots, m$ and that (ii) ensures $\boldsymbol{\pi} > \mathbf{0}$.
From (25) we get easily that for any $j = 1, \dots, m$,

$$-\alpha_j \boldsymbol{A}_j \boldsymbol{\pi} - \boldsymbol{\lambda}_j \in \partial V_j(\boldsymbol{x}_j), \qquad \boldsymbol{\lambda}_j \in \partial\, \mathrm{ind}_{\boldsymbol{R}^n_+}(\boldsymbol{x}_j), \qquad \alpha_j \in \Lambda_j(\boldsymbol{\pi}). \tag{51}$$

First we claim that if (i) holds then for each $j \in \{1, \dots, m\}$, $0 \notin \Lambda_j(\boldsymbol{\pi})$. Indeed, if we assume that for some $j \in \{1, \dots, m\}$, $0 \in \Lambda_j(\boldsymbol{\pi})$ then from (51) we get $-\boldsymbol{\lambda}_j \in \partial V_j(\boldsymbol{x}_j)$. Since $-\boldsymbol{\lambda}_j \ge \mathbf{0}$, the contradiction with (50) follows.
Now suppose that (ii) holds and, on the contrary, for some $i \in \{1, \dots, n\}$ we have $p_i = 0$. From (51) we obtain $\left(\boldsymbol{A}_j^{-1}(-\boldsymbol{\lambda}_j)\right)_i \in \left(\boldsymbol{A}_j^{-1} \partial V_j(\boldsymbol{x}_j)\right)_i$ for each $j = 1, \dots, m$. Thus, in particular, if j corresponds to i as stated in (49), we arrive at the contradiction because $-\boldsymbol{\lambda}_j \ge \mathbf{0}$. The proof is complete. ■

Now we consider the case in which only the trivial solution $\boldsymbol{\pi} = \mathbf{0}$ is available.

COROLLARY 4 *Assume all the hypotheses of Theorem 3. Moreover, suppose that for any $\boldsymbol{x}_{j0} \in \partial \overline{V}_j^\star(\mathbf{0})$, $j = 1, \dots, m$,*

$$\left\langle \boldsymbol{\tau}, \sum_{j=1}^m \boldsymbol{A}_j^T \boldsymbol{x}_{j0} \right\rangle < \Phi(\boldsymbol{\tau}) - \Phi(\mathbf{0}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{R}^n_+ \setminus \{\mathbf{0}\}. \tag{52}$$

Then $\boldsymbol{\pi} = \mathbf{0}$ and $\boldsymbol{x}_j \in \partial \overline{V}_j^\star(\mathbf{0})$, $j = 1, \dots, m$, are the only solutions of $(PM)_j$–(PE).

Proof. On the contrary, suppose that $\pi \neq 0$ and $x_j \in \partial \overline{V}_j^\star(-\Lambda_j(\pi)A_j\pi)$ are solutions of $(PM)_j$–(PE), $j = 1, \dots, m$. If $0 \notin \Lambda_j(\pi)$, then $\langle A_j\pi, x_{j0}\rangle > \phi_j(\pi)$ for some $x_{j0} \in \partial \overline{V}_j^\star(0)$ and $\langle A_j\pi, x_j\rangle = \phi_j(\pi)$. This allows us to define

$$\widetilde{x}_{j0} = \begin{cases} x_j & \text{if } 0 \in \Lambda_j(\pi) \\ x_{j0} & \text{if } 0 \notin \Lambda_j(\pi), \qquad j = 1, \dots, m, \end{cases}$$

with the properties that $\widetilde{x}_{j0} \in \partial \overline{V}_j^\star(0)$ and $\langle A_j\pi, \widetilde{x}_{j0}\rangle \geq \langle A_j\pi, x_j\rangle$ for any $j = 1, \dots, m$. Hence

$$\Big\langle \pi, \sum_{j=1}^m A_j^T \widetilde{x}_{j0} \Big\rangle \geq \Big\langle \pi, \sum_{j=1}^m A_j^T x_j \Big\rangle. \tag{53}$$

Now, from (PE), by substituting $\tau = 0$, we get

$$\Big\langle \pi, \sum_{j=1}^m A_j^T x_j \Big\rangle \geq \Phi(\pi) - \Phi(0).$$

Combining this with (53) yields

$$\Big\langle \pi, \sum_{j=1}^m A_j^T \widetilde{x}_{j0} \Big\rangle \geq \Phi(\pi) - \Phi(0),$$

which, due to $\pi \neq 0$, contradicts the hypothesis (52). The proof is complete. ■

REMARK Notice that the assumption (52) is stronger than $\sum_{j=1}^m A_j^T x_{j0} \in \partial\Phi_+(0)$.

6. Ideal minimum

Let us define a vector function $V : R^n \to (R \cup \{+\infty\})^2$ as

$$V(x) := \big(U(x), W(x)\big), \qquad x \in R^n_+, \tag{54}$$

and let E be a nonempty subset of R^n_+.

Recall that $x^\star \in E$ is said to be Pareto optimal if there are no other $x \in E$ such that (see Aubin, 1993):

$$U(x) \leq U(x^\star) \text{ and } W(x) \leq W(x^\star),$$

with at least one of the foregoing inequalities being strict. An element $x^\star \in E$ is said to be an ideal minimum on E if

$$V(x^\star) \leq V(x), \quad \forall x \in E.$$

Now we consider the problem concerning the existence of an ideal minimum: For a given $\boldsymbol{S} > \mathbf{0}$ find $E \subset \{\boldsymbol{y} \in \boldsymbol{R}^n_+ : \boldsymbol{y} \leq \boldsymbol{S}\}$ and $\boldsymbol{x}^\star \in E$ such that

$$\boldsymbol{V}(\boldsymbol{x}^\star) \leq \boldsymbol{V}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in E, \tag{55}$$

i.e. $\boldsymbol{x}^\star$ is an ideal minimum of the vectorial objective $\boldsymbol{V}$ on E.

In order to get the existence result for (55) an auxiliary problem will be formulated. Let us define

$$V_1(\boldsymbol{x}) := U(\boldsymbol{x}) \quad \text{and} \quad V_2(\boldsymbol{x}) := W(\boldsymbol{S} - \boldsymbol{x}), \qquad \boldsymbol{x} \in \boldsymbol{R}^n_+, \tag{56}$$

and consider the problem: Find $\boldsymbol{\pi} \in \boldsymbol{R}^n_+$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^n_+$ such that

$$V_j(\boldsymbol{x}_j) = \min\{V_j(\boldsymbol{y}) \colon \boldsymbol{y} \in \boldsymbol{R}^n_+ \text{ and } \langle \boldsymbol{\pi}, \boldsymbol{y} \rangle \leq B\}, \qquad j = 1, 2, \tag{57}$$

$$\boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{S}. \tag{58}$$

The existence of solutions for the problem (57)–(58) can be derived from Corollary 3. Indeed, when setting $\boldsymbol{A}_j = \boldsymbol{I}$, $\phi_j(\boldsymbol{\pi}) \equiv B > 0$, $j = 1, 2$, and

$$\Phi(\boldsymbol{\tau}) := \langle \boldsymbol{S}, \boldsymbol{\tau} \rangle, \qquad \boldsymbol{\tau} \in \boldsymbol{R}^n, \tag{59}$$

we get $\Phi^\star_+ = (\Phi + \mathrm{ind}_{\boldsymbol{R}^n_+})^\star = \mathrm{ind}_{\leq \boldsymbol{S}}$, where $\mathrm{ind}_{\leq \boldsymbol{S}}$ stands for the indicator function of $\{\boldsymbol{y} \in \boldsymbol{R}^n_+ : \boldsymbol{y} \leq \boldsymbol{S}\}$. Now from Corollary 3 one can obtain the result..

Proposition 5 *Assume that V_1 and V_2 given by (56) fulfill (29)–(31). Moreover, let $\Phi \colon \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ be defined by (59). If for any $\boldsymbol{y} \in \boldsymbol{R}^n_+$ with $\boldsymbol{y} \leq \boldsymbol{S}$ it holds that*

$$\forall i \in \{1, \dots, n\} \quad \exists j \in \{1, 2\} \ \text{ such that } \ \left(\partial V_j(\boldsymbol{y})\right)_i \cap \boldsymbol{R}_+ = \emptyset, \tag{60}$$

then there exist $\boldsymbol{\pi} > \mathbf{0}$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^n_+$, such that (57)–(58) hold. If, in addition,

$$\forall j \in \{1, 2\} \quad \exists i \in \{1, \dots, n\} \ \text{ such that } \ \left(\partial V_j(\boldsymbol{y})\right)_i \cap \boldsymbol{R}_+ = \emptyset, \tag{61}$$

then the inequality constraints in (57) become the equalities, i.e. $\langle \boldsymbol{\pi}, \boldsymbol{x}_j \rangle = B$, $j = 1, 2$.

Now observe that if $\boldsymbol{\pi}$ and $\boldsymbol{x}_1, \boldsymbol{x}_2$ are the solutions of (57)–(58) then $\boldsymbol{x}^\star := \boldsymbol{x}_1$ is an ideal minimum of $\boldsymbol{V}$ in $E = \{\boldsymbol{y} \in \boldsymbol{R}^n_+ : \langle \boldsymbol{\pi}, \boldsymbol{y} \rangle \leq B \text{ and } \boldsymbol{y} \leq \boldsymbol{S}\}$.

The results obtained allow for the formulation of the following theorem:

Theorem 4 *Assume that $U : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ and $W : \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ are such that the corresponding V_1 and V_2 given by (56) fulfill all the hypotheses of Theorem 3, i.e. (29)–(31) hold (notice that (39) holds immediately). Moreover, let for any $\boldsymbol{y} \in \boldsymbol{R}^n_+$ with $\boldsymbol{y} \leq \boldsymbol{S}$ the following hold*

$$\forall i \in \{1, \dots, n\} \quad \exists j \in \{1, 2\} \ \text{ such that } \ \left(\partial V_j(\boldsymbol{y})\right)_i \cap \boldsymbol{R}_+ = \emptyset, \tag{62}$$

then there exists $\boldsymbol{\pi} > \mathbf{0}$ *such that the problem (55) admits at least one ideal minimum* $\boldsymbol{x}^\star$ *on* $E := \{\boldsymbol{y} \in \boldsymbol{R}_+^n : \langle \boldsymbol{\pi}, \boldsymbol{y} \rangle \leq B \text{ and } \boldsymbol{y} \leq \boldsymbol{S}\}$, *i.e.*

$$\big(U(\boldsymbol{x}^\star), W(\boldsymbol{x}^\star)\big) \leq \big(U(\boldsymbol{y}), W(\boldsymbol{y})\big), \quad \forall\, \boldsymbol{y} \in E. \tag{63}$$

If, in addition, for any $\boldsymbol{y} \in \boldsymbol{R}_+^n$ *with* $\boldsymbol{y} \leq \boldsymbol{S}$,

$$\forall j \in \{1, 2\} \quad \exists i \in \{1, \dots, n\} \text{ such that } \big(\partial V_j(\boldsymbol{y})\big)_i \cap \boldsymbol{R}_+ = \emptyset, \tag{64}$$

then $\langle \boldsymbol{\pi}, \boldsymbol{x}^\star \rangle = B$.

7. Quadratic functions

Consider the case in which V_j are quadratic, $\boldsymbol{A}_j = \boldsymbol{I}$, $j = 1, \dots, m$, $\boldsymbol{I}$ being the identity and $\Phi(\boldsymbol{\tau}) = \langle \boldsymbol{S}, \boldsymbol{\tau} \rangle$ with $\boldsymbol{S} \in \boldsymbol{R}_+^n$ and $\boldsymbol{S} > \mathbf{0}$, $\phi_j(\boldsymbol{\tau}) \equiv B_j > 0$, $j = 1, \dots, m$. We assume V_j to be of the form:

$$V_j(\boldsymbol{x}) := \frac{1}{2} \langle \boldsymbol{C}_j \boldsymbol{x}, \boldsymbol{x} \rangle - \langle \boldsymbol{D}_j, \boldsymbol{x} \rangle, \qquad \boldsymbol{x} \in \boldsymbol{R}_+^n, \tag{65}$$

where $\boldsymbol{C}_j \in \boldsymbol{R}_{sym}^{n \times n}$ is a symmetric $n \times n$, positive definite matrix, $\boldsymbol{D}_j \in \boldsymbol{R}_+^n$.

Let us introduce the notations: for any $\boldsymbol{X} = (X_i) \in \boldsymbol{R}^n$, $[\boldsymbol{X}]^+ = (X_i^+)$, $[\boldsymbol{X}]^- = (X_i^-)$, where

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0, \end{cases} \qquad X^- = \begin{cases} X & \text{if } X \leq 0 \\ 0 & \text{if } X > 0. \end{cases}$$

It is not difficult to check that V_j, $j = 1, \dots, m$, given by (65), fulfill all the requirements of Theorem 3. In this case we have

$$\partial V_j(\boldsymbol{x}) = \boldsymbol{C}_j \boldsymbol{x} - \boldsymbol{D}_j, \qquad \boldsymbol{x} \in \boldsymbol{R}_+^n \tag{66}$$

$$\begin{aligned} \partial \overline{V}_j^\star(\boldsymbol{\mu}) &= \Big(\partial V_j + \partial\, \mathrm{ind}_{\boldsymbol{R}_+^n}\Big)^{-1}(\boldsymbol{\mu}) \\ &= \big[\boldsymbol{C}_j^{-1} \boldsymbol{\mu} + \boldsymbol{C}_j^{-1} \boldsymbol{D}_j\big]^+, \qquad \boldsymbol{\mu} \in \boldsymbol{R}_-^n. \end{aligned} \tag{67}$$

In particular,

$$\partial \overline{V}_j^\star(0) = \big[\boldsymbol{C}_j^{-1} \boldsymbol{D}_j\big]^+.$$

Further, $\Lambda_j(\boldsymbol{\pi}) = \{\alpha_j(\boldsymbol{\pi})\}$, $j = 1, \dots, m$, where $\alpha_j(\cdot)$ are continuous, bounded functions, determined by the conditions:

$$\begin{aligned} &\langle \boldsymbol{\pi}, -[\boldsymbol{C}_j^{-1} \boldsymbol{D}_j]^+ \rangle + B_j \geq 0 \implies \alpha_j = 0, \\ &\langle \boldsymbol{\pi}, -[\boldsymbol{C}_j^{-1} \boldsymbol{D}_j]^+ \rangle + B_j < 0 \implies \langle \boldsymbol{\pi}, [\alpha_j \boldsymbol{C}_j^{-1} \boldsymbol{\pi} - \boldsymbol{C}_j^{-1} \boldsymbol{D}_j]^- \rangle + B_j = 0. \end{aligned} \tag{68}$$

The operator $\mathcal{R}$ takes the form

$$\mathcal{R}(\boldsymbol{\mu}) = \sum_{j=1}^{m} \big[\alpha_j(\boldsymbol{\mu}) \boldsymbol{C}_j^{-1}\boldsymbol{\mu} - \boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{-}, \qquad \boldsymbol{\mu} \in \boldsymbol{R}_{+}^{n}. \tag{69}$$

If $\boldsymbol{\pi} \in \boldsymbol{R}_{+}^{n}$ is a solution of (44) with $\mathcal{R}$ given by (69), then the corresponding $\boldsymbol{x}_j$, $j = 1, \ldots, m$, can be derived from (43), which in the case considered takes the form

$$\boldsymbol{x}_j = -\big[\alpha_j(\boldsymbol{\pi}) \boldsymbol{C}_j^{-1}\boldsymbol{\pi} - \boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{-}, \qquad j = 1, \ldots, m.$$

All these results can be summarized as follows.

THEOREM 5 *Suppose that V_j are defined by (65), where $\boldsymbol{C}_j \in \boldsymbol{R}_{sym}^{n\times n}$ are assumed to be symmetric $n \times n$, positive definite matrices, $\boldsymbol{D}_j \in \boldsymbol{R}_{+}^{n}$, $B_j > 0$, $j = 1, \ldots, m$, $\Phi(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}, \boldsymbol{S}\rangle$, $\boldsymbol{\tau} \in \boldsymbol{R}_{+}^{n}$, where $\boldsymbol{S} > \boldsymbol{0}$, $i = 1, \ldots, n$. Then there exist at least one $\boldsymbol{\pi} \in \boldsymbol{R}_{+}^{n}$ and the corresponding $\boldsymbol{x}_j \in \boldsymbol{R}_{+}^{n}$ fulfilling the conditions:*

$$V_j(\boldsymbol{x}_j) = \min\{V_j(\boldsymbol{y})\colon \boldsymbol{y} \in \boldsymbol{R}_{+}^{n} \text{ and } \langle \boldsymbol{\pi}, \boldsymbol{y}\rangle \le B_j\},\ j = 1, \ldots, m, \tag{70}$$

$$\Big\langle -\sum_{j=1}^{m} \boldsymbol{x}_j + \boldsymbol{S},\, \boldsymbol{\tau} - \boldsymbol{\pi}\Big\rangle \ge 0, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}_{+}^{n}. \tag{71}$$

Moreover, the conditions

$$\begin{aligned} &\langle \boldsymbol{\tau}, -\big[\boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{+}\rangle + B_j \ge 0 \implies \alpha_j = 0, \\ &\langle \boldsymbol{\tau}, -\big[\boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{+}\rangle + B_j < 0 \implies \langle \boldsymbol{\tau}, \big[\alpha_j \boldsymbol{C}_j^{-1}\boldsymbol{\pi} - \boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{-}\rangle + B_j = 0, \end{aligned} \tag{72}$$

determine bounded, continuous functions $\alpha_j(\cdot)$, $j = 1, \ldots, m$, with the property that $\boldsymbol{\pi}$ is a solution of the variational inequality

$$\Big\langle \boldsymbol{\tau} - \boldsymbol{\pi},\, \sum_{j=1}^{m} \big[\alpha_j(\boldsymbol{\pi}) \boldsymbol{C}_j^{-1}\boldsymbol{\pi} - \boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{-} + \boldsymbol{S}\Big\rangle \ge 0, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{R}_{+}^{n}, \tag{73}$$

and $\boldsymbol{x}_j$, $j = 1, \ldots, m$, are given by the formulas

$$\boldsymbol{x}_j = -\big[\alpha_j(\boldsymbol{\pi}) \boldsymbol{C}_j^{-1}\boldsymbol{\pi} - \boldsymbol{C}_j^{-1}\boldsymbol{D}_j\big]^{-}, \qquad j = 1, \ldots, m. \tag{74}$$

Now we consider the question concerning conditions under which a solution $\boldsymbol{\pi} = [p_1, \ldots, p_n] \in \boldsymbol{R}_{+}^{n}$ of (73) has positive coordinates, i.e. $p_i > 0$ for all

$j = 1, \dots, m$. Notice that in such a case instead of variational inequality (73) the equality

$$\sum_{j=1}^{m} \left[\alpha_j(\pi) C_j^{-1} \pi - C_j^{-1} D_j\right]^{-} + S = 0 \tag{75}$$

results, or equivalently, $\sum_{j=1}^{m} x_j = S$.

To formulate the pertinent result, Corollary 3 will be used. For this purpose we have to adopt its hypotheses to our case.

Firstly, we check that $\Phi_+^\star = \mathrm{ind}_{\leq S}$, where $\mathrm{ind}_{\leq S}$ is the indicator function of $\{y \in R^n : y \leq S\}$. Secondly, let us assume that

$$\forall i \in \{1, \dots, n\}\ \forall j \in \{1, \dots, m\} \qquad (D_j)_i > 0, \tag{76}$$

and define $\mathcal{K}$ by setting

$$y \in \mathcal{K} \iff \forall i \in \{1, \dots, n\}\ \forall j \in \{1, \dots, m\} \qquad (C_j^{-1} y - D_j)_i < 0.$$

Further, for any $S > 0$ define $K_S := \{y \in R_+^n : y \leq S\}$. Due to (76), for sufficiently small S we have $K_S \subset \mathcal{K}$. On the basis of Corollary 3 we are allowed to formulate the following result.

COROLLARY 5 *Assume all the hypotheses of Theorem 5 and suppose that (76) holds. Then for any $S > 0$ with $K_S \subset \mathcal{K}$ the problem: Find $\pi > 0$ and $x_j \in R_+^n$, $j = 1, \dots, m$, such that*

$$V_j(x_j) = \min\{V_j(y) : y \in R_+^n \text{ and } \langle \pi, y \rangle \leq B_j\}, \qquad j = 1, \dots, m, \tag{77}$$

and

$$\sum_{j=1}^{m} x_j = S, \tag{78}$$

has at least one solution. Moreover, $\langle \pi, x_j \rangle = B_j$, $j = 1, \dots, m$.

Consider the case in which S is large enough. On the basis of Corollary 4 one can formulate the result.

COROLLARY 6 *Assume that all the hypotheses of Theorem 5 hold. Moreover, suppose that*

$$S > \sum_{j=1}^{m} [C_j^{-1} D_j]^{+}. \tag{79}$$

Then the only solution of (70)–(71) is the system $\pi = 0$ and $x_j = [C_j^{-1} D_j]^{+}$, $j = 1, \dots, m$.

Proof. Recall that $\{x_j\} = \partial\overline{V}_j^\star(0)$, where $x_j = [C_j^{-1}D_j]^+$. Thus (79) can be written as $-\sum_{j=1}^m x_j + S > 0$, or equivalently,

$$\langle S, \tau\rangle > \Big\langle \sum_{j=1}^m x_j, \tau\Big\rangle, \quad \forall \tau \in R_+^n \setminus \{0\}.$$

Therefore (52) is fulfilled and finally, by Corollary 4, the assertion follows. ■

8. Examples

Assume $V^j : R^2 \to R$, $j = 1, 2$, to be of the form

$$V^j(x) := \frac{1}{2}[x_1, x_2]\begin{bmatrix} C_1^j & 0 \\ 0 & C_2^j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [D_1^j, D_2^j]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x \in R_+^2, \tag{80}$$

where $C_1^j, C_2^j > 0$ and $D_1^j, D_2^2 > 0$, $j = 1, 2$. Moreover we assume that $\Phi : R_+^2 \to R$ is given by $\Phi(\tau) = \langle S, \tau\rangle$, $\tau \in R_+^2$, where $S = [S_1, S_2]$, $S_1, S_2 > 0$. Then the corresponding $\alpha^j(\cdot)$, $j = 1, 2$, derived from (72), are given by

$$\alpha^j(\pi) = \begin{cases} 0 & \text{if } p_1C_2^jD_1^j + p_2C_1^jD_2^j \\ & \quad -BC_1^jC_2^j \le 0 \\ \dfrac{p_1D_1^j - B^jC_1^j}{(p_1)^2} & \text{if } \begin{cases} p_2 > \dfrac{D_2^j(p_1)^2}{p_1D_1^j - B^jC_1^j} \\ p_1 > \dfrac{BC_1^j}{D_1^j} \end{cases} \\ \dfrac{p_2D_2^j - B^jC_2^j}{(p_2)^2} & \text{if } \begin{cases} p_1 > \dfrac{D_1^j(p_2)^2}{p_2D_2^j - BC_2^j} \\ p_2 > \dfrac{B^jC_2^j}{D_2^j} \end{cases} \\ \dfrac{p_1C_2^jD_1^j + p_2C_1^jD_2^j - B^jC_1^jC_2^j}{C_2^j(p_1)^2 + C_1^j(p_2)^2} & \text{otherwise.} \end{cases} \tag{81}$$

Governing relation (73) leads to finding $\pi \in R_+^2$ such that

$$\begin{aligned} &(\tau_1 - p_1)\bigg(\bigg[\frac{\alpha^1(\pi)p_1 - D_1^1}{C_1^1}\bigg]^- + \bigg[\frac{\alpha^2(\pi)p_1 - D_1^2}{C_1^2}\bigg]^- + S_1\bigg) + \\ &+(\tau_2 - p_2)\bigg(\bigg[\frac{\alpha^1(\pi)p_2 - D_2^1}{C_2^1}\bigg]^- + \bigg[\frac{\alpha^2(\pi)p_2 - D_2^2}{C_2^2}\bigg]^- + S_2\bigg) \\ &\ge 0, \qquad \forall \tau \in R_+^2. \end{aligned} \tag{82}$$

The strategy for finding solutions of (82) is to consider each of the cases:
Case 1. $\lambda_1 := D_1^1 C_1^2 + D_1^2 C_1^1 - S_1 C_1^1 C_1^2 > 0$ and
$\lambda_2 := D_2^1 C_2^2 + D_2^2 C_2^1 - S_2 C_2^1 C_2^2 > 0$,
Case 2. $\lambda_1 > 0$ and $\lambda_2 < 0$,
Case 3. $\lambda_1 > 0$ and $\lambda_2 = 0$,
Case 4. $\lambda_1 < 0$ and $\lambda_2 < 0$,
separately, and to extend the results by symmetry. In this paper we confine ourselves to the most important cases leaving the remaining ones to the reader.

Solutions:

CASE 1. If $\lambda_1 > 0$ and $\lambda_2 > 0$ then (82) reduces to the system of equations:

$$\begin{aligned} 0 &= \frac{\alpha^1(\pi)p_1 - D_1^1}{C_1^1} + \frac{\alpha^2(\pi)p_1 - D_1^2}{C_1^2} + S_1 \\ 0 &= \frac{\alpha^1(\pi)p_2 - D_2^1}{C_2^1} + \frac{\alpha^2(\pi)p_2 - D_2^2}{C_2^2} + S_2. \end{aligned}$$

A. If we are looking for solutions in $\Gamma_A \subset \boldsymbol{R}_+^2$, where

$$\pi \in \Gamma_A \iff \left(\alpha^1(\pi) = \frac{p_1 D_1^1 - B^1 C_1^1}{(p_1)^2}, \ \alpha^2(\pi) = \frac{p_1 D_1^2 - B^2 C_1^2}{(p_1)^2}\right),$$

then we get

$$\begin{aligned} p_1 &= \frac{B^1 + B^2}{S_1} \\ p_2 &= \frac{(D_2^1 C_2^2 + D_2^2 C_2^1 - S_2 C_2^1 C_2^2)(B^1 + B^2)^2}{S_1 B^1 (D_1^1 C_2^2 + D_1^2 C_2^1 - S_1 C_1^1 C_2^2) + S_1 B^2 (D_1^1 C_2^2 + D_1^2 C_2^1 - S_1 C_1^2 C_2^1)} \\ \alpha^1(\pi) &= \frac{S_1}{(B^1 + B^2)^2}(B^1 D_1^1 + B^2 D_1^1 - B^1 C_1^1 S_1) \\ \alpha^2(\pi) &= \frac{S_1}{(B^1 + B^2)^2}(B^1 D_1^2 + B^2 D_1^2 - B^2 C_1^2 S_1), \end{aligned}$$

provided that $[p_1, p_2] \in \Gamma_A$.

B. If we are looking for solutions in $\Gamma_B \subset \boldsymbol{R}_+^2$, where

$$\pi \in \Gamma_B \iff \left(\alpha^1(\pi) = \frac{p_1 D_1^1 - B^1 C_1^1}{(p_1)^2}, \ \alpha^2(\pi) = 0\right),$$

then we obtain

$$p_1 = \frac{B^1 C_1^2}{S_1 C_1^2 - D_1^2}$$
$$p_2 = \frac{B^1 (C_1^2)^2 \big(C_2^2 D_2^1 - C_2^1 (D_2^2 - S_2 C_2^2)\big)}{C_2^2 (S_1 C_1^2 - D_1^2)\big(C_1^2 D_1^1 - C_1^1 (S_1 C_1^2 - D_1^2)\big)},$$
$$\alpha^1(\boldsymbol{\pi}) = \frac{S_1 C_1^2 - D_1^2}{B^1 (C_1^2)^2} \big(C_1^2 D_1^1 - C_1^1 (S_1 C_1^2 - D_1^2)\big),$$

provided that $[p_1, p_2] \in \Gamma_B$.

C. If we are looking for solutions in $\Gamma_C \subset \boldsymbol{R}_+^2$, where

$$\boldsymbol{\pi} \in \Gamma_C \iff \left(\alpha^1(\boldsymbol{\pi}) = \frac{p_1 D_1^1 - B^1 C_1^1}{(p_1)^2},\ \alpha^2(\boldsymbol{\pi}) = \frac{p_2 D_2^2 - B^2 C_2^2}{(p_2)^2}\right), \quad (83)$$

and, moreover, $\gamma_0 := C_1^2 C_2^1 - C_1^1 C_2^2 \neq 0$, the results are

$$p_1 = \frac{t^2 B^2 C_2^2 \gamma_0}{t D_2^2 \gamma_0 - t \lambda_2 C_1^2 + \lambda_1 C_2^2}$$
$$p_2 = \frac{t B^2 C_2^2 \gamma_0}{t D_2^2 \gamma_0 - t \lambda_2 C_1^2 + \lambda_1 C_2^2},$$

where $t > 0$ is a positive solution of the equation

$$t^3 \lambda_2 B^2 C_1^1 C_2^2 + t^2 B^2 C_2^2 \big(D_1^1 \gamma_0 - \lambda_1 C_2^1\big) - t B^1 C_1^1 \big(D_2^2 \gamma_0 - \lambda_2 C_1^2\big)$$
$$- \lambda_1 B^1 C_1^1 C_2^2 = 0,$$

provided that $[p_1, p_2] \in \Gamma_C$.

D. If we are looking for solutions in $\Gamma_D \subset \boldsymbol{R}_+^2$, where

$$\boldsymbol{\pi} \in \Gamma_D \iff \left(\begin{array}{l} \alpha^1(\boldsymbol{\pi}) = \dfrac{p_1 C_2^1 D_1^1 + p_2 C_1^1 D_2^1 - B^1 C_1^1 C_2^1}{C_2^1 (p_1)^2 + C_1^1 (p_2)^2} \\ \alpha^2(\boldsymbol{\pi}) = \dfrac{p_2 D_2^2 - B^2 C_2^2}{(p_2)^2} \end{array}\right),$$

and, moreover, $\gamma_0 \neq 0$, we get

$$p_1 = \frac{t^2 B^2 C_2^2 \gamma_0}{t D_2^2 \gamma_0 - t \lambda_2 C_1^2 + \lambda_1 C_2^2}$$
$$p_2 = \frac{t B^2 C_2^2 \gamma_0}{t D_2^2 \gamma_0 - t \lambda_2 C_1^2 + \lambda_1 C_2^2}$$

where $t > 0$ is a positive solution of the equation

$$t^3\lambda_2 B^2 C_1^2 C_2^1 C_2^2 + t^2 B^2 C_2^2\big(\lambda_1 (C_2^1)^2 - \gamma_0 C_2^1 D_1^1 - \gamma_0 C_1^1 D_2^1\big) \\ +t\big(\lambda_2 B^2 C_1^2 C_2^2 + B^1 C_1^1 C_2^1(\gamma_0 D_2^2 - \lambda_2 C_1^2)\big) + \lambda_1 (B^1 + B^2) C_1^1 C_2^2 C_2^1 = 0,$$

provided that $[p_1, p_2] \in \Gamma_D$.

E. If we are looking for solutions in $\Gamma_E \subset \boldsymbol{R}_+^2$, where

$$\pi \in \Gamma_E \iff \left(\begin{array}{l} \alpha^1(\pi) = \dfrac{p_1 C_2^1 D_1^1 + p_2 C_1^1 D_2^1 - B^1 C_1^1 C_2^1}{C_2^1 (p_1)^2 + C_1^1 (p_2)^2} \\ \alpha^2(\pi) = \dfrac{p_1 C_2^2 D_1^2 + p_2 C_1^2 D_2^2 - B^2 C_1^2 C_2^2}{C_2^2 (p_1)^2 + C_1^2 (p_2)^2} \end{array} \right),$$

and, moreover, $\gamma_0 \neq 0$, the following results are available:

$$p_1 = \frac{t^2 \gamma_0 B^1 C_1^1 C_2^1}{t^2\gamma_0 (C_2^1 D_1^1 + C_1^1 D_2^1) - (\lambda_1 C_2^1 + t\lambda_2 C_1^1)(t^2 C_2^1 + C_1^1)}$$
$$p_2 = \frac{t \gamma_0 B^1 C_1^1 C_2^1}{t^2\gamma_0 (C_2^1 D_1^1 + C_1^1 D_2^1) - (\lambda_1 C_2^1 + t\lambda_2 C_1^1)(t^2 C_2^1 + C_1^1)}$$

where $t > 0$ is a positive solution of the equation

$$\frac{t^2\gamma_0 (C_2^1 D_1^1 + C_1^1 D_2^1) - (\lambda_1 C_2^1 + t\lambda_2 C_1^1)(t^2 C_2^1 + C_1^1)}{B^1 C_1^1 C_2^1} \\ = \frac{t^2\gamma_0 (C_2^2 D_1^2 + C_1^2 D_2^2) - (\lambda_1 C_2^2 + t\lambda_2 C_1^2)(t^2 C_2^2 + C_1^2)}{B^2 C_1^2 C_2^2},$$

provided that $[p_1, p_2] \in \Gamma_E$.

CASE 2. $\lambda_1 > 0$ and $\lambda_2 < 0$.
In this case we have

$$\left[\frac{\alpha^1(\pi) p_2 - D_2^1}{C_2^1}\right]^- + \left[\frac{\alpha^2(\pi) p_2 - D_2^2}{C_2^2}\right]^- + S_2 > 0, \quad \forall \pi \in \boldsymbol{R}_+^2,$$

therefore p_2 has to be 0. This implies that $p_1 > 0$ and at least one of α's has to be positive in order to fulfill the equation

$$\left[\frac{\alpha^1(\pi) p_1 - D_1^1}{C_1^1}\right]^- + \left[\frac{\alpha^2(\pi) p_1 - D_1^2}{C_1^2}\right]^- + S_1 = 0.$$

A. If $B^2D_1^1 - B^1(S_1C_1^1 - D_1^1) > 0$ and $B^1D_1^2 - B^2(S_1C_1^2 - D_1^2) > 0$ then

$$\begin{aligned}
p_1 &= \frac{B^1 + B^2}{S_1} \\
p_2 &= 0 \\
\alpha^1(\boldsymbol{\pi}) &= \frac{S_1}{(B^1 + B^2)^2}\left(B^2D_1^1 - B^1(S_1C_1^1 - D_1^1)\right) \\
\alpha^2(\boldsymbol{\pi}) &= \frac{S_1}{(B^1 + B^2)^2}\left(B^1D_1^2 - B^2(S_1C_1^2 - D_1^2)\right).
\end{aligned}$$

B. If $B^1 \geq B^2/(S_1C_1^1 - D_1^1) > 0$ and $C_1^1D_1^2 - C_1^2(S_1C_1^1 - D_1^1) > 0$ then

$$\begin{aligned}
p_1 &= \frac{B^2C_1^1}{S_1^1C_1^1 - D_1^1} \\
p_2 &= 0 \\
\alpha^1(\boldsymbol{\pi}) &= 0 \\
\alpha^2(\boldsymbol{\pi}) &= \frac{S_1C_1^1 - D_1^1}{B^2(C_1^1)^2}\left(C_1^1D_1^2 - C_1^2(S_1C_1^1 - D_1^1)\right).
\end{aligned}$$

Case 3. $\lambda_1 > 0$ and $\lambda_2 = 0$.
This case can be treated analogously as the previous one due to the fact that matrices $\boldsymbol{C}^j$, $j = 1, 2$, are diagonal (only $p_2 = 0$ is admissible).

Case 4. $\lambda_1 < 0$ and $\lambda_2 < 0$.
This is a trivial case with the only solution $p_1 = p_2 = 0$.

In order to get $\boldsymbol{x}^j$, $j = 1, 2$, we use the formulas

$$\begin{aligned}
\boldsymbol{x}^1 &= -\left[\frac{[\alpha^1(\boldsymbol{\pi})p_1 - D_1^1]^-}{C_1^1}, \frac{[\alpha^1(\boldsymbol{\pi})p_2 - D_2^1]^-}{C_2^1}\right] \\
\boldsymbol{x}^2 &= -\left[\frac{[\alpha^2(\boldsymbol{\pi})p_1 - D_1^2]^-}{C_1^2}, \frac{[\alpha^2(\boldsymbol{\pi})p_2 - D_2^2]^-}{C_2^2}\right],
\end{aligned}$$

derived from (74).

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